Bianchi Type-I and Type-V spacetimes with evolving gravitational and cosmological 'constants'

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- Modern cosmology based on a maximally symmetric spacetime, FLRW
  - ✓ Homogeneous: all regions of space look alike, no preferred positions
  - ✓ Isotropic: no preferred directions
- Recent cosmological observations have shown that the universe is undergoing a recent epoch of accelerated expansion
- Whereas it is not conclusively known what caused this recent cosmic acceleration, the prevailing argument is that dark energy caused it
- Among the most widely considered candidates of dark energy is the vacuum energy of the cosmological constant Λ
- Some serious problems associated with the cosmological constant, among them the eponymous *cosmological constant problem* [1] and the *coincidence problem* [2]
- Several alternatives proposed, such as:
  - ✓ Interacting vacuum
  - ✓ $\Lambda = \Lambda(t)$
- Of particular interest for us here are those anisotropic models with changing gravitational and cosmological 'constants' - $G(t)$ and $\Lambda(t)$

[1] Weinberg, S. The cosmological constant problem. Rev. Mod. Phys. 1989, 61 (1), 1

[2] Velten, H. E. et al. Aspects of the cosmological "coincidence problem". Eur. Phys. J. C 2014, 74 (11), 1

- Since the isotropy assumption is only an approximation on large scales, and not something explained from first principles, there is the possibility that the spatially homogeneous and anisotropic cosmological modes play a significant role in explaining the evolution of the universe at its early stages. At these times, the universe was full of anisotropies with a highly irregular mechanism that isotropized later
- In fact, there are several claims regarding some degree of anisotropy in the observed universe that necessitates the consideration of a non-FLRW geometry
- There is a need for a detailed study of cosmological models that describe an early-time anisotropy with a proper mechanism to produce [near] isotropy at late times on the one hand, and an accelerated expansion at the present epoch on the other
- Bianchi models to the rescue: homogenous but not [necessarily] isotropic cosmological models
  - ✓ 9 possible cosmological solutions
  - ✓ Bianchi-I and Bianchi-V are the simplest, and probably the most widely explored

Group Class	Group Type	n_1	n_2	n_3
A ($a_i = 0$)	I	0	0	0
	II	+	0	0
	VI_0	0	+	−
	VII_0	0	+	+
	IX	+	+	+
B ($a_i \neq 0$)	V	0	0	0
	IV	0	0	+
	VI_h	0	+	−
	VII_h	0	+	+

Figure 1.1: Bianchi classifications.

Evolving Λ and G solutions

- Dirac's hypothesis [3] that the gravitational constant decreases with time has been a matter of scrutiny for some time, but recent attempts to consider both Λ and the universal gravitational constant G as dynamical quantities, and therefore not as constants, has gained more attention due to the aforementioned not-so-well-explained cosmic acceleration.
- Different forms of changing Λ assumptions exist, such as the one we adopt for our Bianchi Type-I considerations:

$$\Lambda = \frac{\alpha}{a^2} + \beta H^2$$

  - ✓ Predicted by quantum field theoretic considerations
  - ✓ Constants α and β to be determined from both QFT and cosmological observations

[3] $G \propto \frac{1}{t}$; physical constants depend on the age of the universe t.

- The Bianchi type-I is identified by the metric of the form [4]

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)\, dy^2 + C^2(t)\, dz^2$$

where $A(t), B(t)$ and $C(t)$ are the scale factors along $x,\ y$ and z direction

- Perfect-fluid cosmic matter distribution is given by the following energy-momentum tensor:

$$T_{ij} = (p + \rho)u_i u_j + pg_{ij}$$

where ρ is matter density, $u^i = \delta^i_t = (-1, 0, 0, 0)$ is the normalized fluid four-velocity, which is a time-like quantity such that $u^i u_i = -1$, and p is the fluid's isotropic pressure that is related to mater density through the barotropic equation of state (EoS) $p = w\rho$, with

$$w = \begin{cases} 0 & \text{for dust} \\ 1/3 & \text{for radiation} \\ -1 & \text{for dark energy} \end{cases}$$

[4] Alfedeel, A. H., & AA. (2022), The evolution of time-dependent Λ and G in multi-fluid Bianchi-I models, Open Astronomy 31 198 (2022)

