Gauss-Bonnet Gravity: Singularity Analysis

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I. Introduction & Motivation

- One of the important features of GR is the existence of spacetime singularities.

- Singularity theorems (Hawking and Penrose) assured the existence of singularities for any gravitational system starting from generic matter distribution with some energy conditions.

- In the absence of a canonical framework for quantum gravity several authors tried to smooth out these singularities through modifying the EoS or Einstein field equations near singularities!

- E.g., for cosmological singularities we have several choices and it is very difficult to justify the use of a specific EoS or theory of modified gravity.

- *Here we constrain the discussion to two modifications of GR, one is produced by Weyl/conformal anomaly and the other is a curvature correction inspired by string theory, or Gauss-Bonnet gravity.*

- Therefore, it is natural to ask; *Can these higher curvature theories change the nature of some known singular solutions, namely, FLRW and Schwarzschild solutions?*

II. Cosmology with Weyl anomaly

- Consider a theory with massless fields (ϕ,ψ, A_μ) coupled to a background metric $g_{\mu\nu}$.
- A classical theory is called Weyl/Conformally invariant if its action is invariant under Weyl transformation

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \psi' = \Omega^d \psi, \quad \text{d is the conformal weight}$$

- Consequently, we have $T_\mu{}^\mu = 0$.
- Quantum one-loop effects breaks this symmetry. The trace takes the form

$$\langle T^\mu_\mu \rangle = c_1 E_4 + c_2 I_4 + c_3 \Box R. \quad *$$

- The symmetry is anomalous and c's are spin-dependent coefficients, R is the Ricci scalar, E_4 *is* the Euler density and I_4 is the Weyl scalar.

$$E_4 = \frac{1}{64}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$$

$$I_4 = -\frac{1}{64}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right).$$

* D. Capper and Duff, Nouvo Cimento Soc. Ital. Fis. 23A,173 (1974); M. Duff, Nucl. Phys. B125, 334 (1977)

II. Cosmology with Weyl anomaly

FLRW Cosmology:

- Consider a generic conformal field theory of scalars, spinors and gauge fields coupled to a flat FLRW metric

$$ds^2 = -dt^2 + a(t)^2[dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2)]$$

- Here we are interested in regularization-independent back-reaction ($\alpha=0$) of these fields on the geometry at early times. Einstein field equations is given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T^{(m)}_{\mu\nu} + \kappa\,\beta\left[\frac{R^2}{12}g_{\mu\nu} - R^{\rho\sigma}R_{\rho\mu\sigma\nu}\right]$$

- where $T_{\mu\nu}^{(m)}=diag(\rho, P, P, P)$, *EoS consistent with conformal symmetry* $P=1/3\ \rho$,
- $\beta = \frac{-1}{2880\pi^2}(n_s + 11\, n_f + 62\, n_v) < 0$, for the above field theory.

II. Cosmology with Weyl anomaly: FLRW cosmology

- Two independent field eqn.'s are

$$\kappa\rho - 3[1 + \kappa\beta H^2]\, H^2 = 0$$

$$2\ddot{a}a(1 + 2\kappa\beta H^2) + a^2 H^2(1 - \kappa\beta H^2) + \kappa\, a^2 P = 0,$$

- The last eqn. can take the form

$$\dot{H}(1 + 2\kappa\,\beta H^2) = -\frac{\kappa}{2}(\rho + P),$$

- Solving the first eqn. for H , we get

$$H = \pm\sqrt{\frac{-1 \pm \sqrt{1 + \frac{4}{3}\beta\kappa^2\rho}}{2\beta\kappa}}.$$

- β < 0 implies that density and Hubble rate have max. values and scale factor has a min.

II. Cosmology with Weyl anomaly: Solution

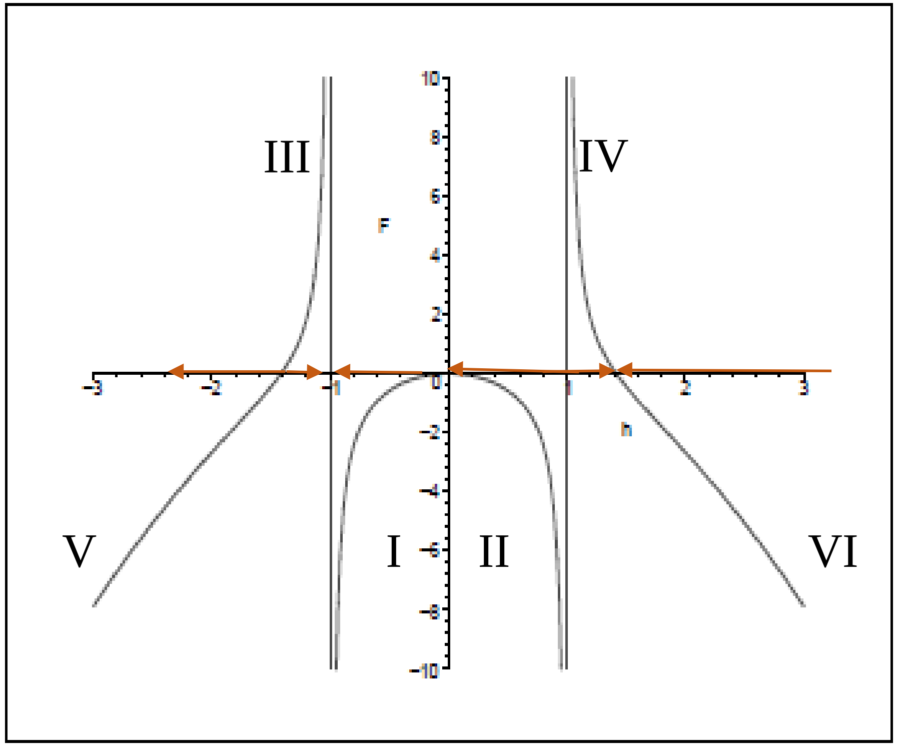
- Taking $t = \tau\sqrt{2\beta\kappa},\ a/a_c = \eta$, and $a_c = 4\beta\kappa^2 c/3.$
- Soln. of the cosmological eqns is

