Neutron Stars & the Nuclear Equation of State

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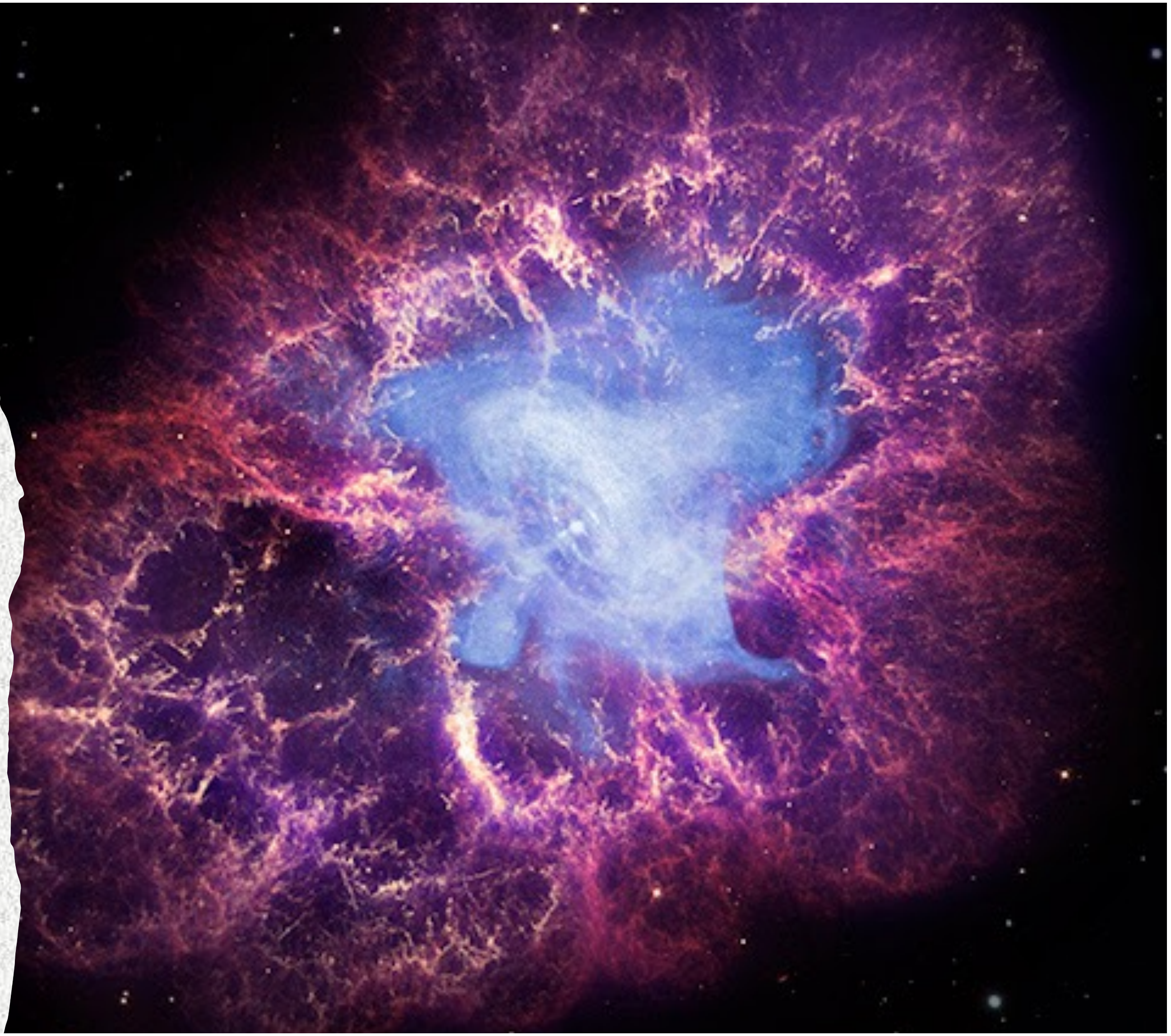
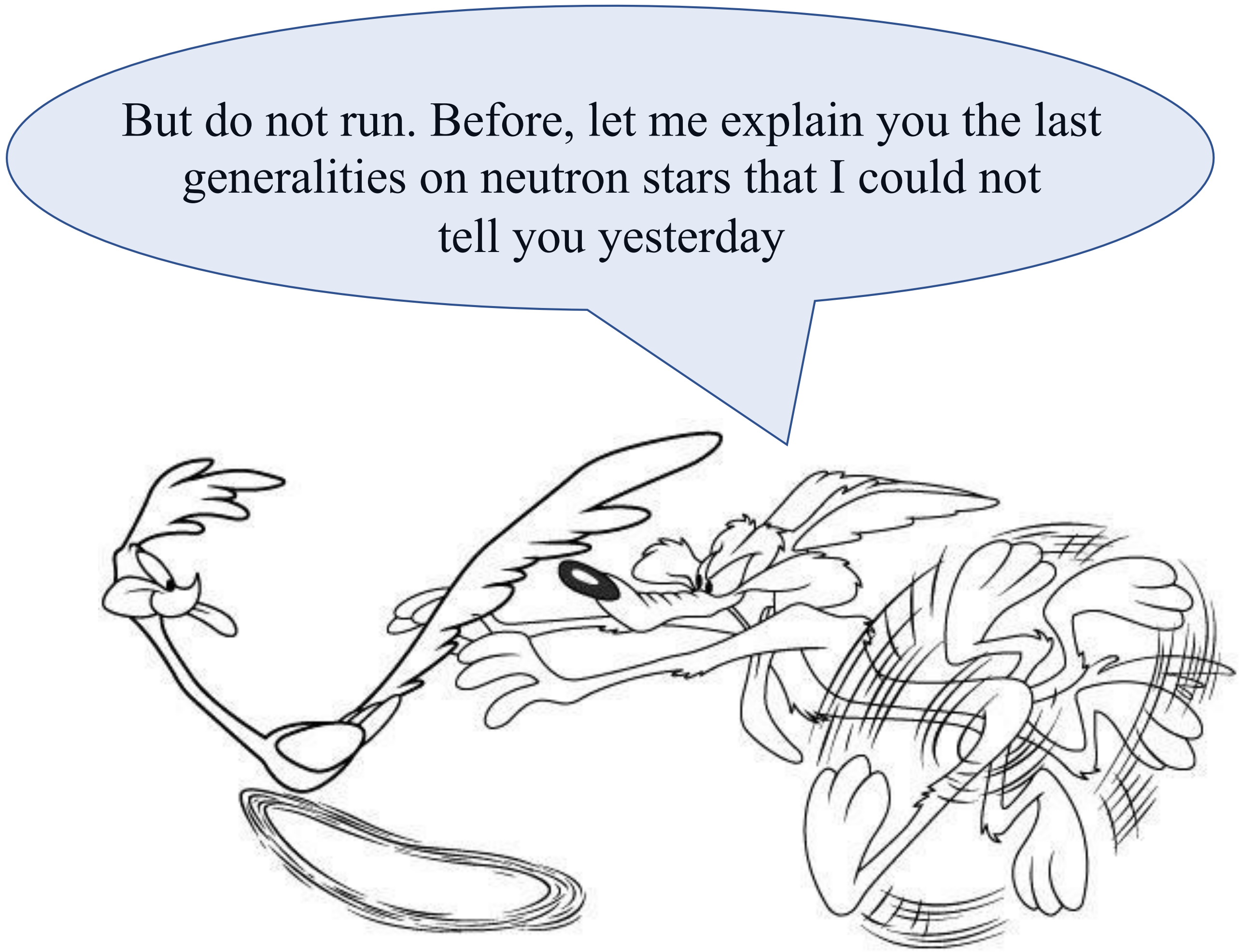


Lecture Program: Part 2

Baryon-baryon interaction

Theoretical approaches of the nuclear EoS


But do not run. Before, let me explain you the last generalities on neutron stars that I could not tell you yesterday

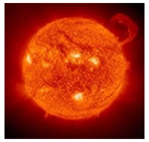
Neutron Star Structure:
General Relativity or Newtonian Gravity ?

Surface gravitational potential tell us how much compact an object is

$$\frac{2GM}{c^2R}$$

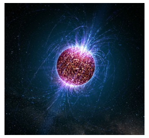
➔ Relativistic effects are very important in Neutron Stars and General Relativity must be used to describe their structure

$\sim 10^{-10}$

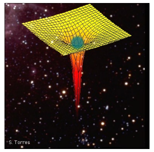


$\sim 10^{-5}$

$\sim 10^{-4} - 10^{-3}$



$\sim 0.2 - 0.4$



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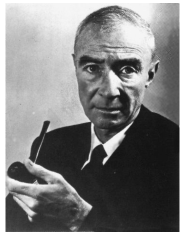
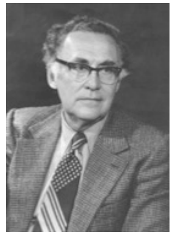
The Tolman-Oppenheimer-Volkoff Equations

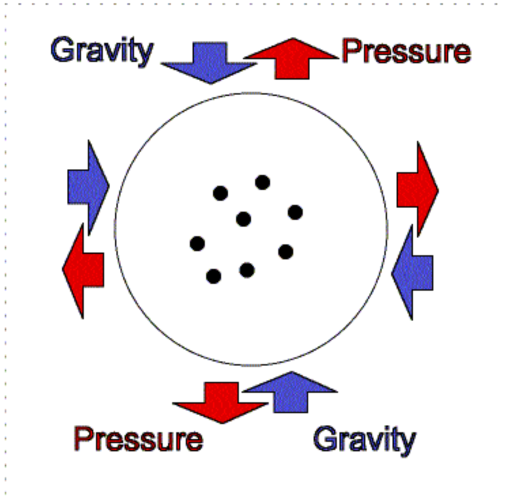
In 1939 Tolman, Oppenheimer & Volkoff obtain the equations that describe the structure of a static star with spherical symmetry in General Relativity (Chandrasekhar & von Neumann obtained them in 1934 but they did not published their work)





Tolman, Phys. Rev. 55, 364 (1939)

Oppenheimer & Volkoff, Phys. Rev. 55, 374 (1939)