- The Einstein Field Equations with time-dependent $\Lambda = \frac{\alpha}{a^2} + \beta H^2$, where $a = (ABC)^{1/3}$ is average scale factor, $H \equiv 1/3 \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right)$ is the average Hubble parameter, and $G = G(t)$ and the conservation of T^{ij} (*i.e.*, $\nabla_j T^{ij} = 0$) are reduced to the energy density evolution

$$\dot{\rho} + (\rho + p) \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0$$

giving a solution for the energy density as

$$\rho = \frac{\rho_0}{a^{3(1+w)}}$$

where ρ_0 corresponds to the current value of the energy density.

- The generalized Friedmann equations read:

$$8\pi G p - \Lambda = (2q - 1)H^2 - \sigma^2$$
$$8\pi G \rho + \Lambda = 3H^2 - \sigma^2$$

where σ is the shear modulus, q is deceleration parameter. Or alternatively

$$\frac{\ddot{a}}{a} + (2 - \beta)\frac{\dot{a}^2}{a^2} - \frac{\alpha}{a^2} = 4\pi G(t)(\rho - p)$$

- In the multi-fluid setting, ρ and p are the total energy density and total pressure of the cosmic fluid, respectively.

- The time evolution equation connecting G and Λ can be given by

$$8\pi\rho\dot{G} + \dot{\Lambda} = 0 \tag{2.1}$$

and the expression for the metric variables A, B and C as

$$A = A_0 \; a \; \exp\left\{ k_1 \int \frac{dt}{a^3} \right\} \tag{2.2}$$

$$B = B_0 \; a \; \exp\left\{ k_2 \int \frac{dt}{a^3} \right\} \tag{2.3}$$

$$C = C_0 \; a \; \exp\left\{ k_3 \int \frac{dt}{a3} \right\} \tag{2.4}$$

where m_1,m_2, m_3, k_1,k_2, k_3 and x_1, x_2 and x_3 are constants of integration satisfying the following relations:

$$A_0 = \sqrt[3]{m_1 m_2} \; , B_0 = \sqrt[3]{m_1^{-1} m_3} \; , C_0 = \sqrt[3]{(m_1 m_3)^{-1}}$$
$$k_1 = \frac{2x_1 + x_3}{3} \; , k_2 = \frac{x_3 - x_1}{3} \; , k_3 = -\frac{x_1 + 2x_3}{3}$$
$$A_0 B_0 C_0 = 1 \; , k_1 + k_2 + k_3 = 0$$

- Dynamical parameters: the deceleration parameter q, the average anisotropy parameter A_p, and shear module

$$
\begin{aligned}
q &= -\frac{\ddot{a}a}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2} \\
A_p &= \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 \\
\sigma^2 &= \frac{1}{3}\left(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}\right) - \frac{1}{3}\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC}\right) \\
&= \frac{1}{2}\left(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}\right) - \frac{3H^2}{2} \implies \sigma = \frac{K}{a^3}
\end{aligned}
$$

where $K = (x_1^2 + x_2^2 + x_1 x_2)/3$ is a numerical constant which is related to the anisotropy of the model

- Let's introduce

$$\rho_{\mathrm{i}0} = \frac{3H_0^2}{8\pi G_0}\,\Omega_{\mathrm{i}0} \implies \rho_{\mathrm{i}} = \frac{3H_0^2}{8\pi G_0}\,\Omega_{\mathrm{i}0}(1+z)^{(1+w)}$$

- Background evolution

$$\frac{da}{dt} = Z \tag{2.5}$$

$$\frac{dZ}{dt} = -\frac{(2-\beta)\,Z^2}{a} + \frac{\alpha}{a} + c_1 SG \tag{2.6}$$

$$\frac{dG}{dt} = c_2\frac{Z}{a^3} + c_3\frac{Z^3}{a^3} - c_4\frac{Z}{a^2}\,G \tag{2.7}$$

where we have used the following short-hands:

$$c_1 \equiv \frac{3H_0^2}{2G_0}\left(\Omega_{\mathrm{m}0}(1+z)^3 + \frac{2\Omega_{\mathrm{r}0}}{3}(1+z)^4 + 2\Omega_{\Lambda 0}\right)$$

$$c_2 \equiv \frac{2\alpha(1-\beta)G_0}{3H_0^2}$$

$$c_3 \equiv \frac{2\beta(3-\beta)G_0}{3H_0^2}$$

$$c_{\equiv}\beta\left(\Omega_{\mathrm{m}0}(1+z)^3 + \frac{2\Omega_{\mathrm{r}0}}{3}(1+z)^4 + 2\Omega_{\Lambda 0}\right)$$

