Isotropic compact star in mimetic gravitational theory coupled with Lagrangian multiplier

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Isotropic solution in mimetic-like gravity coupled with Lagrange multiplier

In the mimetic theory, the metric $g_{\mu\nu}$ is defined in terms of an auxiliary metric $\bar{g}_{\mu\nu}$ and a scalar field η as

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\partial_\alpha\eta\partial_\beta\eta \,. \tag{1}$$

The action of the mimetic-like gravity coupled with the Lagrange multiplier λ and the function ω has the form

$$S = \int \mathrm{d}x^4\sqrt{-g}\,\{R + \lambda\,(g^{\mu\nu}\omega\partial_\mu\eta\partial_\nu\eta + 1)\} + L_{\mathrm{matt}}\,, \tag{2}$$

where L_{matt} is the Lagrangian of the matter field and η is the mimetic scalar field.

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The field equations:

$$0 = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{2} g_{\mu\nu} \left\{ \lambda \left(g^{\rho\sigma} \omega \partial_\rho \eta \partial_\sigma \eta + 1 \right) \right\} - \lambda \partial_\mu \eta \partial_\nu \eta + \frac{1}{2} T_{\mu\nu} \,, \tag{3}$$

where $T_{\mu\nu}$ is the energy-momentum tensor corresponding to the matter field.

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The variation the action (2) w.r.t. the mimetic scalar field η gives:

$$2\nabla^\mu(\lambda\omega\,\partial_\mu\eta) = 0\,. \tag{4}$$

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At the end, the variation of the action (2) w.r.t. the Lagrange multiplier λ yields:

$$g^{\rho\sigma}\omega\partial_\rho\eta\partial_\sigma\eta = -1\,. \tag{5}$$

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Applying the field equations (3) and (4) to the spherically symmetric spacetime

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f_1(r)} - r^2\left(d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \, , \tag{6}$$

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The (t, t)-component of the field equation (3) is:

$$\rho(r) = \frac{1 - f_1 - r f_1'}{r^2}\,, \tag{7}$$

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both of the (θ,θ) and (ϕ,ϕ)-components of the field equation (3) have the form:

$$p(r) = \frac{2\, f_1 f'' f r - f'^2 f_1 r + f\,(2\, f_1 + f_1' r)\, f' + 2\, f_1' f^2}{4 f^2 r}\,, \tag{9}$$

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and the field equation (4) takes the form:

$$0 = 2\lambda' \omega f r + \left[\omega' f r + \omega\left(f' r + 4 f\right)\right]\lambda\,, \tag{10}$$

Isotropic solution in mimetic-like gravity coupled with Lagrange multiplier

The energy-momentum tensor of isotropic fluid has the form:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \,, \tag{11}$$

where ρ is the energy-density, p is the pressure, with u^μ being the time-like vector defined as $u^\mu = [1, 0, 0, 0]$.

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Assuming the fluid under consideration has a perfect fluid($p = p\,(\rho)$).
Using the conservation law of matter gives:

$$0 = \nabla^\mu \, T_{\mu r} = 2f\frac{dp}{dr} + f'\,(\rho + p) \;. \tag{12}$$

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If the form of EoS $\rho = \rho(p)$ is presented, then Eq. (12) yield:

$$\frac{1}{2}\ln f = -\int^{r} dr \frac{\frac{dp}{dr}}{\rho + p} = -\int^{p(r)} \frac{dp}{\rho(p) + p} \,. \qquad (13)$$

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If we consider a compact star like neutron star, one usually consider the EoS as:

1. Energy-polytrope

$$p = k\rho^{1+\frac{1}{s}}\,, \tag{14}$$

where k and s are constants. It is well known that for neutron star, s lies in the interval $s \in [0.5\,,\,1]$.

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2. Mass-polytrope

$$\rho = \rho_m + s_1 p\,, \qquad p = m_m \rho_m^{1+\frac{1}{s_m}}\,, \tag{15}$$

with ρ_m being the rest mass energy density and m_m, s_1, and s_m are constants.