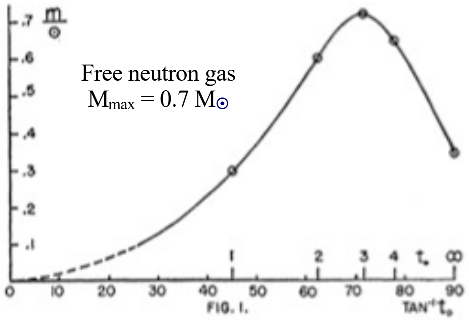


$$\frac{dP}{dr} = -G\frac{m(r)\varepsilon(r)}{r^2}\left(1+\frac{P(r)}{c^2\varepsilon(r)}\right)\left(1+\frac{4\pi r^3 P(r)m(r)}{c^2}\right)\left(1-\frac{2Gm(r)}{c^2 r}\right)^{-1}$$

$$\frac{dm}{dr} = 4\pi r^2\varepsilon(r)$$

boundary conditions

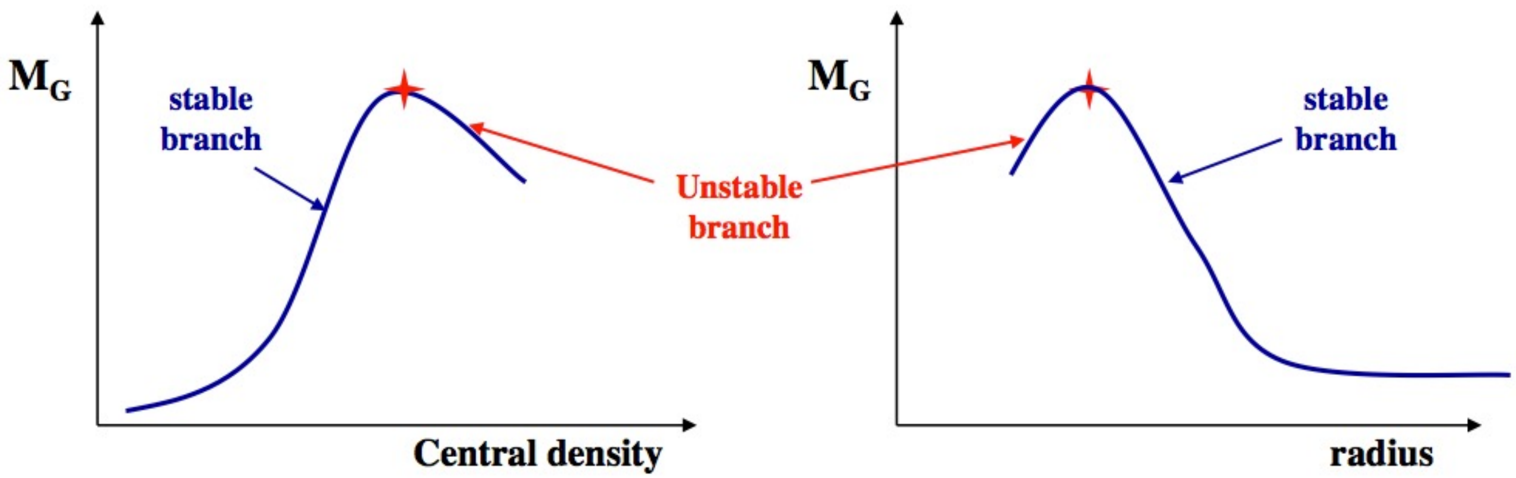
$$P(0) = P_o, \quad m(0) = 0$$
$$P(R) = 0, \quad m(R) = M$$



Stability solutions of the TOV equations

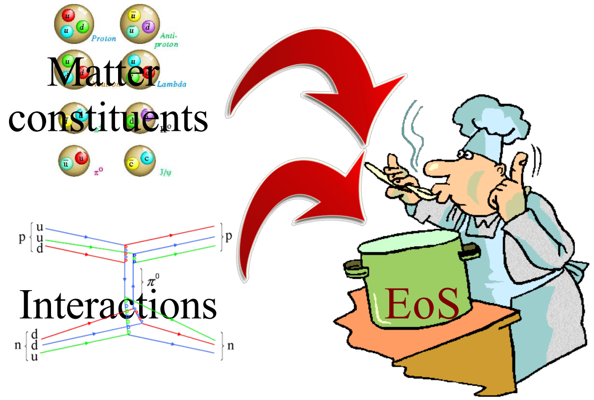
- The solutions of the TOV equations represent static equilibrium configurations
- Stability is required with respect to small perturbations

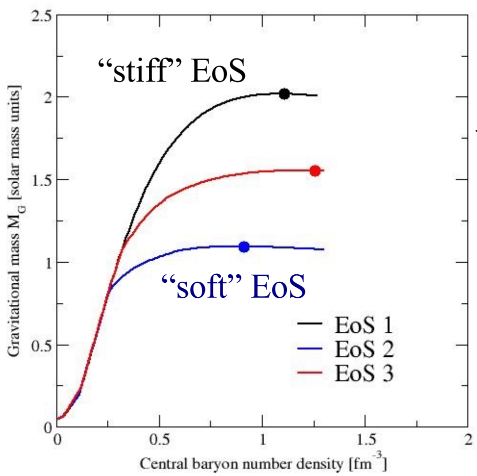
$$\frac{dM_G}{d\rho_c} > 0, \quad or \quad \frac{dM_G}{dr} < 0$$

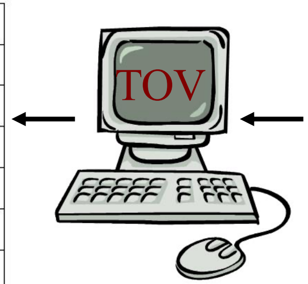


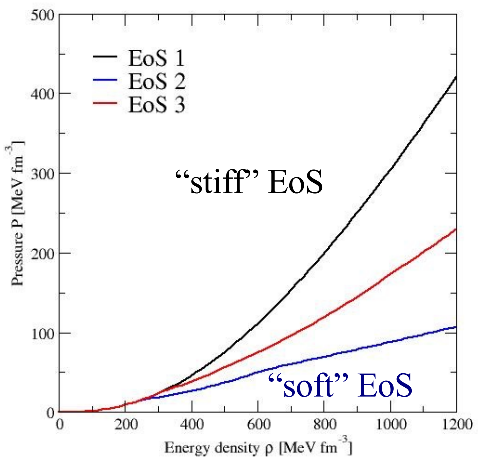
The role of the Equation of State

The only ingredient needed to solve the TOV equations is the (poorly known) EoS (i.e., p(ε)) of dense matter









The Nuclear EoS

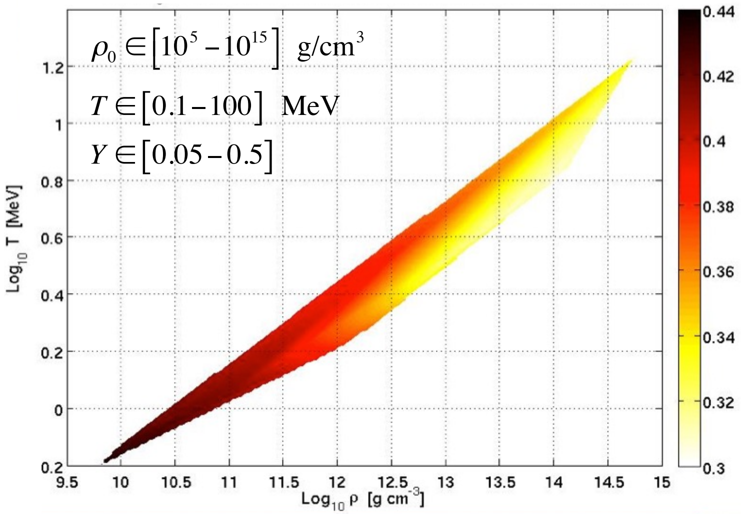
The Nuclear EoS is a fundamental ingredient for the understanding of the static & dynamical properties of NS, core-collapse SN & compact star mergers

However, its determination is very challenging due to the wide range of densities, temperatures & isospin asymmetries found in these astrophysical scenarios.

Main difficulties associated to:

- ✓ Complexity of the bare baryon-baryon interaction
- ✓ Very complicated resolution of the so-called nuclear many-body problem

Conditions in the center of the star from the onset of the collapse up to 25 ms after bounce (15 M_{sun} progenitor)



Baryon-baryon interaction

Few generalities

QCD is commonly recognized as the fundamental theory of strong interactions. It is a non-Abelian gauge theory described by the Lagrangian density

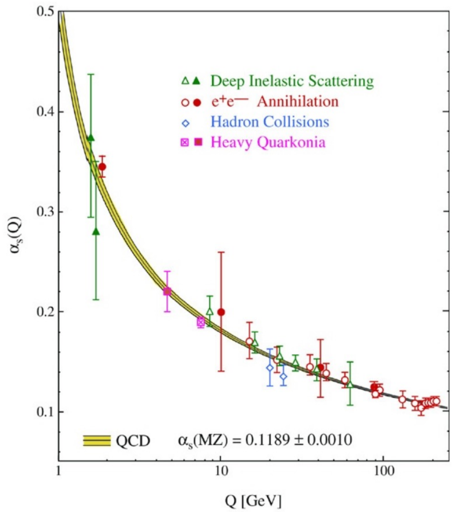
$$\mathcal{L}_s = -\frac{1}{4}\mathcal{G}^a_{\mu\nu}\mathcal{G}_a^{\mu\nu} + \sum_{f=u,d,s,c,b,t} \bar{\Psi}_f\left(i\gamma^\mu\mathcal{D}_\mu - m_f\right)\Psi_f$$

$$\mathcal{G}^a_{\mu\nu} = \partial_\mu\mathcal{A}^a_\nu - \partial_\nu\mathcal{A}^a_\mu + g\sum_{b,c=1}^{8} f^{abc}\,\mathcal{A}^b_\mu\mathcal{A}^c_\nu, \quad \mathcal{D}_\mu = \partial_\mu - ig\frac{\lambda^a}{2}\mathcal{A}^a_\mu$$

- The baryon-baryon interaction can, in principle, be completely determined from the underlying quark-gluon dynamics in QCD

- However, due to the mathematical problems raised by the non-perturbative character of QCD at low & intermediate energies (in this energy range the strong coupling constant becomes too large for perturbative approaches) one is still far from a quantitative understanding of the baryon-baryon interaction from the QCD point of view

- This problem is circumvented by introducing simplified models where hadronic degrees of freedom are assumed to be the only relevant ones



Few generalities

Nowadays, bare baryon-baryon interactions are derived following several approaches

- Phenomenological approaches
  - Meson exchange-models
  - Potential models
- Chiral effective field theory
- Renormalization group methods
- Lattice QCD calculations
- …

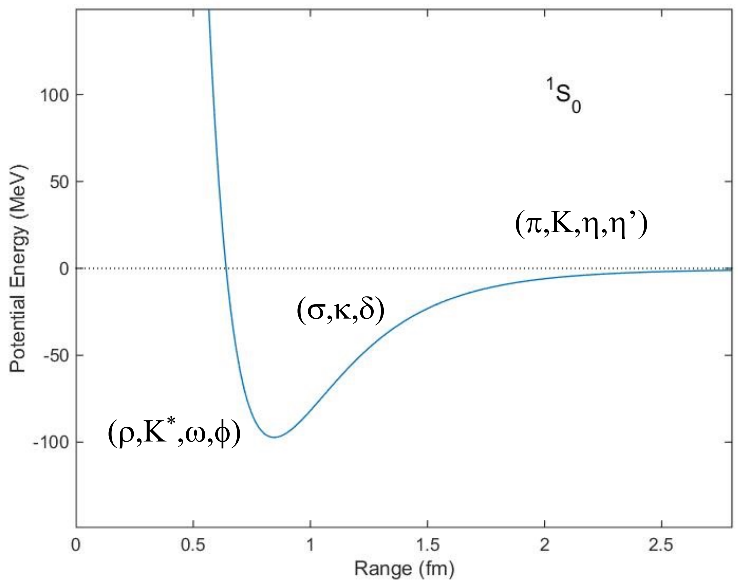
In the next we will describe each one of them

Meson-exchange models

➢ Based on the Yukawa idea

"Baryon-baryon interactions are mediated by the exchange of mesons"

- Long-range*:* pseudoscalar mesons (π,K,η,η')
- Intermediate-range: scalar mesons (σ,κ,δ)
- Short-range: vector mesons (ρ,K^*,ω,ϕ)



➢ Various models differ mainly in the mesonic content & treatment of two meson-exchange contributions. But all them describe successfully NN scattering phase shift & deuteron properties

