Darboux covariance: a hidden symmetry of perturbed Schwarzschild Black Holes

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M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021), M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)

Einstein perturbed equations

$$\delta\, G_{\mu\nu} + \Lambda\, h_{\mu\nu} = 0$$

$\longrightarrow$

Master equation

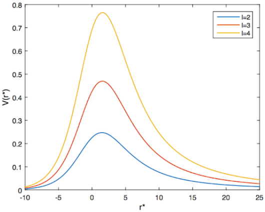
$$(\Box_2 - \Omega)\, \Psi(h, h', ...) = 0$$

Outline

Motivations

Motivations

- Perturbation theory in GR is a fundamental tool to describe physical systems and make predictions
- Scattering from BH
- Quasi-normal modes and gravitational waves (energy and angular momentum fluxes)



Perturbation theory of vacuum spherically symmetric spacetimes

Steps towards decoupling

- Perturbed Einstein equations at linear order

$$g_{\mu\nu} = \widehat{g}_{\mu\nu} + h_{\mu\nu} \quad \longrightarrow \quad \begin{cases} \widehat{G}_{\mu\nu} + \Lambda \widehat{g}_{\mu\nu} = 0 \\ \delta G_{\mu\nu} + \Lambda h_{\mu\nu} = 0 \end{cases}$$

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- Metric splitting reflecting spherical symmetry

$$\widehat{g}_{\mu\nu} = \begin{pmatrix} g_{ab} & 0 \\ 0 & r^2 \Omega_{AB} \end{pmatrix} \quad \longrightarrow \quad \begin{array}{l} g_{ab}\mathrm{d}x^a \mathrm{d}x^b = -f(r)\,\mathrm{d}t^2 + \mathrm{d}r^2/f(r) \\ \Omega_{AB}\mathrm{d}\Theta^A \mathrm{d}\Theta^B = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2 \end{array}$$

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- Harmonics expansion of perturbed metric

$$h_{\mu\nu} = \sum_{\ell,m} h_{\mu\nu}^{\ell m,\mathrm{odd}} + h_{\mu\nu}^{\ell m,\mathrm{even}}$$

Harmonic decoupled Einstein equations

$$\delta G_{ab}^{\ell m}(x^c, \Theta^A) = \mathcal{E}_{ab}^{\ell m}(x^c)\, Y^{\ell m}(\Theta^A)$$

$$\delta G_{aA}^{\ell m}(x^b, \Theta^B) = \mathcal{E}_{a}^{\ell m}(x^b)\, Y_A^{\ell m}(\Theta^B) + \mathcal{O}_{a}^{\ell m}(x^b)\, X_A^{\ell m}(\Theta^B)$$

$$\delta G_{AB}^{\ell m}(x^a, \Theta^C) = \mathcal{E}_{T}^{\ell m}(x^a)\, T_{AB}^{\ell m}(\Theta^C) + \mathcal{E}_{Y}^{\ell m}(x^a)\, Y_{AB}^{\ell m}(\Theta^C) + \mathcal{O}_{X}^{\ell m}(x^a)\, X_{AB}^{\ell m}(\Theta^C)$$

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Known master functions

Odd parity

$$\Psi_{\mathrm{RW}} = \frac{r^a}{r}\tilde{h}_a$$
$$\Psi_{\mathrm{CPM}} = \frac{2r}{(\ell-1)(\ell+2)}\varepsilon^{ab}\left(\tilde{h}_{b:a} - \frac{2}{r}r_a\tilde{h}_b\right)$$

T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063–1069 (1957), C. T. Cunningham et al., Astrophys. J. **224**, 643–667 (1978)

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Even parity

$$\Psi_{\mathrm{ZM}} = \frac{2r}{\ell(\ell+1)}\left\{\tilde{K} + \frac{2}{\lambda}\left(r^a r^b \tilde{h}_{ab} - r r^a \tilde{K}_{:a}\right)\right\}$$

F. J. Zerilli, Phys. Rev. D **2**, 2141–2160 (1970), V. Moncrief, Ann. Phys. (N.Y.) **88**, 323 (1974)

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$\tilde{h}_a = h_a - \frac{1}{2}h_{2:a} + \frac{r_a}{r}h_2\,,$