Isotropic solution in mimetic-like gravity coupled with Lagrange multiplier

It is the time to study the case of the energy-polytrope. Then EoS (14) can be rewritten as:

$$\rho = \tilde{k} p^{(1+\frac{1}{\tilde{s}})}\,, \quad \tilde{k} \equiv k^{-\frac{1}{1+\frac{1}{s}}}\,, \quad \tilde{s} \equiv \frac{1}{\frac{1}{1+\frac{1}{s}} - 1} = -1 - s\,. \qquad (16)$$

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Eq. (13) can take the form:

$$\frac{1}{2}\ln f = -\int^{p(r)} \frac{dp}{\tilde{k} p^{1+\frac{1}{\tilde{s}}} + p} = \frac{c_1}{2} + \tilde{s}\ln\left(1 + \tilde{k}^{-1} p^{-\frac{1}{\tilde{s}}}\right) = \frac{c_1}{2} - (1+s)\ln\left(1 + k\rho^{\frac{1}{s}}\right), \qquad (17)$$

where c_1 is a constant of integration.

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Using the same method of polytrope we get for mass-polytrope the function f as:

$$\frac{1}{2}\ln f = \frac{\tilde{c}}{2} + \ln\left(1 - k_m {\rho_m}^{\frac{1}{s_m}}\right) \,, \tag{18}$$

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Under one of the above equations of state, we may assume the following profile of $\rho = \rho(r)$ just for an example,

$$\rho = \begin{cases} \rho_0 \left(1 - \frac{r}{R}\right) & \text{when } r < R \\ 0 & \text{when } r \geq R \end{cases} . \tag{19}$$

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The mass of the compact star for polytropic EoS is defined as:

$$M = 4\pi \int_0^R y^2 \rho(y) dy = \frac{\pi \rho_0 R^3}{3} \,. \tag{20}$$

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The unknown functions f(r) and f1(r) are defined as

$$f = \frac{\mathrm{e}^{c_1}}{\left(1 + k\rho_0 \left(1 - \frac{r}{R}\right)\right)^4} \,, \tag{21}$$

$$f_1 = 1 - \frac{\rho_0 r^2}{3} + \frac{\rho_0 r^3}{4R} \,. \tag{22}$$

Isotropic solution in mimetic-like gravity coupled with Lagrange multiplier

The Lagrangian multiplier of the above model has the form

$$\lambda(r) = \frac{c_2 \left(R + k\rho_0[R - r]\right)^2}{r^{5/2}} \,. \tag{23}$$

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The form of the function $\omega = c_3 r$ and the mimetic scalar field becomes:

$$\eta(r) = \frac{1}{\sqrt{c_3 r \left(\frac{8\pi r^2}{3} - \frac{2r^3}{R} - 1\right)}} . \tag{24}$$

Necessary conditions for a real physical star

- The components of metric potentials g_{tt} and g_{rr}, and the energy-momentum components ρ, p must be well defined at the center of the star and regular inside the star.

[1] In this study we will take $r = xR$ where R is the radius of the star and x is a dimensionless parameter.

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• The components of metric potentials g_{tt} and g_{rr}, and the energy-momentum components ρ, p must be well defined at the center of the star and regular inside the star.

• In the interior of star $\rho \geq 0$. Additionally, the energy-density has a positive finite value at the center of the star, and $\frac{d\rho}{dr} \leq 0$ at the surface of the star.

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• $p \geq 0$. Additionally, $\frac{dp}{dr} < 0$ in the interior of the stellar. At the same time at $r = R$, the pressure p must be vanishing. [1]

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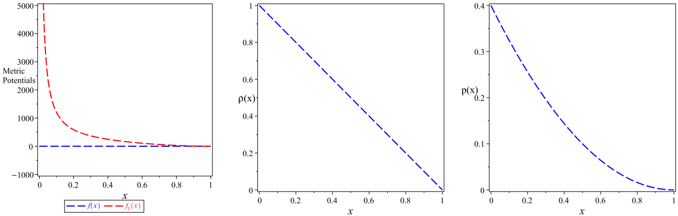
- Finally, the adiabatic index must has a value more than $\frac{4}{3}$.