➢ Some very refined models for NN interaction: Paris, Bonn, Nijmegen potentials

➢ YN & YY meson-exchange models: Juelich, Nijmegen potentials

Machleidt et al., PR. 149, 1 (1987)
Nagels et al., PRD 17, 768 (1978)

Meson-exchange models

Guided by symmetry principles, simplicity & physical intuition, the most general interaction Lagrangian densities that couple meson and baryon fields are the following:

- Scalar mesons: $\mathcal{L}_s = g_s \bar{\psi}\psi\phi^{(s)}$
- Pseudoscalar mesons: $\mathcal{L}_{ps} = g_{ps}\bar{\psi}i\gamma^5\psi\phi^{(ps)}$

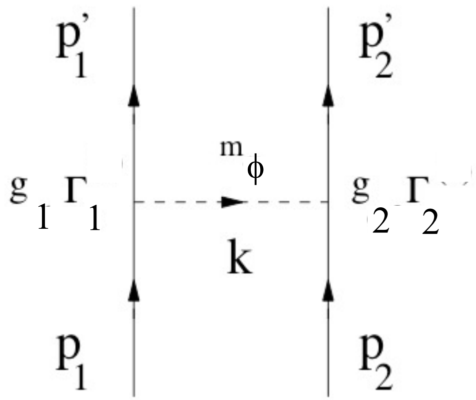
  $\mathcal{L}_{pv} = g_{pv}\bar{\psi}\gamma^5\gamma^\mu\psi\partial_\mu\phi^{(ps)}$ (pseudovector or gradient coupling suggested as an effective coupling by chiral symmetry)
- Vector mesons: $\mathcal{L}_v = g_v\bar{\psi}\gamma^\mu\psi\phi_\mu^{(v)} + g_t\bar{\psi}\sigma^{\mu\nu}\psi\left(\partial_\mu\phi_\nu^{(v)} - \partial_\nu\phi_\mu^{(v)}\right), \quad \sigma^{\mu\nu} = \frac{1}{4i}[\gamma^\mu,\gamma^\nu]$

ψ: spin ½ − baryon fields; $\phi^{(s)}, \phi^{(ps)}, \phi_\mu^{(v)}$: scalar, pseudoscalar & vector meson fields; g's coupling constants to be constrained (if possible) by scattering data

These Lagrangian densities are for isoscalar mesons, those for isovector ones are obtained by replacing $\phi \rightarrow \vec{\tau}\cdot\vec{\phi}$

Meson-exchange models

A typical contribution to the baryon-baryon interaction potential arising from the exchange of a certain meson ϕ is



$$\langle p_1' p_2' | V_\phi | p_1 p_2 \rangle = \frac{\bar{u}_1(p_1') g_1 \Gamma_1 u_1(p_1) P_\phi \bar{u}_2(p_2') g_2 \Gamma_2 u_2(p_2)}{k^2 - m_\phi^2}$$

- $\frac{P_\phi}{k^2 - m_\phi^2}$: meson propagator; $P_\phi = 1$ for scalar & pseudoscalar mesons; $P_\phi \equiv P_{\mu\nu} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_\phi^2}$ for vector mesons
- m_ϕ: mass of the exchanged meson
- u_i & $\bar{u}_i$: Dirac spinor & its adjoint ($\bar{u}u = 1, \bar{u} = u^\dagger \gamma^0$)
- $\Gamma_s = 1, \quad \Gamma_{ps} = i\gamma^5, \quad \Gamma_v = \gamma^\mu, \quad \Gamma_t = \sigma^{\mu\nu}, \quad \Gamma_{pv} = \gamma^5 \gamma^\mu \partial_\mu$: Dirac structures of the vertices

Meson-exchange models

In general, when all types of mesons are included the total baryon-baryon interaction potential is the sum of all the partial contributions

$$\langle p_1' p_2' | V | p_1 p_2 \rangle = \sum_\phi \langle p_1' p_2' | V_\phi | p_1 p_2 \rangle$$

Expanding the Dirac spinor in terms of 1/M (M: baryon mass) to lowest order leads to the familiar non-relativistic expressions of the baryon-baryon potential, which through Fourier transformation give the configuration version of the potential

$$V(\vec{r}) = \sum_\phi \left\{ C_{C_\phi} + C_{\sigma_\phi} \vec{\sigma}_1 \cdot \vec{\sigma}_2 + C_{L_\phi} \left(\frac{1}{m_\phi r} + \frac{1}{(m_\phi r)^2} \right) \vec{L} \cdot \vec{S} + C_{T_\phi} \left(1 + \frac{3}{m_\phi r} + \frac{3}{(m_\phi r)^2} \right) S_{12}(\hat{r}) \right\} \frac{e^{-m_\phi r}}{r}$$

- *C*'s: numerical factors containig all baryon-baryon-meson couplings & baryon masses
- *L*, *S*: total orbital angular momentum & total spin
- $S_{12}(\hat{r}) = 3(\vec{\sigma}_1 \cdot \hat{r})(\vec{\sigma}_2 \cdot \hat{r}) - (\vec{\sigma}_1 \cdot \vec{\sigma}_2)$; $\hat{r} = \frac{\vec{r}}{r}$: tensor operator

Meson-exchange models

Finally, one has to remember that all baryon-baryon-meson vertices must be modified with the introduction of a form factor

Two types of form factors are usually employed:

$$F_\phi\left(|\vec{k}|^2\right) = \left(\frac{\Lambda_\phi^2 - m_\phi^2}{\Lambda_\phi^2 + |\vec{k}|^2}\right)^{n_\phi}$$

: usually n_ϕ takes values 1 (monopole form factor) or 2 (dipole form factor)

or gaussian

$$F_\phi\left(|\vec{k}|^2\right) = exp\left(-\frac{|\vec{k}|^2}{2\Lambda_\phi^2}\right)$$

: in both cases Λ_ϕ is the so called cut − off mass with values between 1.2 − 2 GeV

- Originally form factors were introduced for purely mathematical reasons, namely to avoid divergences in the scattering equation. Our present knowledge of the quark substructure of baryons and mesons provides a physical reason for their introduction
- Meson exchange picture loses its validity in regions where modifications due to the extended structure of hadrons comes into play

Potential models

- Potential models have a complex structure which is expressed via operator invariants consistent with the symmetries of strong interactions:

  - Translational invariance
  - Galilean invariance
  - Rotational invariance
  - Space-reflection invariance
  - Time-reversal invariance
  - Invariance under the interchange of two baryons
  - Isospin symmetry
  - Hermiticity

- The most widely known potential models are the Urbana and the Argonne ones where the NN interaction is given as a sum of several local operators. In the case of the Argonne V18 reads:

$$\hat{V}_{ij}(r_{ij}) = \sum_{p=1}^{18} V_{ij}(r_{ij})\hat{O}_{ij}^{p} \quad \text{with}$$

$$\hat{O}_{ij}^{p=1,\cdots,14} = \left[1, (\vec{\sigma}_i \cdot \vec{\sigma}_j), S_{ij}, \vec{L}\cdot\vec{S}, L^2, L^2(\vec{\sigma}_i \cdot \vec{\sigma}_j), (\vec{L}\cdot\vec{S})^2\right] \otimes \left[1, (\vec{\tau}_i \cdot \vec{\tau}_j)\right]$$

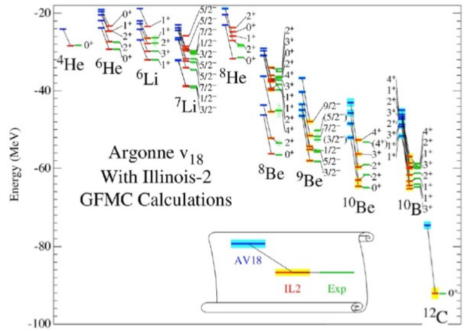
$$\hat{O}_{ij}^{p=15,\cdots,18} = \left[T_{ij}, (\vec{\sigma}_i \cdot \vec{\sigma}_j)T_{ij}, S_{ij}T_{ij}, T_{ij}\left(\tau_{z_i} + \tau_{z_j}\right)\right]$$: charge symmetry breaking

Wiringa et al., PRC 51, 38 (1995)

Some words on the three-nucleon force

Necessary to:

- ✧ Reproduce the spectra of light nuclei
- ✧ Obtain proper saturation properties of symmetric nuclear matter in non-relativistic many-body calculations

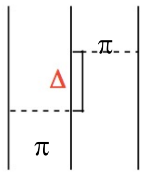


Pieper & Wiringa, ARNPS 51, 53 (2001)

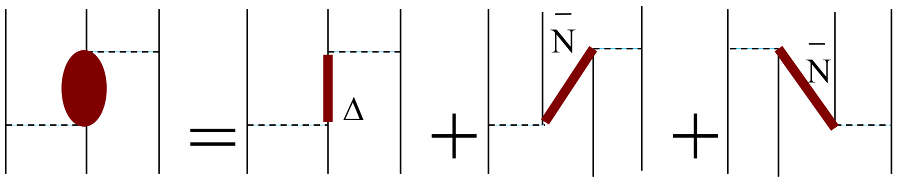
✧ **Urbana-type** $V_{ijk}^{UIX} = V_{ijk}^{2\pi} + V_{ijk}^{R}$

$V_{ijk}^{2\pi}$: Attractive Fujita-Miyazawa force

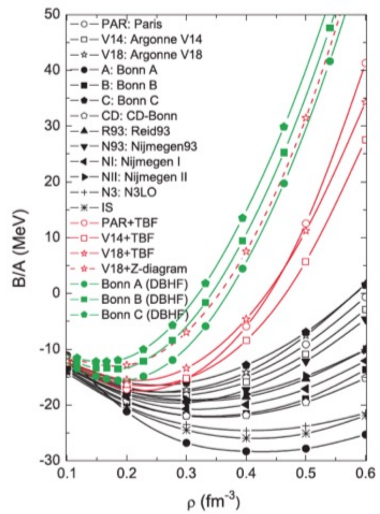
V_{ijk}^{R} : Repulsive & Phenomenological