- $\Omega_{\mathrm{m}0} + \Omega_{\mathrm{r}0} + \Omega_{\Lambda 0} = 1$ by definition

- Rewrite Eqs. (2.6) and (2.7) in redshift space

$$\frac{dH}{dz} = \frac{(3-\beta)}{(1+z)}\,H - \frac{\alpha(1+z)}{H} - \frac{c_1 G}{(1+z)H} \tag{2.8}$$

$$\frac{dG}{dz} = -c_2(1+z) - \frac{c_3 H^2}{(1+z)} + c_4 G \tag{2.9}$$

- Define the dimensionless parameters :

$$h \equiv \frac{H}{H_0}\ , \quad \lambda \equiv \frac{\Lambda}{\Lambda_0}\ , \quad g \equiv \frac{G}{G_0}\ , \quad \gamma \equiv \frac{\alpha}{H_0^2}$$

- Equations in dimensionless parameters

$$\begin{aligned}\frac{dh}{dz} = \frac{(3-\beta)}{(1+z)}h - \frac{\gamma(1+z)}{h} - \frac{3}{2}\frac{g}{(1+z)h}\times \\ \left(\Omega_{\mathrm{m}0}(1+z)^3 + \frac{2\Omega_{\mathrm{r}0}}{3}(1+z)^4 + 2\Omega_{\Lambda 0}\right)\end{aligned} \tag{2.10}$$

$$\begin{aligned}\frac{dg}{dz} = \beta\left(\Omega_{\mathrm{mo}}(1+z)^3 + \frac{2\Omega_{\mathrm{r}0}}{3}(1+z)^4 + 2\Omega_{\sim 0}\right)g \\ - \frac{2\beta}{3}(3-\beta)\frac{h^2}{1+z} - \frac{2\gamma}{3}(1-\beta)(1+z)\end{aligned} \tag{2.11}$$

- The deceleration parameter is then

$$\begin{aligned}q = (2-\beta) - \frac{\gamma(1+z)^2}{h^2} \\ - \frac{3}{2}\left(\Omega_{\mathrm{m}0}(1+z)^3 + \frac{2\Omega_{\mathrm{r}0}}{3}(1+z)^4 + 2\Omega_{\Lambda 0}\right)\frac{g}{h^2}\end{aligned}$$

- Scale factor solutions of the model

$$A = \frac{A_0}{(1+z)} \exp\left\{\kappa_1 \int \frac{(1+z)^2 dz}{h}\right\} \tag{2.12}$$

$$B = \frac{B_0}{(1+z)} \exp\left\{\kappa_2 \int \frac{(1+z)^2 dz}{h}\right\} \tag{2.13}$$

$$C = \frac{C_0}{(1+z)} \exp\left\{\kappa_3 \int \frac{(1+z)^2 dz}{h}\right\} \tag{2.14}$$

where we have defined the new dimensionless parameters

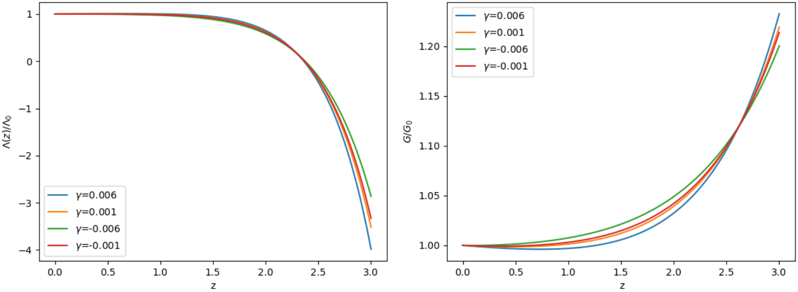
$$\kappa_1 \equiv k_1/H_0 \,, \kappa_2 \equiv k_2/H_0 \,, \quad \kappa_3 \equiv k_3/H_0$$