$$\rho \sim 1/\eta^4$$

$$\tau + c_1 = \pm\left[\frac{\sqrt{2}}{4} tanh^{-1}(2^{-1/2}\sqrt{1 \mp \sqrt{1-\eta^{-4}}}) + \frac{1}{2\sqrt{1 \mp \sqrt{1-\eta^{-4}}}}\right]$$

$$H = \pm\sqrt{\frac{-1 \pm \sqrt{1 + \frac{4}{3}\beta\kappa^2\rho}}{2\beta\kappa}}.$$

$$\dot{H} = -2\left(\frac{1+\kappa\beta H^2}{1+2\kappa\beta H^2}\right)H^2$$



$R \sim \frac{1}{\sqrt{t}}$

(remember, GR FLRW soln. has $R \sim \frac{1}{t^2}$)

dh/dt= F with h

III. Strength of a singularity

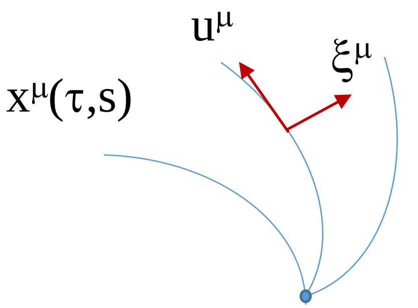
Penrose-Hawking Theorems:

- In this context, geodesic incompleteness is the criterion for a singular spacetime.
- Penrose-Hawking theorems show geodesics inextendibility of spacetimes with certain energy conditions and global properties.
- Energy conditions are needed to show gravitational focusing through Raychaudhuri's eqn.

$$\frac{d\theta}{d\tau} = -R_{lk}u^l u^k - \frac{\theta^2}{3},$$ (expansion parameter θ = - 3 H, for $FLRW$)

- Formation of acoustic (or conjugate points) is an essential ingredient in singularity theorems, where $\theta \rightarrow -\infty$.
- In our model $H = (2\beta\kappa)^{-1/2}$ at the singularity, therefore, this is not a singularity a la Penrose-Hawking.

III. Strength of a singularity



$$u^{\mu} = \frac{\partial x^{\mu}}{\partial \tau}\ ,\ \xi^{\mu} = \frac{\partial x^{\mu}}{\partial s}$$

Tipler's and Krolak's criteria for a strong singularity:

- Soln. of geodesic deviation eqn. are called Jacobi fields

$$\frac{D^2\xi^{\mu}}{d\tau^2} = R^{\mu}_{\alpha\beta\gamma}\, u^{\alpha}u^{\beta}\xi^{\gamma}$$

- *Tipler's criterion for a strong singularity*; a singularity is called strong if the volume spanned by three orthonormal Jacobi fields shrunk to zero size at the singularity.

- *Krolak's criterion for a strong singularity* is similar but based on the rate of change of the volume w.r.t. affine parameter instead of the volume at the singularity.

III. Strength of a singularity

- The singularity is Tipler strong for a null geodesic iff the following integral diverges

$$\lim_{\lambda\to\lambda_0}\int_0^{\lambda} d\lambda' \int_0^{\lambda'} d\lambda'' R_{ab}u^a u^b,$$

- But the singularity is Krolark strong iff the following integral diverges

$$\lim_{\lambda\to\lambda_0}\int_0^{\lambda} d\lambda' R_{ab}u^a u^b,$$

- For our spacetime close to t=0, (int. cond. λ_0= 0 at t=0) solving geodesic eqn.'s we get

$$\frac{dt}{d\lambda} = \pm\sqrt{s + \frac{v^2}{a^2}} = g(t),$$

$$\frac{dx^i}{d\lambda} = \frac{v^i}{a^2} = f^i(\lambda),$$

$$\Longrightarrow \quad a(\lambda) = a_0\left[1 + \chi\, H_0\lambda\right] + O(\lambda^2), \quad t(\lambda) = \chi\,\lambda + O(\lambda^2).$$

- Now evaluating both integrals in the limit $\lambda\to 0$, we find that both integrals vanishes.

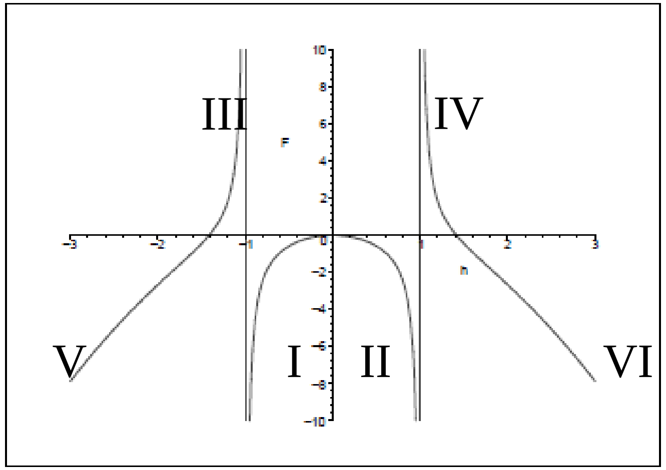
IV. Extending spacetime and singularity crossing

Geodesic extension:

- Geodesic equations for FLRW metric are

$$\frac{dt}{d\lambda} = \pm\sqrt{s + \frac{v^2}{a^2}} = g(t), \qquad \frac{dx^i}{d\lambda} = \frac{v^i}{a^2} = f^i(\lambda),$$

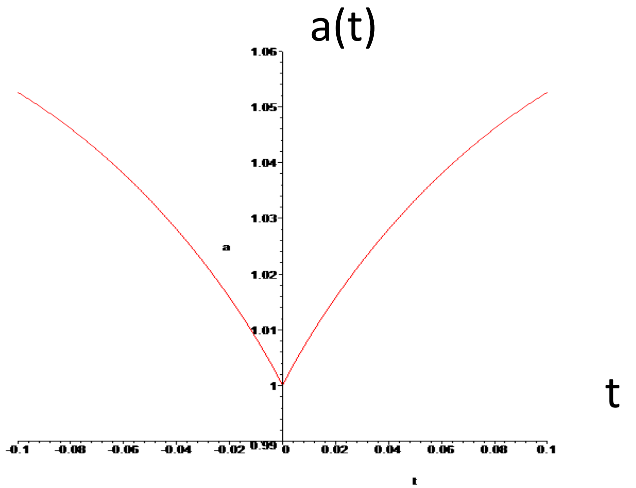
- where s and v^i are integration constants and λ is a non-spacelike affine parameter. s =1,0 for timelike and null affine parameter.