✧ **Microscopic-type**

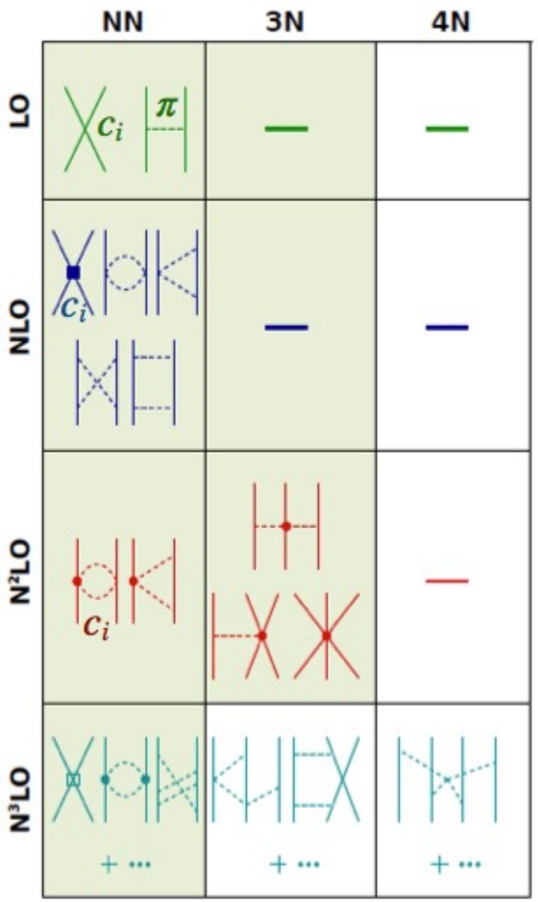


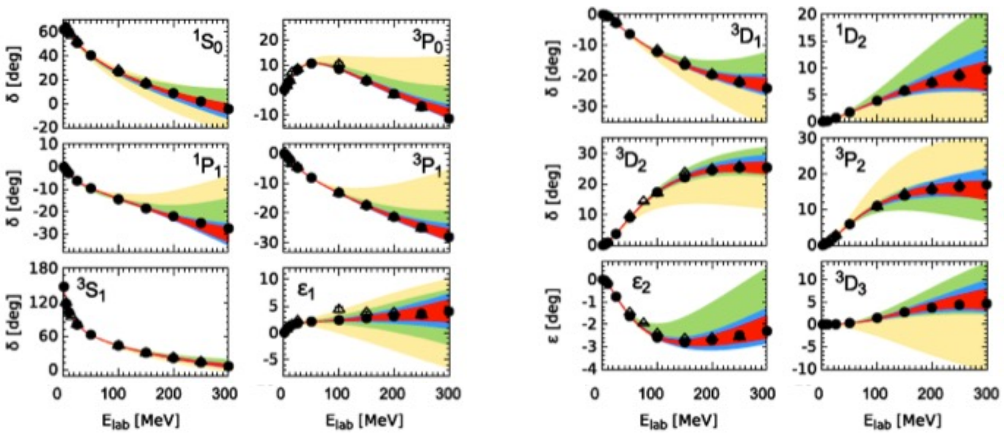
Problem: NNN is not independent of NN



Li et al., PRC 74, 047304 (2006)

Chiral Perturbation Expansion





✧ Starting point: most general chiral effective Lagrangian consistent with the symmetries required by QCD where π & N (recently also Δ) are the relevant degrees of freedom. of the theory

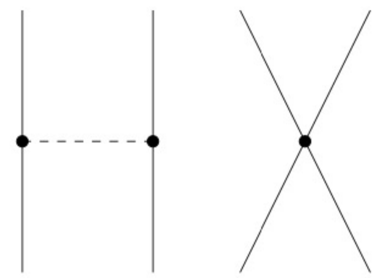
✧ Systematic expansion in powers of Q/Λ_χ [Q=m_π, k; $\Lambda_\chi \sim 1$ GeV]

✧ Consistent derivation of 2N, 3N, 4N, … forces

Weinberg, PLB 251, 288 (1990); NPB 363, 3 (1991)
Entem & Machleidt, PRC 68, 041001(R) (2003)
Epelbaum et al., NPA 747, 363 (2005)

Chiral Perturbation Expansion (LO)

➢ <u>Leading order (LO) contribution</u>



This contribution consist of one pseudoscalar-meson exchange and of four-baryon contact terms each one of them constrained by SU(3)-flavor symmetry

The one pseudoscalar-meson exchange term is obtained from the Lagrangian density

$$\mathcal{L} = \left\langle i\bar{B}\gamma^{\mu}D_{\mu} - M_0\bar{B}B + \frac{D}{2}\bar{B}\gamma^{\mu}\gamma_5\left[u_{\mu}, B\right] + \frac{F}{2}\bar{B}\gamma^{\mu}\gamma_5\left\{u_{\mu}, B\right\}\right\rangle$$

✓ $\langle\cdots\rangle$ denote the trace in flavor space

✓ B is the SU(3) – flavor irreducible representation of the baryon octet

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ -\Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix}$$

Chiral Perturbation Expansion (LO)

$$\mathcal{L} = \left\langle i\bar{B}\gamma^{\mu}D_{\mu} - M_0\bar{B}B + \frac{D}{2}\bar{B}\gamma^{\mu}\gamma_5[u_{\mu}, B] + \frac{F}{2}\bar{B}\gamma^{\mu}\gamma_5\{u_{\mu}, B\}\right\rangle$$

- ✓ D_{μ} is the covariant derivative
- ✓ M_0 is the octet baryon mass in the chiral limit
- ✓ F and D are couping constants satisfying $F + D = g_A \cong 1.26$ (axial-vector strength)
- ✓ $u_{\mu} = i(u^{\dagger}\partial_{\mu}u - u\partial_{\mu}u^{\dagger})$ with $u = exp\left(\frac{iP}{\sqrt{2}F_{\pi}}\right)$ being
  - F_{π}= 92.4 MeV the weak pion decay constant
  - P: SU(3) – flavor irreducible representarion of pseudoscalr meson

$$P = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ -K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}$$

Chiral Perturbation Expansion (LO)

The form of the baryon-baryon potentials obtained from the one pseudoscalar-meson exchange LO contribution is similar to the ones derived from the meson-exchange approach, and in momentum space read

$$V_{OPE}^{BB} = -f_{B_1 B_2 P} f_{B_3 B_4 P} \frac{(\vec{\sigma}_1 \cdot \vec{q})(\vec{\sigma}_2 \cdot \vec{q})}{|\vec{q}|^2 + m_{ps}^2} I_{B_1 B_2 \to B_3 B_4}$$

- ✓ $f_{B_1 B_2 P}, f_{B_3 B_4 P}$ coupling constants of the two vertices
- ✓ m_{ps} mass of the exchanged pseudoscalar meson
- ✓ $\vec{q}$ transferred momemtum
- ✓ $I_{B_1 B_2 \to B_3 B_4}$ isospin factor

Chiral Perturbation Expansion (LO)

The contribution from the four-body contact interaction can be derived from the following minimal set of Lagrangian densities

$$\mathcal{L}^1 = C_i^1 \langle \bar{B}_a \bar{B}_b (\Gamma_i B)_b (\Gamma_i B)_a \rangle \ , \quad \mathcal{L}^2 = C_i^2 \langle \bar{B}_a (\Gamma_i B)_a \bar{B}_b (\Gamma_i B)_b \rangle \ , \quad \mathcal{L}^3 = C_i^3 \langle \bar{B}_a (\Gamma_i B)_a \rangle \langle \bar{B}_b (\Gamma_i B)_b \rangle$$

Here:

- ✓ The labels a and b are the Dirac indices of the particles
- ✓ Γ_i denote the five elements of the Cliffod algebra (usually 3 x 3 matrices in the flavor space)

$$\Gamma_1 = 1, \quad \Gamma_2 = \gamma^\mu, \quad \Gamma_3 = \sigma^{\mu\nu}, \quad \Gamma_4 = \gamma^\mu \gamma^5, \quad \Gamma_5 = \gamma^5$$

- ✓ C_i^1, C_i^2, C_i^3: low − energy constants (LEC). At LO there are 6 independent LEC

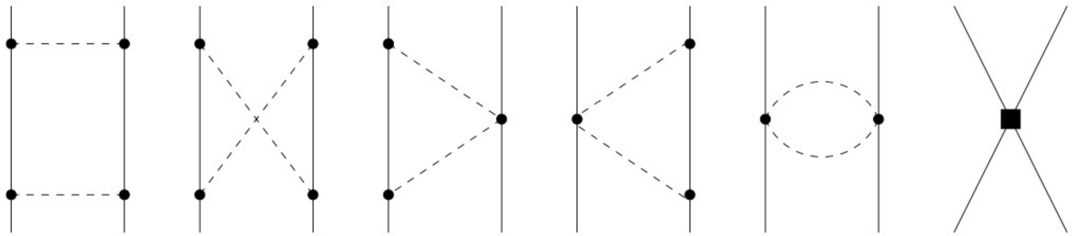
LO contact potential

$$V_{LO}^{BB} = C_C^{BB} + C_S^{BB} (\vec{\sigma}_1 \cdot \vec{\sigma}_2)$$

C_C^{BB} and C_S^{BB} linear combination of the 6 independent LEC

Chiral Perturbation Expansion (NLO)

➢ Next-to-leading-order (NLO) contribution



- Contact terms contribution

$$V_{NLO}^{BB} = C_1 q^2 + C_2 k^2 + (C_3 q^2 + C_4 k^2)(\vec{\sigma}_1 \cdot \vec{\sigma}_2) + i\frac{C_5}{2}(\vec{\sigma}_1 + \vec{\sigma}_2)(\vec{q} \times \vec{k})$$
$$+ C_6(\vec{q} \cdot \vec{\sigma}_1)(\vec{q} \cdot \vec{\sigma}_2) + C_7(\vec{k} \cdot \vec{\sigma}_1)(\vec{k} \cdot \vec{\sigma}_2) + C_8(\vec{\sigma}_1 - \vec{\sigma}_2)(\vec{q} \times \vec{k})$$

- Expressions for two-pseudoscalar meson exchange are rather cumbersome

Chiral Perturbation Expansion

A final comment:

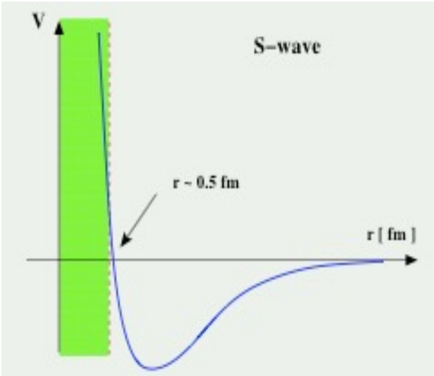
The baryon-baryon potentials constructed in this way are then inserted in the Lippmann-Schwinger equation which is regularized with a cut-off function of the type

$$\mathrm{F}(p, p') = exp\left(-\frac{p^4 + p'^4}{\Lambda^4}\right)$$

in order to remove high-energy components of the baryon and pseudoscalar meson fields. The cut-off Λ is usually taken in the range 450-700 MeV

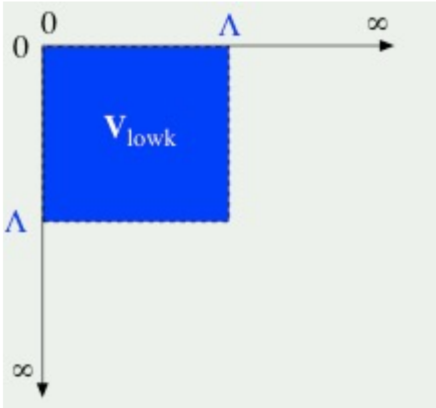
Renormalization Group Method

- The presence of a short–range hard core of the nucleon-nucleon interaction V makes any perturbation expansion in terms of V meaningless



- A possible way to soften it consists in integrating out all the momenta q larger than a certain cut-off Λ obtaining in this way an effective interaction $V_{low\ k}$ that is equivalent to the original one for momenta $q < \Lambda$

This results in a modified Lippmann-Schwinger equation with a cut-off dependent effective potential $V_{low\ k}$



$$T\ (k', k\colon E_k) = V_{low\ k}(k', k) + \frac{2}{\pi}\ P \int_0^{\Lambda} dq q^2 \frac{V_{low\ k}(k', q) T(q, k\colon E_k)}{k^2 - q^2 + i\eta}$$

Bogner et al., Phys. Rep. 386, 1 (2003)

Renormalization Group Method

➢ By demanding $\frac{dT(k',k:E_k)}{d\Lambda} = 0$ one obtains a Renormalization Group equation for $V_{low\ k}$