- Observational values used from Planck-2018

$$\Omega_{\rm m0} = 0.3111 \,, \quad \Omega_{\Lambda 0} = 0.6889 \,, \quad \Omega_{\rm r0} = 1 - \Omega_{\rm m0} - \Omega_{\Lambda 0} \,, \quad H_0 = 67.37 \text{km/s/Mpc}$$

- Initial conditions used

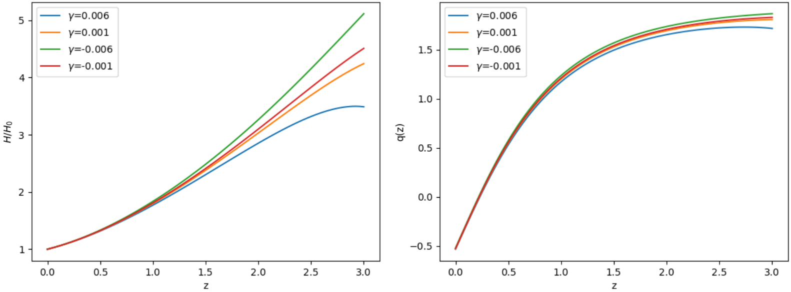
$$h(0) = g(0) = 1 \,, \quad \beta = 0.02$$



(a) Λ vs z

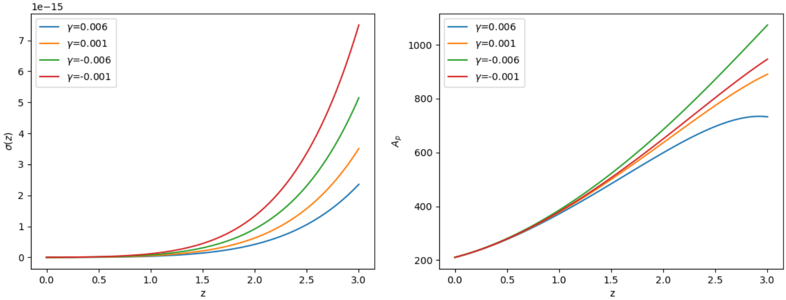
(b) g vs z.

Figure 2.1: Variation of Λ and G with redshift.



(a) h vs z *(b)* q vs z.

Figure 2.2: Variation of h and q with redshift.



(a) σ vs z

(b) A_p vs z.

Figure 2.3: Variation of shear and anisotropy parameters with redshift.

Some highlights of the Bianchi-I solutions

- As predicted from Eq. (2.1), Λ and G evolve in opposite trends: for values of the parameter γ for which G decreases with time (increases with redshift), Λ increases with time (decreases with redshift), and vice versa
- Λ becomes significant today, which is in accordance with the late-time domination of the universe by dark energy
- It is only interesting to note that it could have negative values in the past. In other words, the cosmological parameter Λ as one of the potential candidates for dark energy fits well into this description of dark energy dominating at the present time, but could have been dominated by other component fluids in the cosmological past
- Because gravity is weaker today than in the past, an acceleration in the expansion is expected at late times. That is, since gravity is not be strong enough to keep bound structures, like galaxies, clusters and superclusters together, the spacetime between these structures expands faster
- Each of H, σ and A_p parameters are decreasing functions of redshift for the different values of α considered. This is also consistent with observations as, for example, one has a more or less isotropic universe on large scales today but suggestions of more anisotropy in the past

- Here we consider the Bianchi type$-V$ with spacetime metric of the form[5]

$$ds^2 = dt^2 - A^2 dx^2 - e^{2mx}[B^2 dy^2 + C^2 dz^2] \tag{3.1}$$

  where m is constant

- We assume that the universe is filled by a viscous fluid whose distribution in space is represented by the following energy momentum tensor:

$$T_{ij} = (\rho + \bar{p}) u_i u_j + \bar{p} g_{ij} - 2\eta\sigma_{ij}$$

  where η and ξ are coefficients of shear and bulk viscosity respectively, σ_{ij} is the shear and $\bar{p}$ is the effective pressure which is given by

$$\bar{p} = p - \xi u_{i;i} = p - (3\xi - 2\eta) H$$

- Assume a linear equation of state

$$p = w\rho \, , \qquad -1 \leq w \leq 1$$

- The shear tensor is given by

$$\sigma_{ij} = \left(u_{i;k} h^k_j + \dot{u}_{j;k} h^k_i \right) - \frac{1}{3}\theta h_{ij} \text{ , where } h_{ij} \equiv g_{ij} + u_i u_j \tag{3.2}$$