- Now joining the two branches (soln's for t ≥ 0 and t < 0) together leads to following scale factor

$$a(t) = a_0 \left[1 + |H_0 t| - \frac{2}{3}|H_0 t|^{3/2}\right] + O(t^2).$$

- One can show that it is possible to extend geodesics beyond this singularity, in fact, beyond this type of singularities (Sudden singularities).



IV. Extending spacetime and singularity crossing: geodesic extension

- Now integrating geodesic eqn.'s such that t=0 at λ=0, one obtains

$$t(\lambda) = \chi\,\lambda - sign(\lambda)\frac{H_0 v^2}{2a_0^2}\lambda^2 + O(\lambda^3). \qquad x^i(\lambda) = x_0{}^i + \frac{v^i}{a_0^2}\,\lambda + sign(\lambda)\,\frac{H_0 v^2}{a_0^2}\,\chi\lambda^2 + O(\lambda^3).$$

Notice:

- These geodesics are defined for all values of λ. Therefore, geodesics are complete and nonspacelike test objects does not get destroyed crossing singularity.

- To have a consistent gravitational description it is not enough to have a geodesic extension, we have to check the consistency of this extension with the field equations.

IV. Extending spacetime and singularity crossing

Junction conditions for effective higher-curvature gravity:

- Here we use Gauss-Codazzi equations to derive junction condition for this higher-curvature gravity.
- Let us start with Gauss's normal coordinates near a hypersurface $\sum$ with metric $\tilde{g}$;

$$ds^2 = \epsilon dw^2 + \tilde{g}_{ij} dx^i dx^j ,$$

where n^μ is a normal vector to $\sum$ with n.n = ε = -1, or 1.

- Extrinsic curvature is defined as $K_{ij} = -\frac{1}{2}\ \tilde{g}_{ij},_w$
- Gauss-Codazzi equations reads

$$\begin{aligned} R^l{}_{ijk} &= \tilde{R}^l{}_{ijk} + \epsilon\,(K_{ij}K_k{}^l - K_{ik}K^l{}_j) \\ R^w{}_{ijk} &= -\epsilon\,(K_{ij|k} - K_{ik|j}) \\ R^w{}_{iwj} &= \epsilon\,(K_{ij,w} + K_{il}K^l{}_k). \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} R^i{}_j &= \tilde{R}^i{}_j + \epsilon\,(K^i{}_{j,w} - K\,K^i{}_j) \\ R^w{}_j &= -\epsilon\,(K^i{}_{j|i} - K_{|j}) \\ R^w{}_w &= \epsilon\,(K_{,w} - trK^2), \end{aligned}$$

where $K = K^i{}_i$ and $trK^2 = K_{ij}K^{ij}$.

IV. Extending spacetime and singularity crossing:

Junction conditions for effective higher-curvature gravity:

- Now we want to express everything in terms of K_{ij} and its derivatives. Field eqn.'s

$$G_{\mu\nu} + H_{\mu\nu} = \kappa T^{(m)}_{\mu\nu}, \qquad H_{\mu\nu} = -\kappa\,\beta\left[\frac{R^2}{12}g_{\mu\nu} - R^{\rho\sigma}R_{\rho\mu\sigma\nu}\right].$$

- Let us start with Einstein tensor;

$$\begin{aligned} G^w{}_w &= -\frac{1}{2}\tilde{R} + \frac{1}{2}\epsilon\,[K^2 - trK^2] \\ G^w{}_i &= -\epsilon\,[K_i{}^m{}_{|m} - K_{|i}] \\ G^i{}_j &= \tilde{G}^i{}_j + \epsilon\left[(K^i{}_j - \delta^i{}_j\,K)_{,w} - KK^i{}_j + \frac{1}{2}\delta^i{}_jK^2 + \frac{1}{2}\delta^i{}_j\,trK^2\right] \end{aligned}$$

- Now for joining two spacetimes at the hypersurface w=0, we must have a continues $\tilde{g}_{ij}$.
- Integrating the above eqn.'s one obtains

$$\lim_{\sigma\to 0}\int_{-\sigma}^{\sigma} G^w{}_w\,dw = [\mathbf{G}^w{}_w] = 0 \qquad \lim_{\sigma\to 0}\int_{-\sigma}^{\sigma} G^i{}_j\,dw = [\mathbf{G}^i{}_j] = \epsilon\,([K^i{}_j] - \delta^i{}_j\,[K])$$

$$\lim_{\sigma\to 0}\int_{-\sigma}^{\sigma} G^w{}_i\,dw = [\mathbf{G}^w{}_i] = 0 \qquad \text{or } [K_{ij}] \neq 0$$

IV. Extending spacetime and singularity crossing:

Junction conditions for effective higher-curvature gravity:

- Now let us express $H_{\mu\nu}$ in terms of K_{ij};

$$
\begin{aligned}
H^{w}{}_{w} = \quad & \epsilon\,\beta\,\kappa \hat{K}_{ij,w}\,\tilde{R}^{ij} + \epsilon^2\,\beta\,\kappa \left[\hat{K}_{ij,w}\,\hat{K}^{ij}_{,w} + \hat{K}^{ij}_{,w}\,\left(\frac{4}{3}K\,\hat{K}_{ij} + \hat{K}^{m}_{i}\,\hat{K}_{jm}\right)\right. \\
& \left. -\hat{K}_{ij,w}\,\left(3\,K\,\hat{K}^{ij} + 2\,\hat{K}^{jr}\,\hat{K}^{i}_{r}\right)\right], \\
H^{w}{}_{i} = \quad & \epsilon^2\,\beta\,\kappa \left[\hat{K}_{ij,w}\,\left(\hat{K^{mj}}_{|m} - \frac{2}{3}\,\hat{K}^{j}_{|}\right) - \hat{K}^{rs}_{,w}\,\hat{K}_{ri|s} - \hat{K}^{rs}_{,w}\,\hat{K}_{rs|i}\right], \\
H^{i}{}_{j} = \quad & -\frac{1}{3}\epsilon\,\beta\,\kappa \left[K_{,w}\,\tilde{R}\,\delta^{i}{}_{j} - K_{,w}\,\tilde{R}^{i}_{j} - 3\,\hat{K^{rs}}_{,w}\,\tilde{R}^{i}_{rjs}\right] + \epsilon^2\,\beta\,\kappa \left[\frac{K_{,w}}{3}\left(3\,\hat{K^{i}}_{j,w}\right.\right. \\
& \left. -2\,\hat{K}^{ir}\,\hat{K}_{rj} - \frac{5}{3}K\,\hat{K^{i}}_{j} - \frac{4}{9}K^2\,\delta^{i}{}_{j}\right) - \hat{K^{i}}_{j,w}\left(\frac{2}{9}K^2 + tr\hat{K}^2\right) \\
& \left. +\hat{K^{rs}}_{,w}\,\left(\hat{K_{rj}}\,\hat{K^{i}}_{s} - \hat{K_{rs}}\,\hat{K^{i}}_{j} - \frac{\delta^{i}{}_{j}}{3}K\,\hat{K_{rs}}\right) + \hat{K^{s}}_{j,w}\,\hat{K^{i}}_{s}\,K\right].
\end{aligned}
$$

IV. Extending spacetime and singularity crossing:

Junction conditions for effective higher-curvature gravity:

- Since $H_w{}^w$ and $H_i{}^w$ depends quadratically on $K_{ij,w}$ one might choose $[K_{ij}]=0$, but this doesn't allow for surface layer to form.
- It is more convenient to split K_{ij} into a trace and traceless part (this splitting is adopted in F(R) theories for junction condition too);

$$\hat{K}_{ij} = K_{ij} - \frac{\tilde{g}_{ij}}{3} K.$$

- Now the conditions reads

$$[\hat{K}_{ij}] = 0, \qquad [K] \neq 0.$$

- These conditions leads to

$$\kappa\,[\mathbf{T}^w{}_w] = [\mathbf{G}^w{}_w] + [\mathbf{H}^w_w] = 0,$$
$$\kappa\,[\mathbf{T}^w{}_i] = [\mathbf{G}^w{}_i] + [\mathbf{H}^w_i] = 0,$$
$$\kappa\,[\mathbf{T}^i{}_j] = [\mathbf{G}^i{}_j] + [\mathbf{H}^i_j] = \mathbf{S}^i_j$$

$$\mathbf{S}^i_j = \lim_{\sigma\to 0}\int_{-\sigma}^{\sigma} dw K_{,w}\left[\frac{\epsilon}{3}\left(\beta\kappa[\tilde{R}^i_j - \tilde{R}\delta^i{}_j] - 2\delta^i{}_j\right) + \epsilon^2\beta\kappa\left(\hat{K}^i_{j,w} - \frac{4}{27}K^2\delta^i{}_j\right)\right]$$

IV. Extending spacetime and singularity crossing:

Junction conditions for effective higher-derivative gravity:

- The extended spacetime scale factor is (for all times)

$$a(t) = a_0 \left[1 + |H_0 t| - \frac{2}{3}|H_0 t|^{3/2}\right] + O(t^2),$$

where, $w = t$, $\varepsilon = -1$ and $\tilde{g}_{ij} = a(t)^2 \, \delta_{ij}$ and

$$\hat{K}_{ij} = 0, \qquad K = -3\,H(t), \qquad K_{,t} = -3\,\dot{H}(t).$$

- Calculating S_{ij}

$$\begin{aligned} \mathbf{S}^i_j &= \lim_{\sigma\to 0}\int_{-\sigma}^{\sigma} dt\, K_{,t}\left[\frac{2}{3} - \frac{4\beta\kappa}{27}K^2\right]\delta^i{}_j \\ &= -4\,H_0\left[1 - \frac{H(0)^2}{H_0{}^2}\right]\delta^i_j = 0 \end{aligned}$$

V. Gauss-Bonnet Cosmology

- **Gauss-Bonnet gravity in 5 dimensions** has the action

$$I = \frac{1}{16\pi G}\int d^5x\,\sqrt{-g}\,[R + \alpha L_2] \qquad L_2 = R_{ABCD}R^{ABCD} - 4R_{AB}R^{AB} + R^2.$$

- This is the most general two-derivative gravity theory in 5 dimensions. Also, it is ghost-free when expanded about flat space.
- Let us start with FLRW cosmology in GB gravity

$$ds^2 = -dt^2 + a(t)^2\,[dx^2 + dy^2 + dz^2 + d\xi^2]$$

- Einstein field eqns. reduces to

$$\rho = \frac{6\,H^2}{\kappa}\left[1 + 2\alpha\,H^2\right]$$

$$\frac{\ddot{a}}{a} = -\frac{1}{3}\frac{[\kappa\,P - 3\,H^2]}{(1 + 4\,\alpha H^2)}$$

V. Gauss-Bonnet Cosmology

- After rescaling “H” and “t” the system reduces to

$$\frac{dh}{d\tau} = -\frac{h^2(2-h^2)}{1-h^2}, \qquad \frac{\dot{\eta}(\tau)}{\eta(\tau)} = \gamma h(\tau)$$