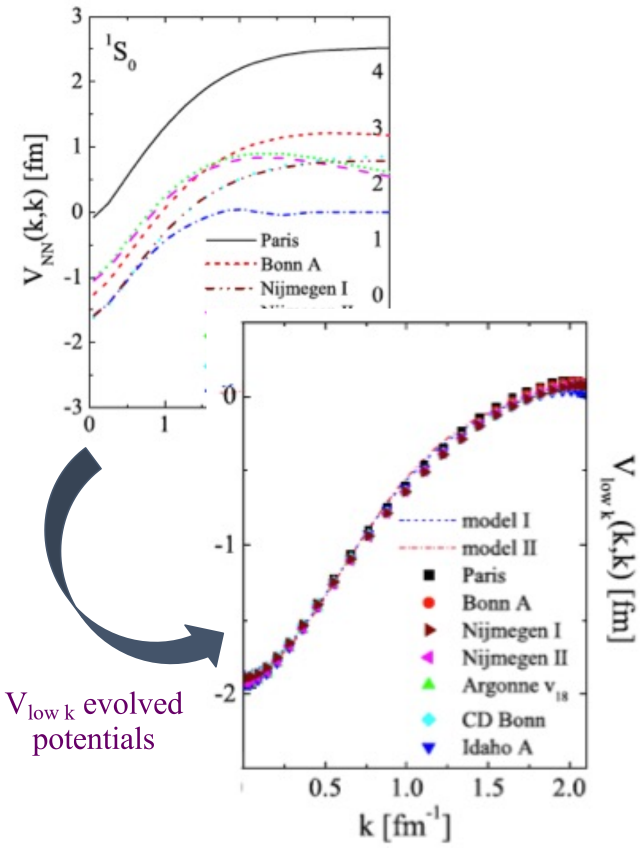
$$\frac{dV_{low\ k}(k',k)}{d\Lambda} = \frac{2}{\pi}\frac{V_{low\ k}(k',k)T(\Lambda,k,\Lambda^2)}{1-k^2/\Lambda^2}$$

➢ Integrating this flow equation one obtains a "universal" nucleon-nucleon low-momentum potential $V_{low\ k}$ that is:

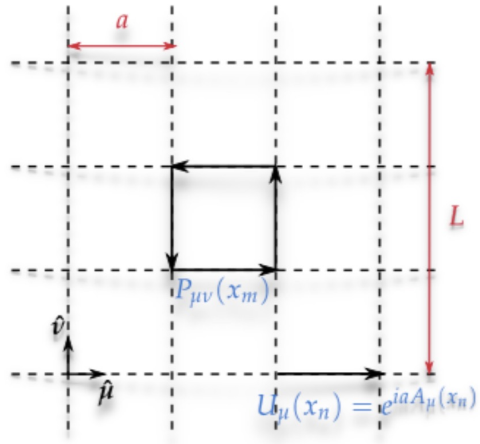
- ✓ phase shift equivalent
- ✓ energy independent
- ✓ softer (no hard core)
- ✓ hermitian

➢ Having a much softer core the $V_{low\ k}$ potential can be used in perturbation expansions and nuclear structure calculations in a more efficient way

➢ The method has been applied also to the hyperon-nucleon case. The results seem to indicate a similar convergence to a "universal" softer low-momentum hyperon-nucleon interaction

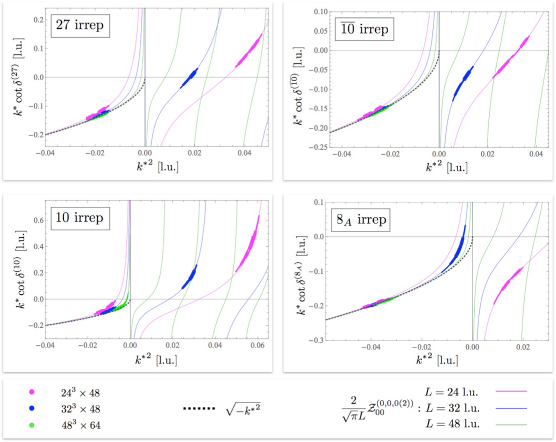


Baryon-baryon interactions from Lattice QCD



- The key idea behind lattice QCD is to replace the infinite four-dimensional space-time continuum with a finite hypercubic lattice
  - Quark fields are defined on the lattice sites
  - Gluon fields live on the links
  - The quantum field theory is mapped into a classical statistical system
  - Computer simulations use methods analogous to those of statistical mechanics to calculate correlation functions of hadronic operators & matrix elements of any operator between hadronic states in terms of fundamental quark and gluon degrees of freedom
  - Extremely expensive from a numerical point of view
- A big progress has been made by the NPLQCD & the HALQCD collaborations to derive baryon-baryon interactions from lattice QCD

Baryon-baryon interactions from Lattice QCD



NPLQCD & the HALQCD strategies

- NPLQCD

Combines calculations of correlation functions of two-baryon systems at several light-quark-mass values with low-energy effective field theory to extract scattering phase-shifts

- HALQCD

  - Determine the Nambu-Bethe-Salpeter wave function on the lattice

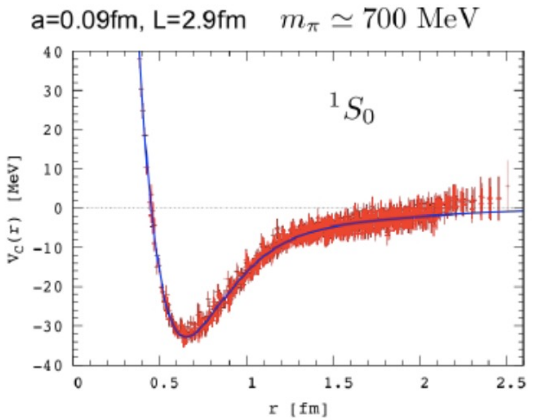
$$\varphi_{E(r)} = \langle 0|N((x+r,0)N(x,0)|6q,E\rangle\,, N(x) = \varepsilon_{abc}q^a(x)q^b(x)q^c(x)$$

  - Define a local potential $U(x,y)$ from $\varphi_{E(r)}$

$$\left[E - \frac{\hbar^2\nabla^2}{2\mu_N}\right]\varphi_{E(x)} = \int d^3y U(x,y)\varphi_{E(y)}\,, \qquad U(x,y) = V(x,\nabla)\delta\,(x-y)$$

$$V(x,\nabla) = V_c(x) + V_T(x)S_{12} + V_{LS}(x)\vec{L}\cdot\vec{S} + \{V_D,\nabla^2\} + \cdots$$

  - Calculate observables (phase shifts, binding energies, …)



Theoretical approaches to the nuclear EoS

Approaches to the Nuclear EoS: "Story of Two Philosophies"

Ab-initio Approaches

Based on two- & three-nucleon realistic interactions which reproduce scattering data & the deuteron properties. The EoS is obtained by "solving" the complicated many-body problem

- ✧ Brueckner-Bethe-Goldstone theory
- ✧ Self Consistent Green's Function formalism
- ✧ Variational Approach
- ✧ Quantum Monte Carlo Methods

Phenomenological Approaches

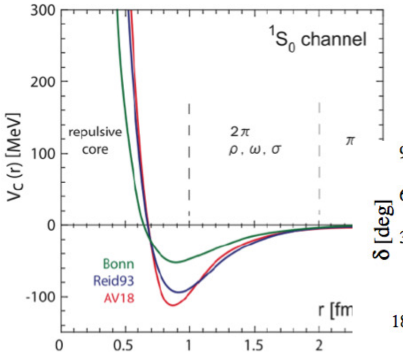
Based on effective density-dependent interactions with parameters adjusted to reproduce nuclear observables & compact star properties.

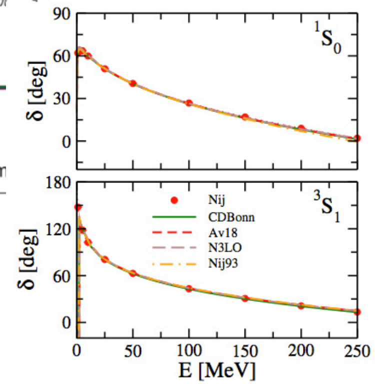
- ✧ Non-relativistic approaches: Skyrme & Gogny
- ✧ Relativistic Mean Field Theory
- ✧ Others: QMC, BCPM

Ab-initio approaches

Difficulties of ab-initio approaches

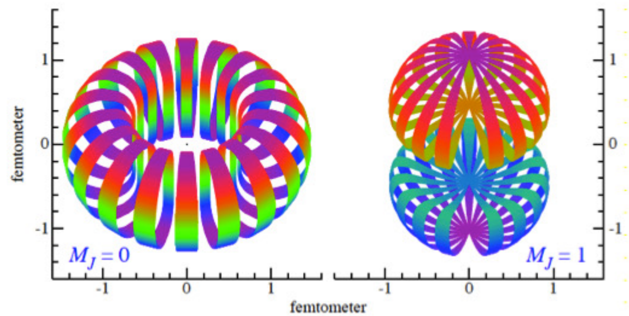
✧ Different NN potentials in the market … but all are phase-shift equivalent





✧ Short range repulsion makes any perturbation expansion in terms of V meaningless. Different ways of treating short range correlations

✧ Complicated channel & operatorial structure (central, spin-spin, spin-isospin, tensor, spin-orbit, …)



Brueckner-Bethe-Goldstone theory

Consider a system of A fermions described by the hamiltonian

$$H = \sum_{i=1}^{A} K_i + \sum_{i<j}^{A} V_{ij}$$

→ **Ground State** $H|\psi\rangle = E|\psi\rangle$

UNSOLVABLE because of the short-range hard core of BB interaction

- **Idea:** introduce an auxiliary single-particle potential U_i

$$H = \sum_{i=1}^{A}(K_i + U_i) + \sum_{i<j}^{A} V_{ij} - \sum_{i=1}^{A} U_i$$

H_o unperturbed — H_1 perturbation

$$E = E_0 + \Delta E$$

$$H_0|\phi_0\rangle = E_0|\phi_0\rangle$$

ΔE → perturbation theory

Brueckner-Bethe-Goldstone theory

Perturbation theory $\longrightarrow$ $$\Delta E = \langle \Phi_o | H_1 | \Phi_o \rangle + \left\langle \Phi_o \middle| H_1 \frac{1 - |\Phi_0\rangle\langle\Phi_0|}{E_0 - H_0} H_1 \middle| \Phi_o \right\rangle + \cdots$$

- The correlated wave function Ψ & the uncorrelated one Φ_0 satisfy: $|\Psi\rangle = |\Phi_0\rangle + \frac{1 - |\Phi_0\rangle\langle\Phi_0|}{E_0 - H_0} H_1 |\Psi\rangle$

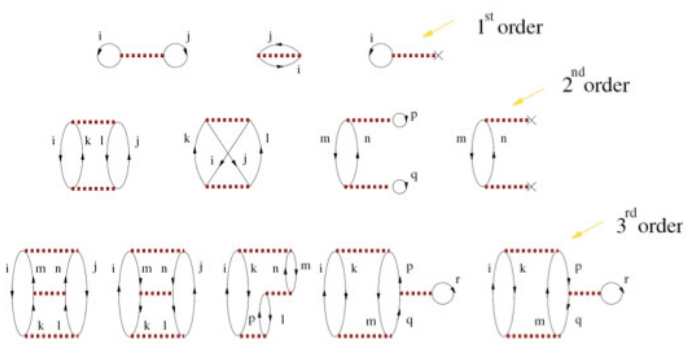
$$\Rightarrow \quad \Delta E = \frac{\langle \Phi_0 | H_1 | \Psi \rangle}{\langle \Phi_0 | \Psi \rangle}$$