[5] Tiwari, R.K., Alfedeel, A. H., Sofuoğlu, D., AA, Eltagani, I.H., & Shukla, B. H. (2022), A cosmological model with time-dependent Λ, G, and viscous fluid in General Relativity, *Front. Astron. Space Sci.* **9** 965652 (2022)

- The Einstein field equations with time-varying cosmological constant (Λ) in geometrical units where $c = 1$ [6] are given by

$$R_{ij} - \frac{1}{2} g_{ij} R = -\kappa G T_{ij} + \Lambda g_{ij} \tag{3.3}$$

- Using Eqs. (5)-(3.2), the EFEs in (3.3) for a viscous fluid distribution reduce to the following set of pdes:

$$\frac{m^2}{A^2} - \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} - \frac{\dot{B}}{B}\frac{\dot{C}}{C} + 2\eta\frac{\dot{A}}{A} = \kappa G \left[p - \left(\xi - \frac{2}{3}\eta \right) \theta \right] - \Lambda \tag{3.4}$$

$$\frac{m^2}{A^2} - \frac{\ddot{A}}{A} - \frac{\ddot{C}}{C} - \frac{\dot{A}}{A}\frac{\dot{C}}{C} + 2\eta\frac{\dot{B}}{B} = \kappa G \left[p - \left(\xi - \frac{2}{3}\eta \right) \theta \right] - \Lambda \tag{3.5}$$

$$\frac{m^2}{A^2} - \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} + 2\eta\frac{\dot{C}}{C} = \kappa G \left[p - \left(\xi - \frac{2}{3}\eta \right) \theta \right] - \Lambda \tag{3.6}$$

$$-\frac{3m^2}{A^2} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} = \kappa G \rho + \Lambda \,, \tag{3.7}$$

$$\frac{\dot{B}}{B} + \frac{\dot{C}}{C} - 2\frac{\dot{A}}{A} = 0 \tag{3.8}$$

[6] But $8\pi \equiv \kappa$ in this work

- The $\nabla_j T_{ij} = 0$ equation leads to the fluid continuity equation:

$$\kappa G \left[\dot{\rho} + (\overline{p} + \rho)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right] + \kappa\rho\dot{G} + \dot{\Lambda} - 4\kappa G\eta\sigma^2 = 0$$

- Split this into the following two equations, using $\overline{p} = p - (3\xi - 2\eta)H$:

$$\dot{\rho} + 3H\left[p + \rho - (3\xi - 2\eta)H\right] = 0 \tag{3.9}$$
$$\kappa\rho\dot{G} + \dot{\Lambda} - 4\kappa G\eta\sigma^2 = 0 \tag{3.10}$$

- The field equations (3.4)-(3.8) can be integrated to give

$$\frac{\dot{A}}{A} = \frac{\dot{a}}{a} \implies A = a \tag{3.11}$$
$$\frac{\dot{B}}{B} - \frac{\dot{C}}{C} = \frac{k_1}{a^3} e^{-2\int \eta dt} \tag{3.12}$$
$$\frac{\dot{C}}{C} - \frac{\dot{A}}{A} = \frac{k_2}{a^3} e^{-2\int \eta dt} \tag{3.13}$$

- Plugging Eq. (3.11) into Eq. (3.12) and (3.13) produces

$$\frac{\dot{B}}{B} = \frac{\dot{a}}{a} + \frac{k_1}{a^3} e^{-2\int \eta dt} \tag{3.14}$$
$$\frac{\dot{C}}{C} = \frac{\dot{a}}{a} + \frac{k_2}{a^3} e^{-2\int \eta dt} \tag{3.15}$$

- Integrating gives an expression for the metric function B and C as

$$B = d_1 a \exp\left[\int \left(\frac{k_1}{a^3} e^{-2\int \eta dt}\right) dt\right] \tag{3.16}$$

$$C = d_2 a \exp\left[\int \left(\frac{k_2}{a^3} e^{-2\int \eta dt}\right) dt\right] \tag{3.17}$$

  where k_1, k_2, d_1 and d_2 are constants of integration.

