Action-Angle formalism for geodesic motion in Kerr spacetime

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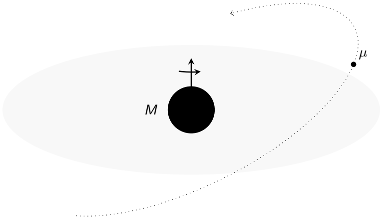
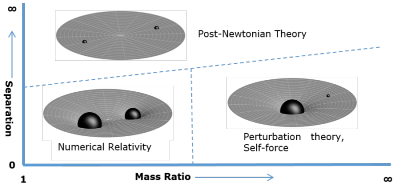
Outline

- What is the EMRI?
- Two timescale analysis.
- Action-angle (AA) formalism.
- Canonical perturbation theory.
- Kerr geodesic Hamiltonian in the AA variables.



What is an EMRI? and why EMRI's?

- *Extreme mass ratio inspirals* (EMRI's): a primary supermassive BH M, and a secondary much lighter compact stellar object μ.



- The mass ratio $q = \frac{\mu}{M} \sim (10^{-7} - 10^{-4})$.
- Inspirals occurs slowly, on the time scale $\tau \sim 1/q$.
- Secondary spends $\sim 10^4 - 10^5$ orbits within **LISA** band.
- EMRI's are one the most promising sources of gravitational for LISA.
- Detailed map of the supermassive BH.

Modeling the EMRI's

To model an EMRI, we use the two time scale approximation. (T. Hinderer & E E. Flanagan, PRD, 2008)

- The slow time scale: evolution of the constants of motion $\{E, L_z, Q\}$.
- The fast time scale: orbital phases of the secondary.

In the action-angle variables

- Constants of motion $\boldsymbol{J_i} = \{J_t, J_r, J_\theta, J_\phi\}$
- Fundamental frequencies of geodesic motion $\Omega_r, \Omega_\theta, \Omega_\phi$.

From canonical perturbation theory, we derived a **closed form** *expressions for the actions!*

Hamiltonian in action-angle variables

For a bounded motion when the system is integrable, there exist a canonical transformation

$$(\boldsymbol{q}, \boldsymbol{p}) \xrightarrow{\text{CT}} (\boldsymbol{\psi}, \boldsymbol{J}) \qquad \Longrightarrow \qquad H(\boldsymbol{q}, \boldsymbol{p}) \to H_{AA}(\boldsymbol{J})$$

where

$$J_i = \frac{1}{2\pi} \oint p_i dq_i$$

From Hamilton equations:

$$\dot{J}_i = -\frac{\partial H_{AA}}{\partial \psi_i} = 0 \qquad \Longrightarrow \qquad J_i(\lambda) = Const.$$

$$\dot{\psi}_i = \frac{\partial H_{AA}}{\partial J_i} = \Omega_i \qquad \Longrightarrow \qquad \psi_i(\lambda) = \Omega_i \lambda + \psi_i(0)$$

Geodesic motion in Kerr background in AA variables

- Hamiltonian In the Boyer–Lindquist coordinates & Mino time

$$H_\lambda = \frac{1}{2}\left(\Delta p_r^2 - \frac{\left(\left(r^2+a^2\right)p_t + aL_z\right)^2}{\Delta} + r^2\right)$$
$$+\frac{1}{2}\left(p_\theta^2 + a^2\cos^2\theta + \frac{\left(L_z + a\sin^2\theta p_t\right)^2}{\sin^2\theta}\right)$$

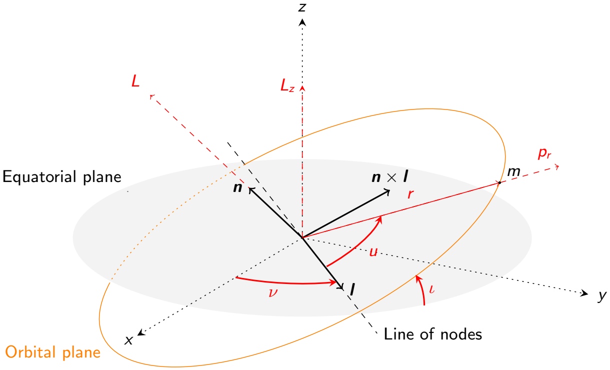
- Mino time is applied to separate the radial and angular parts.