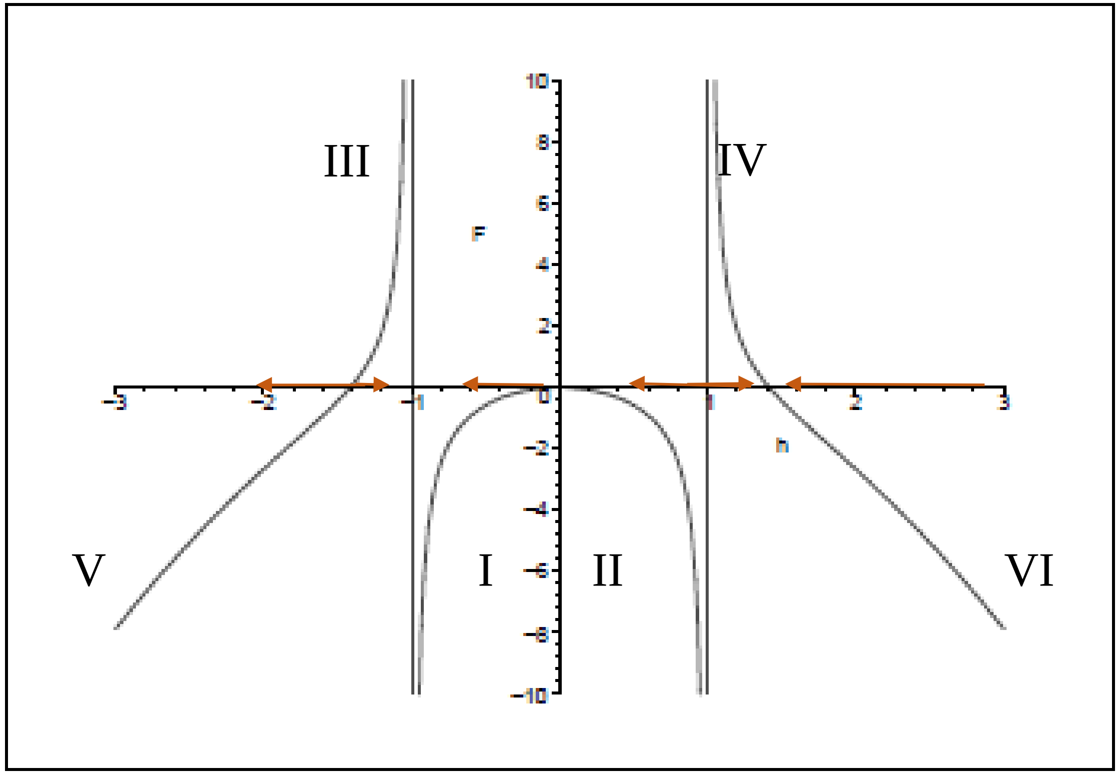
- Which has the following soln.

$$t = \frac{1}{2}\frac{1}{h(\tau)} + \frac{\sqrt{2}}{4}\tanh^{-1}\left(\frac{h(\tau)}{\sqrt{2}}\right) + C_1.$$

$$h(\tau) = \pm\sqrt{1 \mp \sqrt{1-\eta^{-3\bar{\omega}}}}.$$

- Joining two region I&II

$$\eta(\tau) = \left[1 + \gamma|\tau| - \frac{2}{3}\gamma|\tau|^{3/2} + O\left(\tau^2\right)\right]$$



dh dτ = F versus h & different branches of solutions

V. Gauss-Bonnet Cosmology

- Geodesics equations are

$$\frac{dt}{d\lambda} = \pm\sqrt{s + \frac{v^2}{a^2}} = g(t), \qquad \frac{dx^i}{d\lambda} = \frac{v^i}{a^2} = f^i(\lambda),$$

- Solving these eqns. We get

$$t(\lambda) \;=\; \chi\lambda - sign(\lambda)\frac{v^2\gamma H_0}{2a_0^2}\lambda^2 + O(\lambda^2) \qquad x^i(\lambda) \;=\; x_0^i + \frac{v^i}{a_0^2}\lambda - sign(\lambda)\frac{v^i\gamma\chi H_0}{a_0^2}\lambda^2 + O(\lambda^2)$$

- These geodesics are defined for all values of λ. Therefore, geodesics are complete and non-spacelike test objects do not get destroyed upon crossing the singularity.

- To have a consistent gravitational description it is not enough to have a geodesic extension, we have to check the consistency of this extension with the field equations.

V. Gauss-Bonnet Cosmology

- Junction conditions for GB gravity, where $K_{\mu\nu} = -\frac{1}{2}\ \tilde{g}_{\mu\nu,w}$ *

$$H_{\mu\nu} = 2\left[R_{\mu\lambda\sigma\rho}R_{\nu}^{\lambda\sigma\rho} - 2R_{\mu\sigma\nu\rho}R^{\sigma\rho} - 2R_{\mu\sigma}R_{\nu}^{\sigma} + RR_{\mu\nu}\right] - \frac{1}{2}g_{\mu\nu}\left(R_{\sigma\rho\alpha\beta}R^{\sigma\rho\alpha\beta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2\right)$$

$$\begin{aligned} H^{\mu}{}_{\nu} = & 4\frac{\partial}{\partial w}\left\{KK^{\mu}{}_{\alpha}K^{\alpha}_{\nu} - K^{\mu}{}_{\alpha}K^{\alpha\beta}K_{\beta\nu} + \frac{1}{2}K^{\mu}{}_{\nu}Tr(K^2) - \frac{1}{2}K^{\mu}{}_{\nu}K^2\right\} \\ & + 4\frac{\partial}{\partial w}\left\{-\delta^{\mu}{}_{\nu}\frac{1}{2}KTr(K^2) + \delta^{\mu}{}_{\nu}\frac{1}{3}Tr(K^3) + \delta^{\mu}{}_{\nu}\frac{1}{6}K^3\right\} \\ & + 4\left(-{}^4R^{\mu}{}_{\alpha\nu}{}^{\beta}\frac{\partial K^{\alpha}{}_{\beta}}{\partial w} - {}^4R^{\alpha}_{\nu}\frac{\partial K^{\mu}{}_{\alpha}}{\partial w} - {}^4R^{\alpha\mu}\frac{\partial K_{\nu\alpha}}{\partial w}\right) \\ & + 4\left({}^4R^{\mu}{}_{\nu}\frac{\partial K}{\partial w} + \frac{1}{2}{}^4R\frac{\partial K^{\mu}{}_{\nu}}{\partial w} + \delta^{\mu}_{\nu}\,{}^4R^{\alpha\beta}\frac{\partial K_{\alpha\beta}}{\partial w} - \frac{1}{2}\delta^{\mu}_{\nu}\,{}^4R\frac{\partial K}{\partial w}\right) \\ & + \dots, \end{aligned}$$