- Meanwhile, Eqs. (3.4)-(3.8) can be written in terms of H, σ and q as

$$\kappa G\overline{p} - \Lambda = H^2(2q-1) - \sigma^2 + \frac{m^2}{A^2} \tag{3.18}$$

$$\kappa G\rho + \Lambda = 3H^2 - \sigma^2 - \frac{3m^2}{A^2} \tag{3.19}$$

  giving the generalized Friedmann equations for Bianchi type-V spacetimes endowed with the viscous-fluid model

- The generalized Raychaudhuri equation reads:

$$\dot{H} + 3H^2 - \frac{2m^2}{a^2} - \Lambda + \frac{\kappa G}{2}(p-\rho) - \kappa G\left(\frac{3\xi}{2} - \eta\right) H = 0 \tag{3.20}$$

- Dividing (3.19) throughout by $3H^2$, we can write

$$1 = \Omega_m + \Omega_\Lambda + \Omega_\sigma + \Omega_\chi \tag{3.21}$$

where we have the dimensionless "energy densities"

$$\Omega_m \equiv \frac{\kappa G \rho_m}{3H^2}\ , \qquad \Omega_\Lambda \equiv \frac{\kappa G \rho_\Lambda}{3H^2}\ , \qquad \Omega_\sigma \equiv \frac{\sigma^2}{3H^2}\ , \qquad \Omega_\chi \equiv \frac{3m^2}{3H^2 a^2}$$

with present-day values given by

$$\Omega_{m_0} = \frac{\kappa G_0 \rho_{m_0}}{3H_0^2}\ , \qquad \Omega_{\Lambda_0} = \frac{\kappa G_0 \rho_{\Lambda_0}}{3H_0^2}\ , \qquad \Omega_{\sigma_0} = \frac{\sigma_0^2}{3H_0^2}\ , \qquad \Omega_{\chi 0} = \frac{3m^2}{3H_0^2 a_0^2}$$

- Evolutions in terms of the dimensionless parameters:

$$\dot{\Omega}_m + \left(2\frac{\dot{H}}{H} - \frac{\dot{G}}{G}\right)\Omega_m + 3H\left[(1+w_m)\Omega_m - \frac{\kappa G}{3H}(3\xi - 2\eta)\right] = 0$$

$$\dot{\Omega}_\Lambda + \left(2\frac{\dot{H}}{H} - \frac{\dot{G}}{G}\right)\Omega_\Lambda + \frac{\dot{G}}{G} - 12\kappa G \eta \Omega_\sigma = 0$$

$$\dot{\Omega}_\chi + 2\left(H + \frac{\dot{H}}{H}\right)\Omega_\chi = 0$$

$$\dot{\Omega}_\sigma + \left(6H + 2\frac{\dot{H}}{H}\right)\Omega_\sigma = 0$$

- To complete the solutions processe, extra equation or assumption required
- Dirac ansatz:

$$G(t) = \frac{G_0}{a} \implies \dot{G} = -G_0 \frac{H}{a} \tag{3.22}$$

- More dimensionless parameters (α , β dimensionless constants, $0 \leq n \leq \frac{1}{2}$):

$$h \equiv \frac{H}{H_0} \ , \qquad \xi = \alpha H_0 (\rho_{\rm m}/\rho_{\rm m0})^n \ , \qquad \eta = \beta H$$