$\lambda(r) = (\ell+2)(\ell-1) - \Lambda r^2 - 3\,(f-1)$

$\tilde{h}_{ab} = h_{ab} - \kappa_{a:b} - \kappa_{b:a}\,,$

$\tilde{K} = K + \frac{\ell(\ell+1)}{2}G - 2\frac{r^a}{r}\kappa_a$

Known master equations

Master equation

$$\left(\Box_2 - \Omega_{\text{even/odd}}\right) \Psi_{\text{even/odd}} = 0$$

Known master equations

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- In Schwarzschild background

$$\Omega_{\text{odd}}(r) = \frac{\ell(\ell+1)}{r^2} - \frac{3r_s}{r^3}$$
$$\Omega_{\text{even}}(r) = \frac{1}{\lambda^2}\left[\frac{(\ell-1)^2(\ell+2)^2}{r^2}\left(\ell(\ell+1) + \frac{3r_s}{r}\right) + \frac{9r_s^2}{r^4}\left((\ell-1)(\ell+2) + \frac{r_s}{r}\right)\right]$$

- Tortoise coordinate $dx/dr = 1/f(r)$

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - V_{\text{even/odd}}\right)\Psi_{\text{even/odd}} = 0\,, \qquad V_{\text{even/odd}} = f\,\Omega_{\text{even/odd}}$$

Construction of master functions and equations

Assumptions

1. Linear in the metric perturbations and first-order derivatives

$$\begin{aligned}\Psi^{\ell m}_{\rm odd} &= C^\ell_0\, h^{\ell m}_0 + C^\ell_1\, h^{\ell m}_1 + C^\ell_2\, h^{\ell m}_2 \\ &+ C^\ell_3\, \dot h^{\ell m}_0 + C^\ell_4\, h'^{\ell m}_0 + C^\ell_5\, \dot h^{\ell m}_1 \\ &+ C^\ell_6\, h'^{\ell m}_1 + C^\ell_7\, \dot h^{\ell m}_2 + C^\ell_8\, h'^{\ell m}_2\end{aligned}$$

2. Time independent coefficients

$$C^\ell_i = C^\ell_i(r)$$

3. Satisfies a wave equation and the potential is left arbitrary

$$(\Box_2 - \Omega)\,\Psi = 0$$

4. Arbitrary perturbative gauge

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 084053 (2021)

Procedure

Odd parity

$$K_1 \left[\Omega^{\text{odd}}(r) - \Omega_*^{\text{odd}}(r)\right] = 0$$

- Standard branch

$$\Omega^{\text{odd}}(r) = \Omega_*^{\text{odd}}(r)$$

- Darboux branch

$$K_1 = 0$$

Even parity

$$K_{13} \left[\Omega^{\text{even}}(r) - \Omega_*^{\text{even}}(r)\right] = 0$$

- Standard branch

$$\Omega^{\text{even}}(r) = \Omega_*^{\text{even}}(r)$$

- Darboux branch

$$K_{13} = 0$$

Odd parity sector: standard branch

$$\left(\Box_2 - \Omega_{\rm odd}\right) {}_S\Psi_{\rm odd} = 0$$

- A single potential selected $\Omega^{\rm odd}(r) = \Omega^{\rm odd}_*(r)$

$$\Omega^{\rm odd}_*(r) = \Lambda + \frac{\ell(\ell+1) + 3(f-1)}{r^2} \quad (= \Omega_{\rm RW} \quad \text{in Sch})$$

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- Master function is a linear combination of RW and CPM

$${}_S\Psi_{\text{odd}}(t,r) = K_1\, \Psi_{\text{RW}}(t,r) - \frac{(\ell+2)(\ell-1)}{4} K_0\, \Psi_{\text{CPM}}(t,r)$$

$$t^a \Psi_{\text{CPM}\,:a} = 2\, \Psi_{\text{RW}}$$

Odd parity sector: Darboux branch

$$\left(\Box_2 - \Omega_{\rm odd}\right) {}_D\Psi_{\rm odd} = 0$$

- Potential dependent master functions

$${}_D\Psi_{\rm odd}(t,r) = -\frac{(\ell+2)(\ell-1)}{4}\left(K_0 + \hat{K}_7\,\Xi(r)\right)\Psi_{\rm CPM}(t,r) + K_7\,\Phi_{\rm ON}(t,r)$$

with

$$\hat{K}_7 = \frac{2K_7}{(\ell+2)(\ell-1)}$$
$$\Xi(r) = \Xi(r,\Omega(r))$$
$$\Phi_{\rm ON}(t,r) = \varepsilon^{ab}\tilde{h}_{a:b}$$