- Goldstone (Proc. Roy. Soc. A 293, 267 (1957)) showed :

$$\Delta E = \langle \phi_0 | H_1 \sum_{n=0}^{\infty} \left[\frac{1 - |\phi_0\rangle\langle\phi_0|}{E_0 - H_0} H_1 \right]^n |\phi_0\rangle_l$$

$\langle \Phi_0 | H_1 | \Psi \rangle$ factorizes into the product of $\langle \Phi_0 | \Psi \rangle$ & a quantity that contains only linked diagrams

(e.g., those which cannot be separated in two pieces by a vertical cut without crossing a line)



Goldstone Expansion

Brueckner-Bethe-Goldstone theory

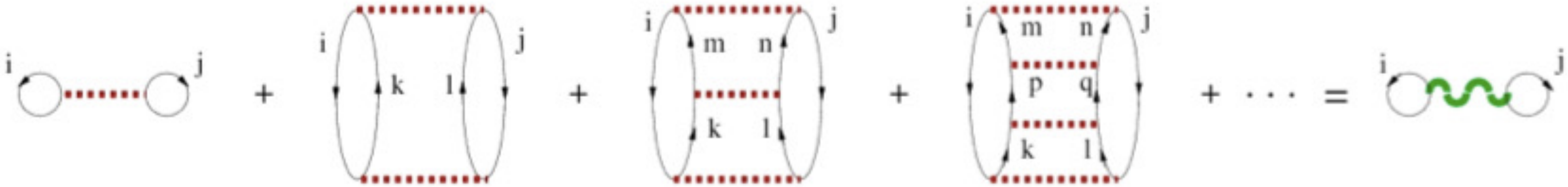
- The Goldstone expansion provides a simple & explicit prescription for calculating every order of perturbation theory

- However, it cannot be used in its present form for nuclear matter calculations because the short-range repulsion of the BB interaction makes all matrix elements very large and the perturbation series does not converge

- The solution is provided by the Brueckner theory in which the perturbation expansion in terms of the bare potential is replaced by another on in terms of the so-called Brueckner's reaction matrix. All the terms in this new perturbation series (Brueckner-Goldstone expansion) are finite and of reasonable size.

Brueckner-Bethe-Goldstone theory

The Brueckner's reaction matrix (or G-matrix) is obatined by performing a partial (infinite) summation of the set of particle-particle ladder diagrams



which defines the so-called Bethe-Goldstone equation

$$G = V + V\frac{Q}{\omega - H_0 + i\eta}V + V\frac{Q}{\omega - H_0 + i\eta}V\frac{Q}{\omega - H_0 + i\eta}V + \cdots$$

$$= V + V\frac{Q}{\omega - H_0 + i\eta}\left[V + V\frac{Q}{\omega - H_0 + i\eta}V + V\frac{Q}{\omega - H_0 + i\eta}V\frac{Q}{\omega - H_0 + i\eta}V + \cdots\right]$$

G

Brueckner-Bethe-Goldstone theory

Then:

$$G = V + V \frac{Q}{\omega - H_0 + i\eta} G$$

Note that the Bethe-Goldstone equation is formally identical to the Lippmann-Schwinger equation describing the scattering of two particles in free space

$$T = V + V \frac{1}{\omega - K + i\eta} T$$

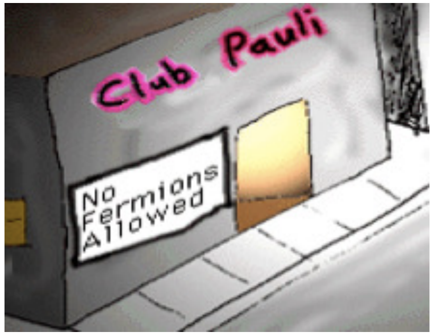
→ "The G-matrix describes the scattering of two particles in the presence of a surrounding medium"

Brueckner-Bethe-Goldstone theory

- Medium Effects

  - ✓ Pauli blocking of intermediate states

    The Pauli operator Q prevents the scattering to any occupied state, limiting the phase space of intermediate states



  - ✓ Dressing of intermediate particles

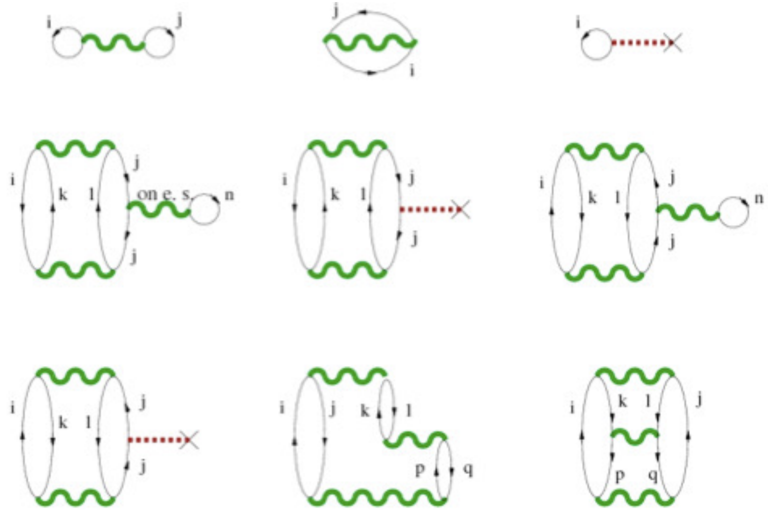
    The s.p. spectrum is modified by U which represents the average potential "felt" by a particle due to the presence of the medium

Brueckner-Bethe-Goldstone theory

- Hole-line expansion & the Brueckner-Hartree-Fock approximation

Goldstone expansion in terms of G

➡ Brueckner-Goldstone expansion

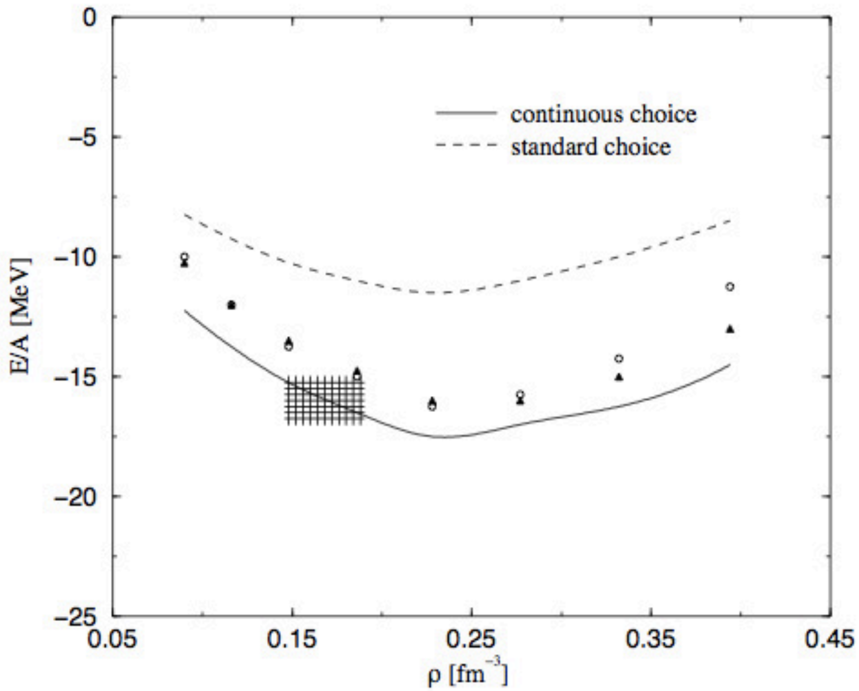


Grouping by number of hole lines ($c/r_0<1$) hole-line or Brueckner-Bethe-Goldstone expansion. Leading term: two-hole line or BHF approximation

$$E_{BHF} = \sum_{i \le A} \langle \alpha_i | K | \alpha_i \rangle + \frac{1}{2} \mathrm{Re} \left[\sum_{i,j \le A} \langle \alpha_i \alpha_j | G(\omega) | \alpha_i \alpha_j \rangle \right]$$

Brueckner-Bethe-Goldstone theory

The convergence of the hole-line expansion depends on the choice of the auxiliary potential U



H. Q. Song et al.,, PRL 81, 1584 (1998)

- Standard or Gap Choice
  - $k < k_F$

$$U_B(k) = \sum_{B'} \sum_{k' \le k_{F_{B'}}} \left\langle \vec{k}\vec{k}' \middle| G\left(\omega = E_B(k) + E_{B'}(k')\right) \middle| \vec{k}\vec{k}' \right\rangle$$

  - $k > k_F$

$$U_B(k) = 0$$

- Continuous Choice

$$U_B(k) = \sum_{B'} \sum_{k' \le k_{F_{B'}}} \left\langle \vec{k}\vec{k}' \middle| G\left(\omega = E_B(k) + E_{B'}(k')\right) \middle| \vec{k}\vec{k}' \right\rangle$$

Self Consistent Green's Function formalism

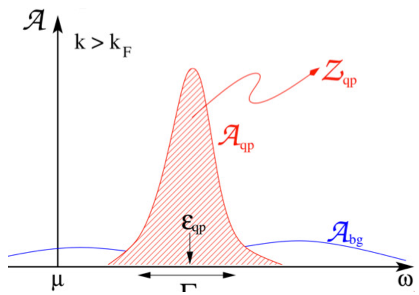
In the Self Consistent Green's Function (SCGF) approach the energy per particle of nuclear matter is obtain through the so-called Galitskii-Migdal-Koltum (GMK) sum-rule

$$E = \frac{\nu}{\rho}\int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{1}{2}\left\{\frac{\hbar^2k^2}{2m}+\omega\right\}A\left(\vec{k},\omega\right)f\left(\omega\right)$$

single-particle spectral function   Fermi-Dirac distribution

The key quantity of this approach is the one-body spectral function A(k,ω) which represents the probability density of removing from or adding to the system a nucleon with momentum k and energy ω. It gives access to the calculation of all the one-body properties of the system and can be obtained from the proper or irreducible self-energy

$$A(\vec{k},\omega) = \frac{-2\,\mathrm{Im}\,\Sigma(\vec{k},\omega)}{\left[\omega - \frac{\hbar^2k^2}{2m} - \mathrm{Re}\,\Sigma(\vec{k},\omega)\right]^2 + \left[\mathrm{Im}\,\Sigma(\vec{k},\omega)\right]^2}$$



Self Consistent Green's Function formalism

The computational implemention of the SCGF method requieres:

1. Calculate the effective interaction (T-matrix) describing the in medium scattering of two nucleons

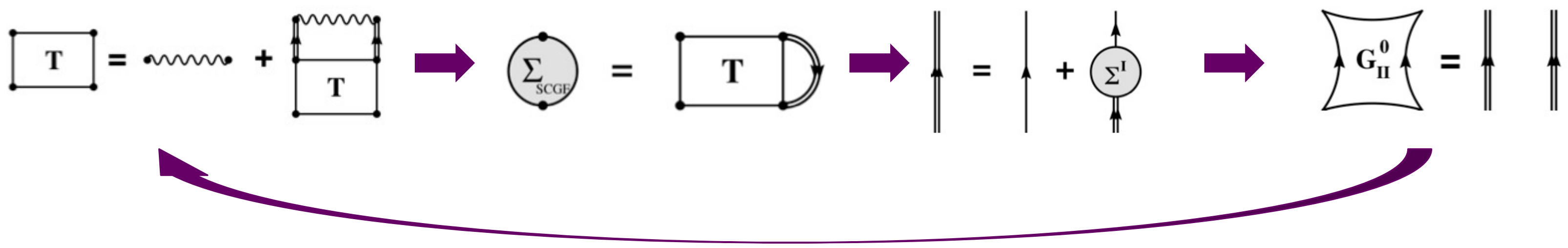
2. Extract the self energy $\Sigma(k,\omega)$ to obtain the one-body propagator $G(k,\omega)$ by solving the Dyson equation which is then inserted in the scattering equation, repeating these steps till a self-consisten solution is achieved.