- And thus the system of 5 odes in fully dimensionless forms [7]

$$h' = \frac{h}{(1+z)} \left[3 - 2\Omega_\chi - 3\Omega_\Lambda - \frac{3}{2}(1 - w_m)\Omega_m \right] - \frac{3\alpha\kappa G_0}{2} \left(\frac{h^2 \Omega_m}{\Omega_{m0}} \right)^n + \kappa G_0 \beta h \tag{3.23}$$

$$\Omega_m' = -\frac{2h'}{h}\Omega_m + \frac{1}{1+z}(4 + 3w_m)\Omega_m - \frac{3\alpha\kappa G_0}{h} \left(\frac{h^2 \Omega_m}{\Omega_{m0}} \right)^n + 2\kappa G_0 \beta \tag{3.24}$$

$$\Omega_\Lambda' = -\frac{2h'}{h}\Omega_\Lambda + \frac{1}{1+z}(\Omega_\Lambda - 1) - 12\beta\kappa G_0 \Omega_\sigma \tag{3.25}$$

$$\Omega_\chi' = -\frac{2h'}{h}\Omega_\chi + \frac{2\Omega_\chi}{1+z} \tag{3.26}$$

$$\Omega_\sigma' = -\frac{2h'}{h}\Omega_\sigma + \frac{6\Omega_\sigma}{1+z} \tag{3.27}$$

[7] In units where $\kappa G_0 = 1$

- The deceleration parameter, the metric variables, and the cosmic volume V are then given by:

$$q = 2 - 2\Omega_\chi - 3\Omega_\Lambda - \frac{3}{2}(1 - w_m)\Omega_m - \frac{3\alpha\kappa G_0(1+z)}{2h}\left(\frac{h^2\Omega_m}{\Omega_{m0}}\right)^n + \beta\kappa G_0(1+z) \tag{3.28}$$

$$B = \frac{d_1}{(1+z)} \exp\left\{\frac{\kappa_1}{H_0}\int \frac{(1+z)^{2+2\alpha}}{h} dz\right\} \tag{3.29}$$

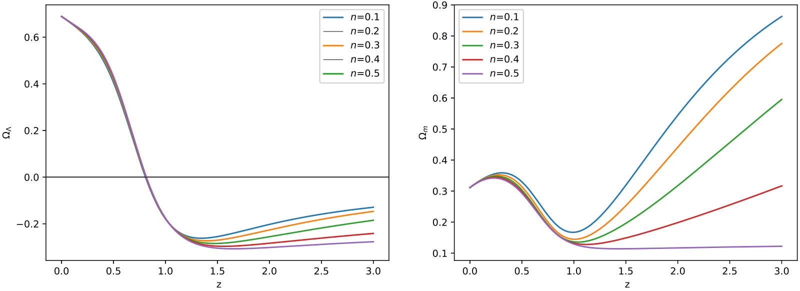
$$C = \frac{d_2}{(1+z)} \exp\left\{\frac{\kappa_2}{H_0}\int \frac{(1+z)^{2+2\alpha}}{h} dz\right\} \tag{3.30}$$

$$V = ABC = \frac{d_3}{(1+z)^3} \exp\left\{\frac{\kappa_3}{H_0}\int \frac{(1+z)^{2+2\alpha}}{h} dz\right\} \tag{3.31}$$

  where $d_3 = d_1 d_2$ and $\kappa_3 = \kappa_1 + \kappa_2$ are numerical constants
- For our numerical integration, we use the following initial conditions[8]:

$$h(0) = 1, \Omega_0 = 0.3111\,, \Omega_{\Lambda 0} = 0.6889\,, \Omega_{\chi 0} = -0.03\,, \Omega_{\sigma 0} = 1 - \Omega_{\mathrm{m}0} - \Omega_{\Lambda 0} - \Omega_{\chi 0}$$

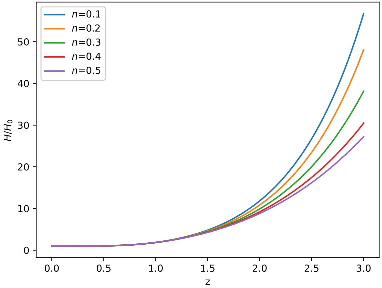
[8] And assumptions: $\alpha = \beta = 1$



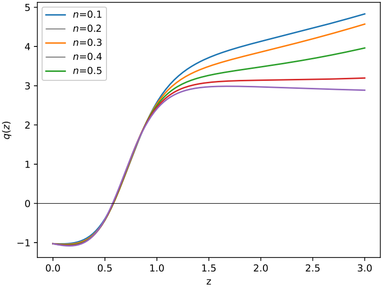
(a) Ω_Λ vs z.