$$J_t = \frac{1}{2\pi}\oint p_t dt = -E$$
$$J_r = \frac{1}{2\pi}\oint p_r dr = ?$$
$$J_\theta = \frac{1}{2\pi}\oint p_\theta d\theta = ?$$
$$J_\phi = \frac{1}{2\pi}\oint p_\phi d\phi = L_z$$

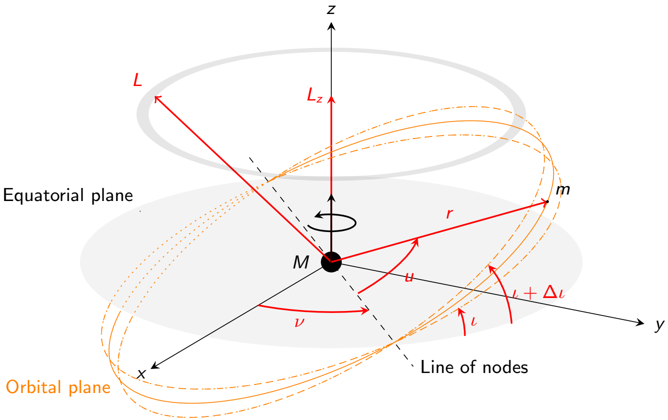
Polar-nodal coordinates

Canonically transformed to **polar-nodal coordinate**

$$\{r, \theta, \phi, p_r, p_\theta, p_\phi\} \to \{r, u, \nu, p_r, L, L_z\}$$



Kerr geodesics in polar-nodal coordinates



Canonical perturbation theory

- In the Birkhoff normal form theory we assume:

$$H^{(0)}(\psi_i, J_i) = Z_0(J_i) + \sum_{k=1}^{N} \epsilon^k H_k^{(0)}(\psi_i, J_i),$$

- We apply the Lie series to the Hamiltonian:

$$H^{(1)} = L_\chi H^{(0)} = Z_0(J_i) + \epsilon Z_1^{(0)}(J_i) + \epsilon^2 H_2^{(0)}(\psi_i, J_i) + \mathcal{O}(\epsilon^3),$$

where $L_\chi f = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \mathcal{L}_\chi^k f$ and $\mathcal{L}_\chi f = \{f, \chi\}$.

- In a similar fashion, we can apply n Lie series:

$$\begin{aligned} H^{(n)} &= L^n_{\chi_n} L^{n-1}_{\chi_{n-1}} \dots L^2_{\chi_2} L_\chi H^{(0)} \\ &= Z_0(J_i) + \epsilon Z_1^{(n)}(J_i) + \dots \epsilon^n Z_n^{(n)}(J_i). \end{aligned}$$

Why Lie series?

- Lie series is are canonical transformations.
- We can express the old variables in terms of the new variables:

$$\psi_i^{(0)} = L_{\chi_n}^{n}\, L_{\chi_{n-1}}^{n-1} \ldots L_{\chi_2}^{2}\, L_{\chi}\psi_i^{(n)},$$
$$J_i^{(0)} = L_{\chi_n}^{n}\, L_{\chi_{n-1}}^{n-1} \ldots L_{\chi_2}^{2}\, L_{\chi}J_i^{(n)},$$

- New variables in terms of the old ones:

$$\psi_i^{(n)} = L_{-\chi}\, L_{-\chi_2}^{2} \ldots L_{-\chi_{n-1}}^{n-1}\, L_{-\chi_n}^{n}\, \psi_i^{(0)},$$
$$J_i^{(n)} = L_{-\chi}\, L_{-\chi_2}^{2} \ldots L_{-\chi_{n-1}}^{n-1}\, L_{-\chi_n}^{n}\, J_i^{(0)}.$$

- The resulting relations from the Lie series are in **closed form**! There are simple analytical relations without any integral, Hypergeometric function and etc.

Results

- We applied 10 canonical transformation for the radial part and 7 canonical transformation for the angular part.
- We derived the new Hamiltonian in the action-angle variables:

$$H_{AA}(J_i) = C_1 J_t + C_2 J_r + C_3 J_L + C_4 J_z$$

- From the Hamilton equations the radial and angular frequencies are obtained:

$$\Omega_r = \frac{\partial H_{AA}}{\partial J_r}, \qquad \Omega_L = \frac{\partial H_{AA}}{\partial J_L}.$$

- We derived the relative errors of the new actions and frequencies with their numerical values.