$$Tr(K^3) \equiv K^{\alpha}{}_{\beta}K^{\beta}{}_{\gamma}K^{\gamma}{}_{\alpha}$$

Junction condition
$[K_{\mu\nu}] \neq 0$

$$\lim_{\sigma\to 0}\int_{-\sigma}^{\sigma} H^{\mu}_{\nu}dw = [\mathbf{H}^{\mu}_{\nu}].$$

* N. Deruelle and T. Dolezel, Phys.Rev. D62 (2000) 103502

V. Gauss-Bonnet Cosmology

- The junction conditions are

$$\kappa \mathbf{T}_{\mu\nu} = [K^{\mu}_{\nu}] - \delta^{\mu}_{\nu}[K] + \mathbf{H}_{\mu\nu} = 4H_0 a_0^2$$

which are satisfied by the two solutions.

VI. Gauss-Bonnet Black Hole

- GB gravity has a Schwarzschild-like solution that has been introduced by Boulware and Deser in 1985.

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + sin(\theta)^2\,d\phi^2 + cos(\theta)^2\,d\psi^2\right)$$

- Where *f* (r) has two branches

$$f(r) = \frac{r^2 + 4\alpha \pm \sqrt{r^4 + 16\,\alpha\, m}}{4\alpha}$$

- The “-” soln is stable while the “+” is unstable (this was shown by Boulware and Deser). Furthermore, the “-” sign soln. goes to Schwarzschild solution in large radial distance “r” and in the limit that $\alpha \rightarrow 0$

$$f(r) = 1 - \frac{2m}{r^2} + O(\frac{1}{r^6})$$

VI. Gauss-Bonnet Black Hole

- We are going to consider the “-” sign soln. for the cases of α > 0 and α < 0.

<u>Black Hole with α > 0 :</u>

- In this case the horizon is at $r_h = \sqrt{2}\,\sqrt{m-\alpha}$
- Curvature is blowing up at *r*= 0, Ricci Scalar goes as*

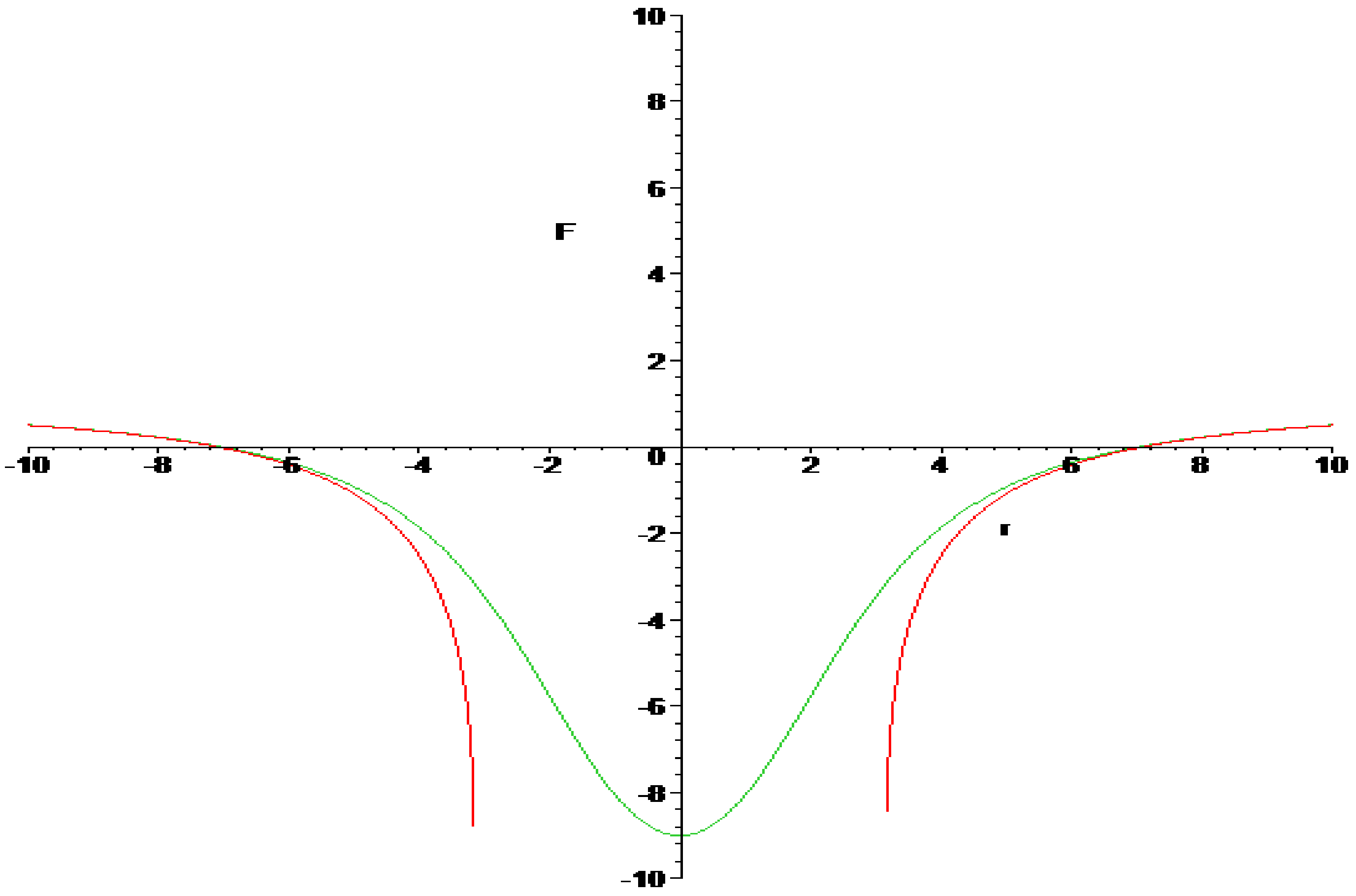
$$R \sim \sqrt{\frac{m}{\alpha}}\,\frac{1}{r^2}$$