(b) Ω_m vs z.

Figure 3.1: The variation of the fractional energy densities of dark energy Ω_Λ and matter Ω_m with redshift.

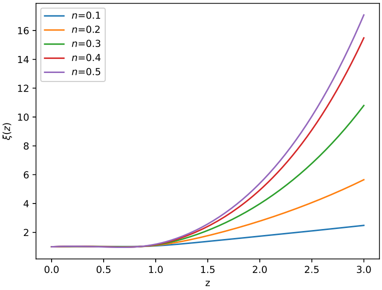


(a) h vs z.

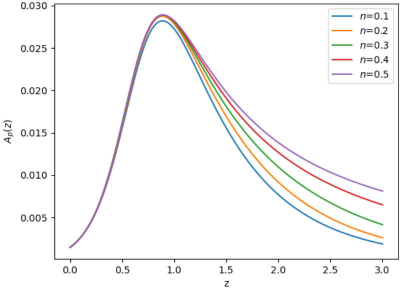


(b) q vs z.

Figure 3.2: The variations of the normalized expansion rate h and the deceleration parameter q with redshift.



(a) ξ vs z.



(b) A_p vs z.

Figure 3.3: The variation of the bulk viscosity ξ and anisotropy A_p parameters with redshift.

Some highlights of the Bianchi-V solutions

- Ω_m starts evolving with redshift from having large value at early stage of cosmic evolution gradually decreasing to its minimum value around $z \sim 1$, reaching a local maximum at about $z \sim 0.5$ and then decreasing to its current value Ω_{m0} at $z = 0$, whereas Ω_Λ grew from a small negative value at the early times to its current positive value at $z = 0$
- In agreement with results from ΛCDM model?
- The normalized Hubble parameter h, the bulk viscosity ξ and the anisotropy parameter A_p have become smaller today compared to their values at larger redshifts, for all vales of n considered. It appears from our analysis, however, that the anisotropy term at about $z \sim 1$ (when the fractional energy density was at its minimum) reaches a maximum value before it decreases to its minimum value today
- The deceleration parameter changes sign at small redshift values, from negative $q > 0$ at the early times to $q < 0$ at the present time for all different values of n considered. The change in q indicates that the universe expansion in this model has gone through a phase transition from slowing (decelerating) early epoch on to a speeding up (accelerating) universe now, with the transition from deceleration to acceleration happening at $z \sim 0.5$, as predicted by observations as well
- Beyond the background expansion history, this model should also be tested for its cosmological viability at perturbative levels, to see, e.g., the effect of viscosity on large-scale structure formation [9]

[9] AA, Alfedeel, A. H., Sofuoğlu, D., Eltagani, I.H., & Tiwari, R.K.. (2022), Perturbations in Bianchi - V spacetimes with varying Λ, G and viscous fluids (under review)

Summary

- Generic solutions for the Bianchi type$-I$ cosmological model with time-varying Newtonian and cosmological 'constants' for realistic multi-component perfect-fluid scenarios obtained
  - ✓ The predicted evolution of the different cosmological parameters for the Bianchi-I spacetimes endowed with multi-component fluids shows that for each value of the defining parameter α in $\Lambda = \alpha/a^2 + \beta H^2$ the Newtonian gravitational factor G decreases with time as Dirac would have it, and the opposite effect is observed for the evolution of Λ, the interesting aspect being both of them asymptoting towards constant values today.
- Also investigated: Bianchi type$-V$ cosmological model in the presence of shear η and bulk ξ viscosities in the cosmic fluids for time-varying G and Λ
  - ✓ The model describes a universe that starts off with a negative cosmological term, dominated by non-relativistic matter and decelerated, that eventually becomes dark energy-dominated and hence expanding with acceleration, in concordance with current observations. Our future endeavour in this direction will involve a more rigorous data analysis to observationally constrain the different assumed parameters of the model.
- Both of the homogeneous and anisotropic spacetime models describe a universe that starts with some anisotropic universe in the past, and isotropic at late times, possibly indistinguishable from the FLRW universe.
- Future directions: putting more stringent constraints on the values of the defining parameters of the model, with more rigorous data and statistical analysis – using existing and upcoming cosmological data– including large-scale structure formation scenarios