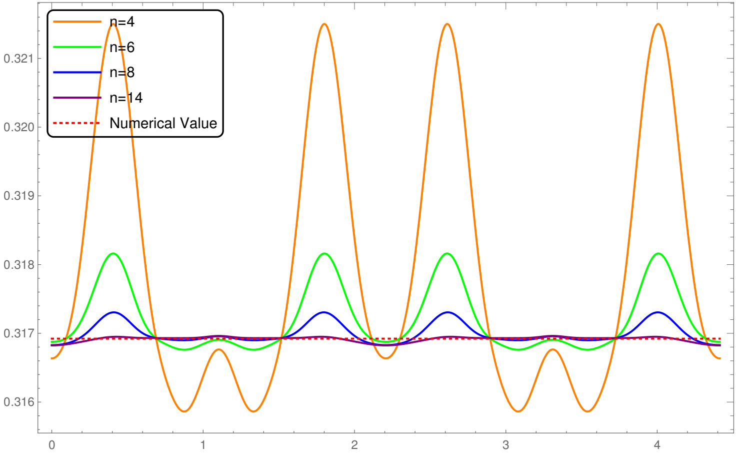


Figure: The radial action J_r in terms of the old variables for different numbers of the CT for a system with $a = 0.99$, $e = 0.4$, $p = 10$, and the initial inclination $\iota_0 = \pi/8$.

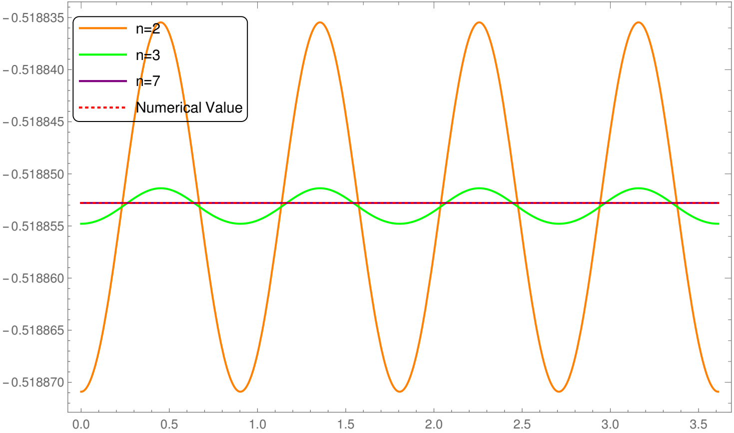


Figure: The angular action J_L in terms of the old variables for different numbers of the CT for a system with $a = 0.99$, $e = 0.4$, $p = 10$, and the initial inclination $\iota_0 = \pi/8$.

Accuracy of the new system

- The relative errors for a system $\{p, \iota_0\} = \{10, \pi/8\}$ for the Kerr parameter $0 < a \leq 0.99$ and the eccentricity $0 < e \leq 0.5$ are

$$\text{maximum error} \implies 0.1 \leq e \leq 0.5$$
$$\mathcal{O}(\Delta J_r)_{max} \implies \{10^{-10}, 10^{-4}\}$$
$$\mathcal{O}(\Delta \Omega_r)_{max} \implies \{10^{-12}, 10^{-4}\}$$
$$\mathcal{O}(\Delta J_L)_{max} \implies \{10^{-11}, 10^{-9}\},$$
$$\mathcal{O}(\Delta \Omega_L)_{max} \implies \{10^{-12}, 10^{-8}\}$$

$$(\text{phase shift error} \times \text{number of cycles}) \sim 0.1\text{radian}$$

Further researches & summary

- EMRI's can be detected by LISA.
- Two timescale approximation provides efficient framework for generation waveforms.
- From Lie series we derived the Hamiltonian in the action variables.
- Our expressions are in closed forms with sufficient accuracy for LISA.

work in progress

- Use Teukolsky equation to generates the fluxes.
- Study the whole inspirals in the LISA band.

Thanks for your attention!