Odd parity sector: Darboux branch

$$\left(\Box_2 - \Omega_{\text{odd}}\right) {}_D\Psi_{\text{odd}} = 0$$

- Family of potentials

$$\left(\frac{\delta V_{,x}}{\delta V}\right)_{,x} + 2\left(\frac{V^{\text{odd}}_{,x}}{\delta V}\right)_{,x} - \delta V = 0$$

where

$$\delta V = V - V^{\text{odd}}$$

$$V^{\text{odd}} = f\Omega_*^{\text{odd}}$$

Even parity sector: standard branch

$$\left(\Box_2 - \Omega_{\text{even}}\right) {}_S\Psi_{\text{even}} = 0$$

- A single potential selected $\Omega^{\text{even}}(r) = \Omega^{\text{even}}_*(r)$

$$\begin{aligned}\Omega^{\text{even}}_* &= \frac{\lambda^3(r) - 2\Lambda r^2[\lambda(r) - (\ell+2)(\ell-1)]^2 + 2\,(\ell+2)^2(\ell-1)^2\left(\ell^2+\ell+1\right)}{3r^2\lambda^2(r)} \\ &= \left(\Omega_{\text{Z}} \quad \text{Sch}\right)\end{aligned}$$

Even parity sector: standard branch

$$\left(\Box_2 - \Omega_{\text{even}}\right) {}_S\Psi_{\text{even}} = 0$$

- A single potential selected $\Omega^{\text{even}}(r) = \Omega_*^{\text{even}}(r)$

$$\Omega_*^{\text{even}} = \frac{\lambda^3(r) - 2\Lambda r^2 [\lambda(r) - (\ell+2)(\ell-1)]^2 + 2\,(\ell+2)^2(\ell-1)^2\left(\ell^2+\ell+1\right)}{3r^2\lambda^2(r)}$$
$$= \left(\Omega_{\text{Z}} \quad \text{Sch}\right)$$

- Master function is a linear combination of ZM and a new one

$${}_S\Psi_{\text{even}}(t,r) = \frac{\ell(\ell+1)}{4} K_2\, \Psi_{\text{ZM}}(t,r) - K_{13}\, \Psi_{\text{NE}}(t,r)$$

$$\Psi_{\text{NE}}(t,r) = \tfrac{1}{\lambda(r)} t^a \left(r\tilde{K}_{:a} - \tilde{h}_{ab} r^b\right)$$

$$t^a \Psi_{\text{ZM}\,:a} = 2\,\Psi_{\text{NE}}$$

Even parity sector: Darboux branch

$$\left(\Box_2 - \Omega_{\text{even}}\right) {}_D\Psi_{\text{even}} = 0$$

- Potential dependent master functions

$${}_D\Psi_{\text{even}}(t,r) = \left(K_2 + K_{24}\,\Sigma(r)\right)\Psi_{\text{ZM}}(t,r) - \frac{K_{24}}{2}\,\tilde{K}(t,r)$$

with

$$\Sigma(r) = \Sigma(r,\Omega(r))$$

Even parity sector: Darboux branch

$$\left(\Box_2 - \Omega_{\text{even}}\right)\, {}_D\Psi_{\text{even}} = 0$$

- Family of potentials

$$\left(\frac{\delta V_{,x}}{\delta V}\right)_{,x} + 2\left(\frac{V^{\text{even}}_{,x}}{\delta V}\right)_{,x} - \delta V = 0$$

where

$$\delta V = V - V^{\text{even}}$$

$$V^{\text{even}} = f\Omega_*^{\text{even}}$$

Darboux covariance and KdV hierarchy in Schwarzschild BH

Full space of master equations

$$\left(-\partial_t^2 + L_V\right)\Psi = \left(-\partial_t^2 + \partial_x^2 - V\right)\Psi = 0$$

Standard branch

$$_S\Psi^{\text{even}}_{\text{odd}} = \mathcal{C}_1 \Psi^{\text{ZM}}_{\text{CPM}} + \mathcal{C}_2 \Psi^{\text{NE}}_{\text{RW}}$$

$$V = V^{\text{Z}}_{\text{RW}}$$

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Darboux branch

$${}_D\Psi^{\text{even}}_{\text{odd}} = \mathcal{C}_1 \Psi^{\text{ZM}}_{\text{CPM}} + \mathcal{C}_2 \left(\Sigma^{\text{even}}_{\text{odd}} \Psi^{\text{ZM}}_{\text{CPM}} + \Phi^{\text{even}}_{\text{odd}}\right)$$

$$\left(\frac{\delta V_{,x}}{\delta V}\right)_{,x} + 2\left(\frac{V^{\text{even/odd}}_{,x}}{\delta V}\right)_{,x} - \delta V = 0$$

Standard view on DTs

- Isospectrality of V_{RW} and V_{Z}

S. Chandrasekhar, Proc. Roy. Soc. Lond. A **369**, 425–433 (1980), K. Glampedakis et al., Phys. Rev. D **96**, 024036 (2017)