<u>Black Hole with α < 0:</u>

- The horizon is at $r_h = \sqrt{2}\,\sqrt{m+\alpha}$
- Curvature is blowing up at *r*= 2(mα)*$^{1/4}$ *, Ricci Scalar goes as*

$$R \sim \frac{(m\,\alpha)^{3/8}}{\alpha}\,(r - 2(m\alpha)^{1/4})^{-3/2}$$

$f(r)$ for $\alpha > 0$ and $\alpha < 0$ branches



VI. Gauss-Bonnet Black Hole

Tipler's and Krolak's criteria for a strong singularity:

- *Krolak's criterion :*

i) For $\alpha > 0$, singularity at $\boldsymbol{r^* = 0}$, *or at* $\lambda_0 = 0$,

$$\lim_{\lambda \to \lambda_0} \int_0^{\lambda} d\lambda' R_{ab} u^a u^b = \lim_{\lambda_0 \to 0} c_3 \lambda_0^{-2}$$

ii) For $\alpha < 0$, singularity at $\boldsymbol{r^* = 2(\alpha\, m)^{1/4}}$, *or at* $\lambda = \lambda_0$,

$$\lim_{\lambda \to \lambda_0} \int_0^{\lambda} d\lambda' R_{ab} u^a u^b = \lim_{\lambda \to \lambda_0} c_4 (\lambda - \lambda_0)^{-3/2}$$

VI. Gauss-Bonnet Black Hole

Penrose-Hawking Theorems:

- Raychaudhuri's eqn.

$$\frac{d\theta}{d\tau} = -R_{lk}u^l u^k - \frac{\theta^2}{3},$$

- Formation of acoustic (or conjugate points) is an essential ingredient in singularity theorems, where $\theta \to -\infty$.

- Calculating $\theta = \nabla_\alpha u^\alpha$, for the GB-BH

i) For $\alpha > 0$, singularity at $r^* = 0$, $\theta = c_1 r^{-2}$

ii) For $\alpha < 0$, singularity at $r^* = 2(\alpha m)^{1/4}$, $\theta = c_2 (r - r^*)^{-1/2}$

VI. Gauss-Bonnet Black Hole

Junction Conditions:

- Since we have $K_{\mu\nu} = -\frac{1}{2}\ \tilde{g}_{\mu\nu,w}$

$$H^{\mu}{}_{\nu} = 4\frac{\partial}{\partial w}\left\{KK^{\mu}{}_{\alpha}K^{\alpha}{}_{\nu} - K^{\mu}{}_{\alpha}K^{\alpha\beta}K_{\beta\nu} + \frac{1}{2}K^{\mu}{}_{\nu}Tr(K^2) - \frac{1}{2}K^{\mu}{}_{\nu}K^2\right\}$$
$$+ 4\frac{\partial}{\partial w}\left\{-\delta^{\mu}{}_{\nu}\frac{1}{2}KTr(K^2) + \delta^{\mu}{}_{\nu}\frac{1}{3}Tr(K^3) + \delta^{\mu}{}_{\nu}\frac{1}{6}K^3\right\}$$
$$+ 4\left(-\ {}^4R^{\mu}{}_{\alpha\nu}{}^{\beta}\frac{\partial K^{\alpha}{}_{\beta}}{\partial w} - \ {}^4R^{\alpha}{}_{\nu}\frac{\partial K^{\mu}{}_{\alpha}}{\partial w} - \ {}^4R^{\alpha\mu}\frac{\partial K_{\nu\alpha}}{\partial w}\right)$$
$$+ 4\left(\ {}^4R^{\mu}{}_{\nu}\frac{\partial K}{\partial w} + \frac{1}{2}\ {}^4R\frac{\partial K^{\mu}{}_{\nu}}{\partial w} + \delta^{\mu}{}_{\nu}\ {}^4R^{\alpha\beta}\frac{\partial K_{\alpha\beta}}{\partial w} - \frac{1}{2}\delta^{\mu}{}_{\nu}\ {}^4R\frac{\partial K}{\partial w}\right)$$
$$+ \dots ,$$

$$Tr(K^3) \equiv K^{\alpha}{}_{\beta}K^{\beta}{}_{\gamma}K^{\gamma}{}_{\alpha}$$

$$\lim_{\sigma\to 0}\int_{-\sigma}^{\sigma} H^{\mu}_{\nu}dw = [\mathbf{H}^{\mu}_{\nu}].$$