PHYSICAL BEHAVIOR OF THE MODEL



(a) Metric potentials, $f(r)$ and $f_1(r)$ given by Eq. (21) and (22)

(b) Density

(c) Pressure

Figure: Schematic plot of the metric potentials (21), and (22) vs. the dimensionless x; (b) the profile of density; and (c) profile of pressure. We have put $\rho_0 = 1$ and $K = 0.4$.

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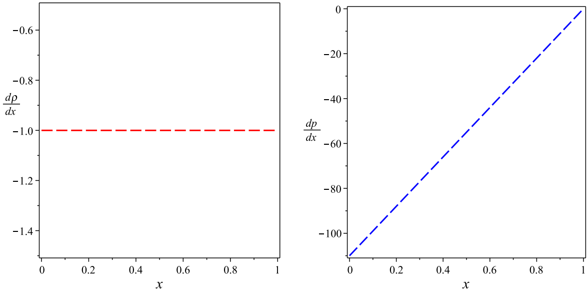
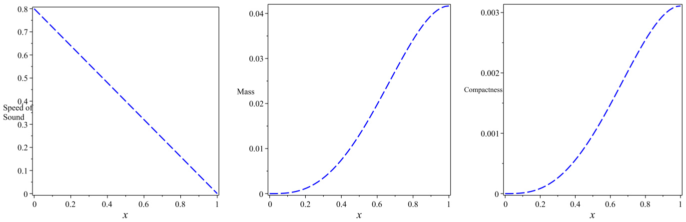


(a) Gradient of density (b) Gradient of pressure

Figure: Plot of the gradient of density and pressure vs. the dimensionless x.

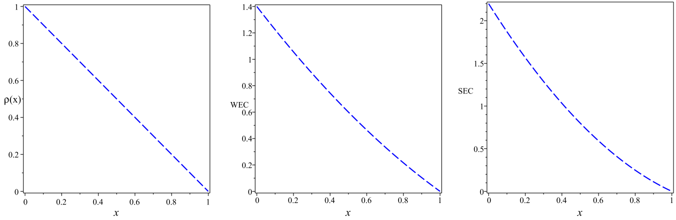
PHYSICAL BEHAVIOR OF THE MODEL



(a) Speed of sound (b) Mass-radius relation (c) Compactness

Figure: Plot of the speed of sound (a), mass-radius relation (b), and compactness of the stellar (c) via the dimensionless x.

PHYSICAL BEHAVIOR OF THE MODEL



(a) Null energy conditions (b) Weak energy conditions (c) Strong energy condition

Figure: Plot of the null, week and strong energy conditions vs. the dimensionless x.

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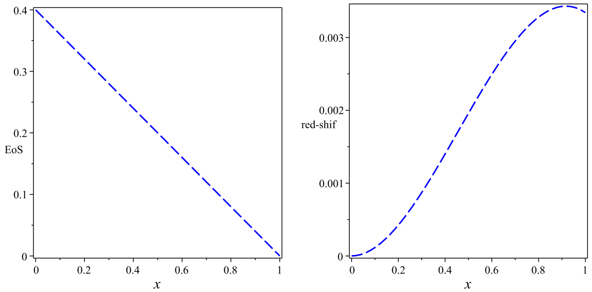

Figure: Plot of the EoS vs. the radial coordinate r (a) and the red shift (b).

Stability of the model

The adiabatic index Γ, is defined as

$$\Gamma = \left(\frac{\rho + p(x)}{p(x)} \right) \left(\frac{dp(x)}{d\rho(x)} \right) . \tag{25}$$

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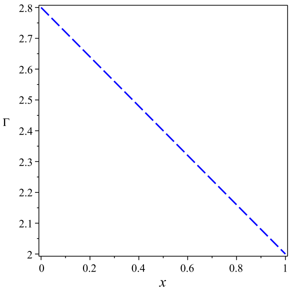

Figure: Plot of the gravitational, and the hydrostatic forces vs. the dimensionless x.

Stability of the model

Stability in the static state:
The mass of the central density, represented as

$$M(\rho_0) = \pi \int_0^R y^2 \rho(y) dy = \frac{\pi \rho_0 R^3}{3} . \tag{26}$$

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Equations (26) and (27) ensure the stability of the model.

Thanks to All