Standard view on DTs

- Isospectrality of V_{RW} and V_{Z}

DT in frequency domain $\Psi(t,r) = e^{i\omega t}\psi(x;\omega)$

$$L_V\psi(x;\omega) = -\omega^2\psi(x;\omega)$$

$$L_V\psi_o = -\omega_o^2\psi_o \longrightarrow g(x) = -(\ln\psi_o)_{,x} \longrightarrow \begin{cases} L_v\phi = -\omega^2\phi \\ \phi = W[\psi,\psi_o]/\psi_o \end{cases}$$

Standard view on DTs

- Isospectrality of $V_{\rm RW}$ and $V_{\rm Z}$

DT in frequency domain $\Psi(t,r) = e^{i\omega t}\psi(x;\omega)$

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- Darboux transformation between RW and ZM

$$\psi_o = \frac{\lambda(r)}{2} \mathrm{e}^{-i\omega_* x} \qquad \qquad G(x) = \frac{6M\, f(r)}{\lambda(r) r^2}$$

$$\omega_* = -i\alpha = -i\frac{(\ell+2)!}{6M\,(\ell-1)!} \qquad \longrightarrow \qquad V_{\rm RW}^{\rm Z} = \pm G_{,x} + \alpha G + G^2$$

Darboux transformation

- Darboux transformation between (v, φ) and (V, Ψ)

$$(\Box_2 - v)\,\varphi = 0 \quad \longrightarrow \quad \begin{cases} \Psi = \varphi_{,x} + g\,\varphi \\ V = v + 2\,g_{,x} \\ g_{,x} - g^2 + v = \mathcal{C} \end{cases} \quad \longrightarrow \quad (\Box_2 - V)\,\Psi = 0$$

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$$\longrightarrow \quad \boxed{\left(\frac{\delta V_{,x}}{\delta V}\right)_{,x} + 2\left(\frac{v_{,x}}{\delta V}\right)_{,x} - \delta V = 0}$$

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$$(\Box_2 - v)\,\varphi = 0 \quad \longrightarrow \quad \begin{cases} \Psi = \varphi_{,x} + g\,\varphi \\ V = v + 2\,g_{,x} \\ g_{,x} - g^2 + v = \mathcal{C} \end{cases} \quad \longrightarrow \quad (\Box_2 - V)\,\Psi = 0$$

$$\longrightarrow \quad \boxed{\left(\frac{\delta V_{,x}}{\delta V}\right)_{,x} + 2\left(\frac{v_{,x}}{\delta V}\right)_{,x} - \delta V = 0}$$

- Darboux covariance of perturbations of spherically-symmetric BHs

$$g^{\mathrm{even}}_{\mathrm{odd}} = \frac{1}{2}\int dx\,\left(V - V^{\mathrm{Z}}_{\mathrm{RW}}\right)$$
$$g^{\mathrm{RW}\to\mathrm{Z}} = \frac{1}{2}\int dx\,\left(V_{\mathrm{Z}} - V_{\mathrm{RW}}\right)$$

Inverse scattering methods in BH perturbation theory

- Scattering through a potential barrier

$$\psi \to \begin{cases} e^{i\omega x} & \text{for } x \to -\infty\,, \\ a(\omega)e^{i\omega x} + b(\omega)e^{-i\omega x} & \text{for } x \to +\infty\,, \end{cases}$$

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- Bogoliubov coefficients completely determine the problem
  - Reflection and transmission coefficients

$$R(\omega) = \frac{b(\omega)}{a(\omega)}\,, \quad T(\omega) = \frac{1}{a(\omega)}$$

  - QNMs ω_i are the zeros of $a(\omega)$

$$a(\omega_i) = 0$$

KdV hierarchy in BH perturbation theory

- The logarithm of $a(\omega)$ can be expanded in inverse powers

$$\ln\left[a(\omega)\right] = \sum_{n=0}^{\infty} \frac{I_n}{(2\,i\,\omega)^n} \quad \Longrightarrow \quad I_n = \int_{-\infty}^{+\infty} \mathrm{d}x P_n(V, V', \ldots)$$

- I_n are the KdV integrals associated to the infinite symmetries of the KdV equation

L. D. Faddeev and V. E. Zakharov, Funct. Anal. Appl. **5**, 280–287 (1971)

KdV hierarchy in BH perturbation theory

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- I_n are the KdV integrals associated to the infinite symmetries of the KdV equation

The KdV integrals I_n are equal for all the Darboux related potentials

M. Lenzi and C. F. Sopuerta, Phys. Rev. D **104**, 124068 (2021)

Conclusions

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- First-time complete picture
- Computational advances (QNMs, scattering data..)
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Thank you!