In-medium interaction

Ladder self-energy

Dyson equation

Free two-particle propagator



Figures adapted from A. Rios

Carbone et al., PRC 88, 054326 (2013)

Variational Approach

The variational approach to the nuclear EoS is based on the Ritz-Raleight variational principle

$$E \leq min\left\{\frac{\langle\Psi_T|\hat{H}|\Psi_T\rangle}{(\Psi_T|\Psi_T)}\right\} \text{ with } \Psi_T(r_1, r_2, \cdots) = \prod_{i<j} f(r_{ij})\,\Phi(r_1, r_2, \cdots)$$

✓ $\Phi(r_1, r_2, \cdots)$: uncorrelated ground-state wave function properly antisymmetrized and product of all possible pairs of particles (i.e., Slater Determinant)

✓ $f(ij)$: correlator factors take into account the correlations of the system. Are determined by means of the Ritz-Raleight variational principle, i.e. by assuming that the mean value of the Hamiltonian reaches a minimum

$$\frac{\delta}{\delta f}\left(\frac{\langle\Psi_T|\hat{H}|\Psi_T\rangle}{(\Psi_T|\Psi_T)}\right) = 0$$

→ The main task of the variational method is to find a suitable ansatz for the correlation factors f

Variational Approach

- For nuclear matter it is necessary to introduce channel-dependent correlation factors. This is equivalent to assume that the f's are actually two-body operators $\hat{F}$ which one assumes can be expanded in the same type of operators of the nucleon-nucleon interaction

$$\hat{F} = \prod_{i<j} \sum_{p} f^{(p)}(r_{ij}) \hat{O}_{ij}^{(p)}$$

- Due to the formal structure of the Argonne NN potential, most variational calculations have been done with this class of interactions supplemented by the Urbana three-nucleon forces.

- The best know and most used variational nuclear matter EoS is the one of Akmal, Pandharipande & Ravenhall (APR) (PRC 85, 1804 (1998))

- Other methods based on the variational approach are the Coupled-Cluster theory (Coester NPA 7, 421 (1958)). or the Variational Monte Carlo (VMC) (Wiringa et al., PRC 89, 024305 (2014))

Quantum Monte-Carlo Methods

✧ VMC:

Evaluate energy & other observables using the Metropolis method

$$\langle \hat{O} \rangle = \frac{\sum_i \langle \Psi(\vec{R}_i) | \hat{O} | \Psi(\vec{R}_i) \rangle / W(\vec{R}_i)}{\sum_i \langle \Psi(\vec{R}_i) | \Psi(\vec{R}_i) \rangle / W(\vec{R}_i)}$$

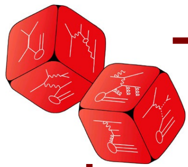
Wiringa et al., PRC 62, 014001 (2000)

✧ DMC:

Model a diffusion process rewriting the Schoedinger equation in imaginary time

$$i \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle \Rightarrow -\frac{\partial}{\partial \tau} |\Psi\rangle = \hat{H} |\Psi\rangle$$

Anderson, J. Chem. Phys. 63, 1499 (19755)



✧ GFMC:

Sample a trial wave function by evaluating path integrals of the form

$$|\Psi(\tau)\rangle = \prod \exp\left[-\left(\hat{H} - E_0\right)\Delta\tau\right] |\Psi_V\rangle$$

$$|\Psi(\tau)\rangle \underset{n\to\infty}{\longrightarrow} |\Psi_0\rangle$$

Carlson et al., PRC 68, 025802 (2003)

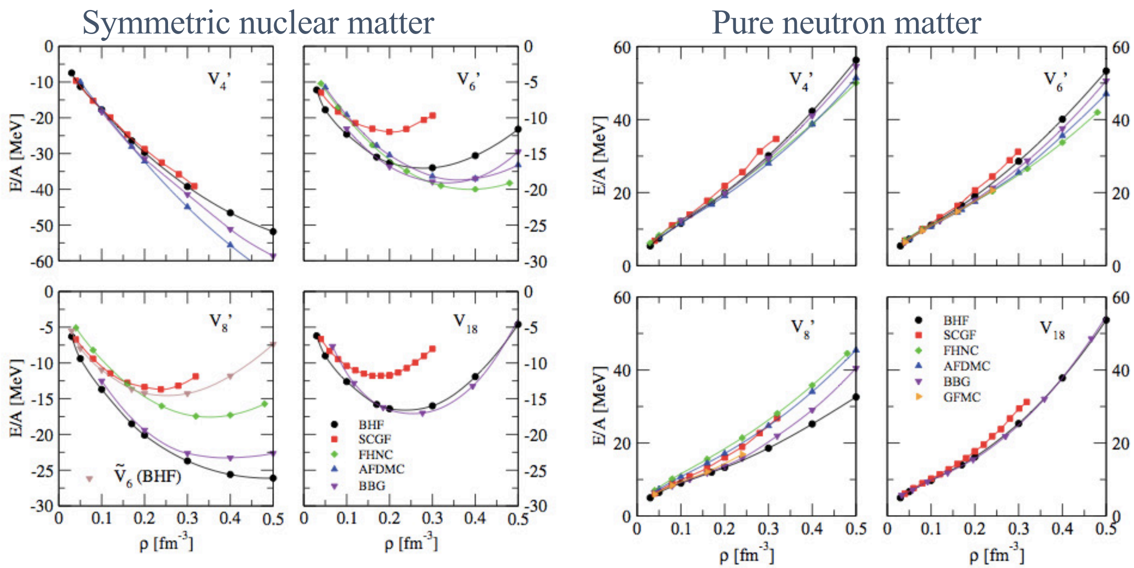
✧ AFDMC:

Rewrite Green's function in order to change the quadratic dependence on spin & isospin operators to a linear one by introducing Hubbard-Stratonovich auxiliary fields

Gandolfi et al., PRC 79, 054005 (2009)

A comparison of some ab-initio approaches

Compare different many-body techniques using the same NN interaction (Argonne family) to find the sources of discrepancies & ultimately determine “systematic error” associated to the nuclear EoS predicted by many-body theory



Tensor & spin-orbit and their in-medium treatment are at the heart of most of the observed discrepancies

M. Baldo, A. Polls, A. Rios, H.-J. Schulze & I.Vidaña, PRC 86, 064001 (2012)

Phenomenological approaches

Skyrme & Gogny interactions

✧ Skyrme interactions:

Effective zero-range density dependent interaction

$$\hat{V}(\vec{r}_1,\vec{r}_2) = t_0\left(1+x_0\hat{P}_\sigma\right)\delta(\vec{r}_{12}) + \frac{t_1}{2}\left(1+x_1\hat{P}_\sigma\right)\left[\hat{k}^{'}\delta(\vec{r}_{12}) + \delta(\vec{r}_{12})\hat{k}^2\right]$$

$$+t_2\left(1+x_2\hat{P}_\sigma\right)\hat{k}'\delta(\hat{r}_{12})\hat{k} + \frac{t_3}{6}\left(1+x_3\hat{P}_\sigma\right)\rho^\alpha(\vec{R}_{12})\delta(\hat{r}_{12})$$

$$+iW_0\left(\hat{\sigma}_1+\hat{\sigma}_2\right)\left[\hat{k}^{'}\times\delta(\hat{r}_{12})\hat{k}\right]$$

Evaluation of the energy density in the HF approximation yields for nuclear matter a simple EDF in fractional powers of the number densities. Many parametrizations exist

Skyrme, Nucl. Phys. 9, 615 (1959)

✧ Gogny interactions:

Effective finite-range density dependent interaction

$$\hat{V}(\vec{r}_1,\vec{r}_2) = \sum_{j=1,2}\exp\left(-\frac{r_{12}^2}{\mu_j^2}\right)\left(W_j + B_j\hat{P}_\sigma - H_j\hat{P}_\tau - M_j\hat{P}_\sigma\hat{P}_\tau\right)$$

$$+t_0\left(1+x_0\hat{P}_\sigma\right)\rho^\alpha(\vec{R}_{12})\delta(\hat{r}_{12})$$

$$+iW_0\left(\hat{\sigma}_1+\hat{\sigma}_2\right)\left[\hat{k}^{'}\times\delta(\hat{r}_{12})\hat{k}\right]$$

Due to the finite-range terms the evaluation of the energy density is numerically more involved. Less number of parametrizations in the market

Brink & Boeker, NPA 91, 1 (1967)

Relativistic Mean Field Theory Approach to the nuclear EoS

RMF models are based on effective Lagrangian densities in which the baryon-baryon interactions are described in terms of meson exchanges. Considering only σ, ω & ρ mesons, e.g.,

$$
\begin{aligned}
L = & \sum_B \bar{\psi}_B \left(i\gamma_\mu \partial^\mu - m_B + g_{\sigma B}\sigma - g_{\omega B}\gamma_\mu \omega^\mu - \frac{1}{2} g_{\rho B}\gamma_\mu \vec{\tau}\cdot\vec{\rho}^\mu \right)\psi_B \\
& + \frac{1}{2}\left(\partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2\right) - \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu \\
& - \frac{1}{4}\vec{\rho}_{\mu\nu}\cdot\vec{\rho}^{\mu\nu} + \frac{1}{2} m_\rho^2 \vec{\rho}_\mu \cdot \vec{\rho}^\mu - \frac{1}{3} b m_N (g_{\sigma N}\sigma)^3 - \frac{1}{4} c (g_{\sigma N}\sigma)^4 \\
& + \sum_\lambda \bar{\psi}_\lambda \left(i\gamma_\mu \partial^\mu - m_\lambda \right)\psi_\lambda
\end{aligned}
$$

Hadron contribution

Lepton contribution
(for neutron star matter)

$$
\omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu; \quad \vec{\rho}_{\mu\nu} = \partial_\mu \vec{\rho}_\nu - \partial_\nu \vec{\rho}_\mu
$$

$$
B = n, p, \Lambda, \Sigma^-, \Sigma^0, \Sigma^+, \Xi^-, \Xi^0; \quad \lambda = e^-, \mu^-
$$

Relativistic Mean Field Theory Approach to the nuclear EoS

The first step is to derive the Euler-Lagrangian equations of motion of the baryon & meson fields

- Baryon field equations of motion

$$\left[\gamma_\mu\left(i\partial^\mu - g_{\omega B}\omega^\mu - \frac{1}{2}g_{\rho B}\vec{\tau}\cdot\vec{\rho}^\mu\right) - \left(m_B - g_{\sigma B}\sigma\right)\right]\psi_B = 0$$

- Meson field equations of motion

$$\left(\partial_\nu\partial^\nu + m_\sigma^2\right)\sigma = \sum_B g_{\sigma B}\overline{\psi}_B\psi_B$$

$$\left(\partial_\nu\partial^\nu + m_\omega^2\right)\omega_\mu - \partial_\mu\partial^\nu\omega_\mu = \sum_B g_{\omega B}\overline{\psi}_B\gamma_\mu\psi_B$$

$$\left(\partial_\nu\partial^\nu + m_\rho^2\right)\rho_\mu^i - \partial_\mu\partial^\nu\rho_\nu^i = \sum_B g_{\rho B}\overline{\psi}_B\gamma_\mu\psi_B$$

Relativistic Mean Field Theory Approach to the nuclear EoS

The next step is to solve the Euler-Lagrange equations. This is done in the mean field approximation which consist in rerplacing the meson fields σ, ω, ρ by their expectation values <σ>, <ω> , <ρ> and the baryon currents by their ground state expectations generated by the presence of mean meson fields

- Baryon field equations of motion

$$\left[i\gamma_\mu \partial^\mu - g_{\omega B}\gamma_0 \left\langle \omega_0 \right\rangle + \frac{1}{2} g_{\rho B}\gamma_0 \left\langle \rho^{03} \right\rangle - m_B + g_{\sigma B} \left\langle \sigma \right\rangle \right] \psi_B = 0$$

- Meson field equations of motion

$$\left\langle \sigma \right\rangle = -b m_N g_{\sigma N}^3 \left\langle \sigma \right\rangle^2 - c m_N g_{\sigma N}^4 \left\langle \sigma \right\rangle^3 + \sum_B \frac{2J_B+1}{2\pi^2} g_{\sigma B} \int_0^{k_{F_B}} \frac{m_B - g_{\sigma B}\left\langle \sigma \right\rangle}{\sqrt{k^2 + (m_B - g_{\sigma B}\left\langle \sigma \right\rangle)^2}} k^2\, dk$$

$$\left\langle \omega_0 \right\rangle = \sum_B \frac{g_{\omega B}}{m_\omega^2} \frac{2J_B+1}{6\pi^2} b_B k_{F_B}^3; \;\; \left\langle \omega_k \right\rangle = 0$$

$$\left\langle \rho_{03} \right\rangle = \sum_B \frac{g_{\rho B}}{m_\rho^2} I_{3B} \frac{2J_B+1}{6\pi^2} b_B k_{F_B}^3; \;\; \left\langle \rho_{k3} \right\rangle = 0$$

Relativistic Mean Field Theory Approach to the nuclear EoS

The EoS (energy density & pressure) can then be obtained from the energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial L}{\partial\left(\partial_\mu \phi_i\right)} \partial^\nu \phi_i - \eta^{\mu\nu} L$$

whose expectation value in the rest mass frame is diagonal

$$T^{\mu\nu} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

being

$$\varepsilon = -\langle L \rangle + \langle \bar{\psi} \gamma_0 k^0 \psi \rangle$$

$$p = \langle L \rangle + \frac{1}{3} \langle \bar{\psi} \gamma_i k^i \psi \rangle$$

Relativistic Mean Field Theory Approach to the nuclear EoS

Using the Lagrangian density of the present theory, we have

- Energy density

$$\varepsilon = \frac{1}{3} b m_N \left(g_{\sigma N} \langle \sigma \rangle \right)^3 + \frac{1}{4} c m_N \left(g_{\sigma N} \langle \sigma \rangle \right)^4 + \frac{1}{2} m_\sigma^2 \langle \sigma \rangle^2 + \frac{1}{2} m_\omega^2 \langle \omega_0 \rangle^2 + \frac{1}{2} m_\rho^2 \langle \rho_{03} \rangle^2$$
$$+ \sum_B \frac{2J_B + 1}{2\pi^2} \int_0^{k_{F_B}} \sqrt{k^2 + \left(m_B + g_{\sigma B} \langle \sigma \rangle \right)^2} k^2 \, dk + \sum_\lambda \frac{1}{\pi^2} \int_0^{k_{F\lambda}} \sqrt{k^2 + m_\lambda^2} k^2 \, dk$$

- Pressure

$$p = -\frac{1}{3} b m_N \left(g_{\sigma N} \langle \sigma \rangle \right)^3 - \frac{1}{4} c m_N \left(g_{\sigma N} \langle \sigma \rangle \right)^4 - \frac{1}{2} m_\sigma^2 \langle \sigma \rangle^2 + \frac{1}{2} m_\omega^2 \langle \omega_0 \rangle^2 + \frac{1}{2} m_\rho^2 \langle \rho_{03} \rangle^2$$
$$+ \frac{1}{3} \sum_B \frac{2J_B + 1}{2\pi^2} \int_0^{k_{F_B}} \frac{k^4 dk}{\sqrt{k^2 + \left(m_B + g_{\sigma B} \langle \sigma \rangle \right)^2}} + \frac{1}{3} \sum_\lambda \frac{1}{\pi^2} \int_0^{k_{F\lambda}} \frac{k^4 dk}{\sqrt{k^2 + m_\lambda^2}}$$

Relativistic Mean Field Theory Approach to the nuclear EoS

A final comment on the coupling constants

- The nucleon coupling constants $g_{\sigma N}$, $g_{\omega N}$, $g_{\rho N}$, b & c are constrained by the empirical values of density ρ_0, energy per particle E/A, incompressibility modulus K, symmetry energy E_{sym} & effective mass m^* at saturation

- The hyperon coupling constants $g_{\sigma Y}$, $g_{\omega Y}$, $g_{\rho Y}$ are constrained by: the binding energy of Λ hyperon in nuclear matter, hypernuclear levels & compact star properties (mass)

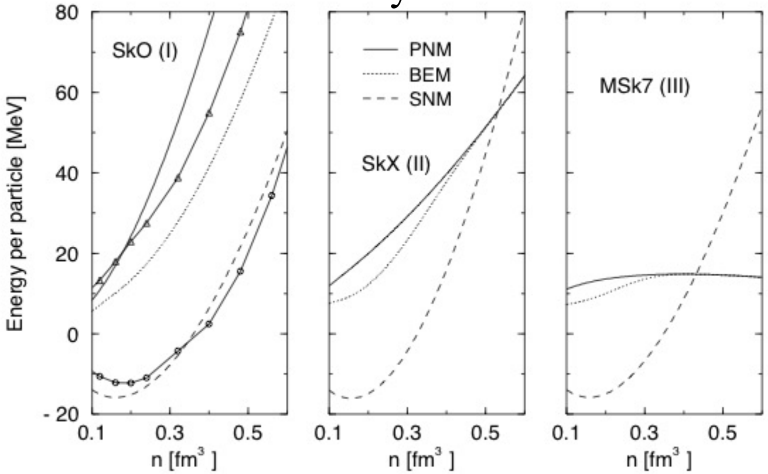
Assuming that all hyperons in the baryon octet have the same coupling, the hyperon couplings can be expressed as:

$$x_\sigma = \frac{g_{\sigma Y}}{g_{\sigma N}}, \quad x_\omega = \frac{g_{\omega Y}}{g_{\omega N}}, \quad x_\rho = \frac{g_{\rho Y}}{g_{\rho N}}$$

A comparison of phenomenological models

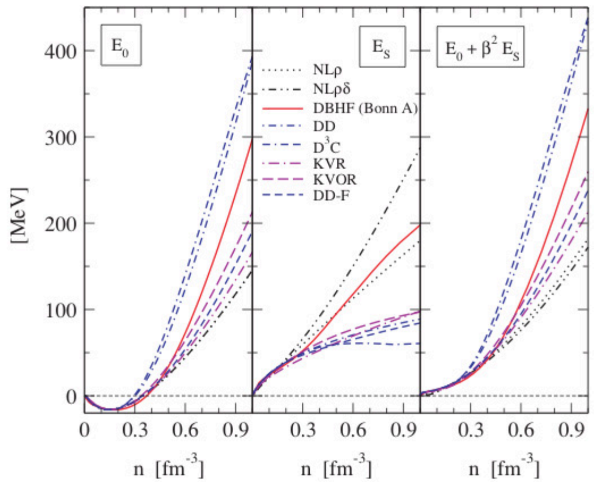
Proliferation of phenomenological models predicting different SM & NM EoS

Skyrme



J. R. Stone et al., PRC 68, 034324 (2003)

RMF



T. Klahn et al., PRC 74, 035802 (2006)

Few years ago M. Dutra et al., (PRC 90, 055203 (2014)) have analyzed 263 parametrizations of 7 different types of RMF imposing constraints from SM, PNM & Symmetry Energy and its derivatives. Similar analysis was done for 240 Skyrme forces by M. Dutra et al., (PRC 85, 035201 (2012)). In both cases a few number of parametrizations passed the stringent tests imposed

Other phenomenological models

✧ **Quark Meson Coupling model:**

Closely related with the RMF. Nucleons are considered a bound states of quarks which couple with mesons in the surrounding medium

Downnum et al., Phys. Lett. B 638, 455 (2006)

✧ **Barcelona-Catania-Paris-Madrid EDF:**

EDF constructed by parametrizing BHF results obtained with realistic NN interactions. The addition of appropiate surface & spin-orbit contributions proves an excellent description of finite nuclei

Baldo et al., PRC 87, 064305 (2013)

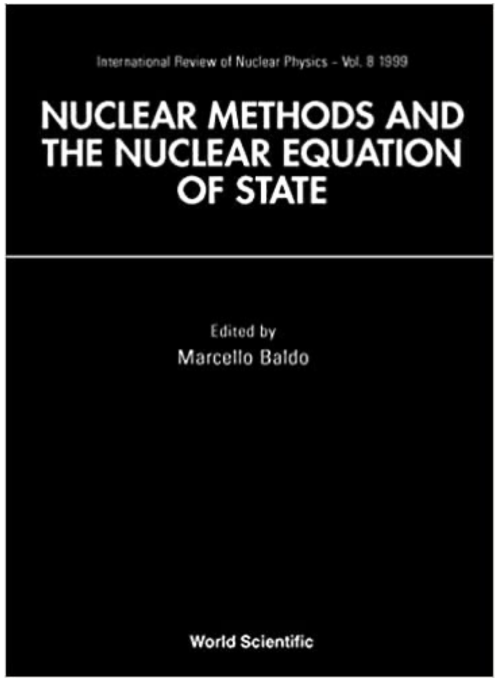
✧ **Other:**

- ✓ Density-dependent separable model (SMO)
- ✓ Three-range Yukawa (M3Y) interactions

Rikovska Stone, PRC 65, 064312 (2002)
Nakada, PRC 68, 014316 (2003)

For further reading

An excellent monographs on this the nuclear methods and the nuclear EoS and for interested readers is:



Other interesting reviews are:

- Oertel, Hempel, Klahn & Typel, Rev. Mod. Phys. 89, 015007 (2017)
- Burgio & Fantina, in “The Physics & Astrophysics of Neutron Stars”, Springer-Verlag 2018
- Burgio, Schulze, I.V. & Wei, Prog. Part. Nucl. Phys. 120, 103879 (2021)

- You for your time & attention
- The organizers for their invitation & support