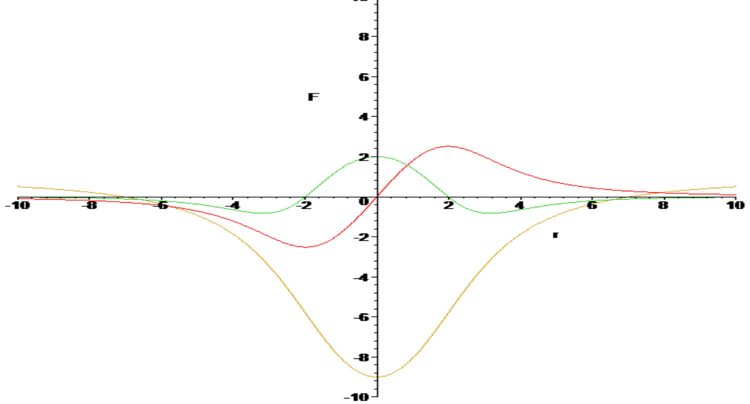
- *The junction conditions are not satisfied since* $K_{\mu\nu}$ *tensor is divergent. For example in the* $\alpha < 0$, case with a singularity at $r^* = 2(\alpha m)^{1/4}$, *one of the components goes as*

$$K_{00} = \frac{C_3}{\sqrt{r - 2\,(m\alpha)^{1/4}}}$$

VII. Conclusion

Here we considered GR corrections due to Weyl anomaly and Gauss Bonnet in modifying FLRW cosmology at early times and showed the following;

- *Curvature corrections changed the nature of the singularity from big bang to sudden singularity.*
- *In is case the singularity is weak, and it admits geodesic extension.*
- *Joining the two disjoint branches of soln.'s provides us with a* C^2 *extension to geodesics that leave the spacetime geodesically complete!*
- *Using Gauss-Codazzi eqn.'s one can obtain junction conditions for these higher-curvature gravity which are consistent with the geodesic extension.*
- For the Gauss-Bonnet theory, the derivative curvature terms were not able to smooth out or modify the singularity Schwarzschild-like solution in a way that enables us to extend geodesics beyond this point.
- Higher-curvature gravity theories are important laboratories for understanding singularities more as well as testing our ideas of how to extend geodesics beyond these points.



VI. Gauss-Bonnet Black Hole

Black Hole with α > 0

- Geodesics:

$$\mathcal{L} = \frac{1}{2} g_{AB} \frac{dx^A}{d\lambda}\frac{dx^B}{d\lambda} = -f\,\dot{t}^2 + f^{-1}\,\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\,sin(\theta)^2\,\dot{\phi}^2 + r^2\,cos(\theta)^2\,\dot{\psi}^2$$

- Geodesics equations:

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -f\,\dot{t} = E \qquad \frac{r^2}{2}\dot{\phi} = L_\phi \qquad \frac{r^2}{2}\dot{\psi} = L_\psi \qquad \dot{r} = \sqrt{-f(\delta + \frac{L^2}{r^2}) + E^2} \qquad L^2 = \frac{1}{2}(L_\phi^2 + L_\psi^2)$$

- Around r = 0, we have

$$\dot{r} = (\sqrt{m/\alpha} - 1)\,\frac{L}{r} + O(r) \qquad f(r) = (1 - \sqrt{m/\alpha}) + \frac{r^2}{4\alpha} + O(r^4)$$

- $\dot{r}$ is not continues at $r = 0$, for $L \neq 0$, therefore, no extension is possible!

II. Cosmology with Weyl anomaly: a mechanical analogue

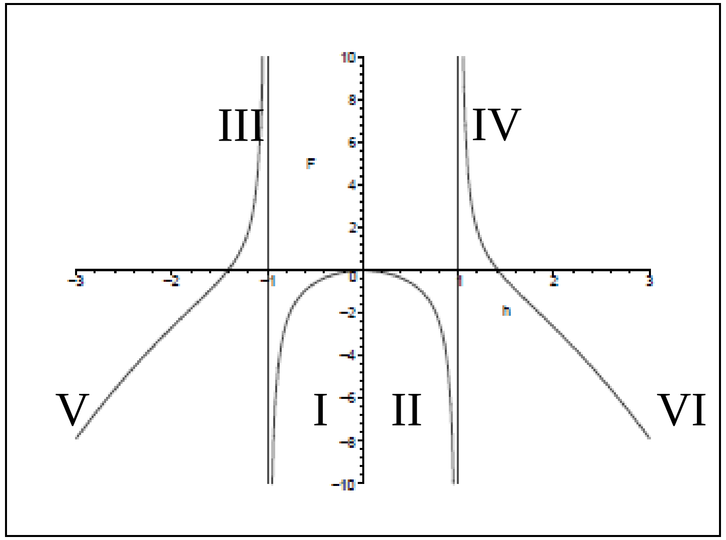
- Notice that the validity of the semi-classical approximation requires that

$$R << R_p = l_P^{-2}$$

- But

$$H = \pm\sqrt{\frac{-1 \pm \sqrt{1 + \frac{4}{3}\beta\kappa^2\rho}}{2\beta\kappa}}.$$

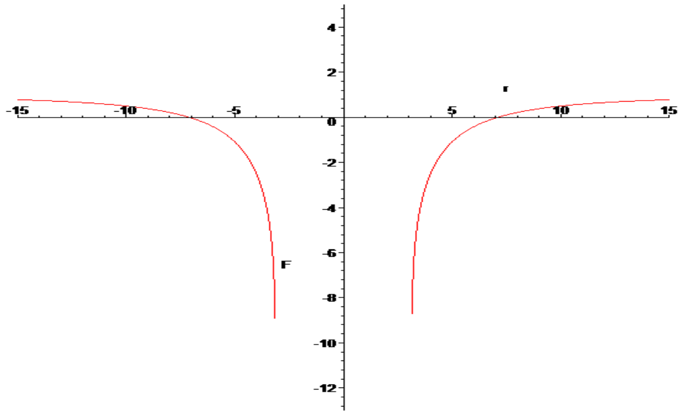
- Therefore, we are considering branches with $H \leq (2\kappa|\beta|)^{-1/2}$. *
- In this model *H and* ρ are bounded, but $R \sim \frac{1}{\sqrt{t}}$ near t=0.
- This singularity is milder than big bang singularity ($R \sim \frac{1}{t^2}$).
- Although the force/acceleration is divergent at t=0, the mechanical system needs only a finite amount of work to go from the singularity to any closeby point since pot. energy at t=0 is finite.



* This shows that $H_{max} \sim \frac{M_P}{\sqrt{\beta}}$, is the effective cutoff scale of gravity (also predicted by Antoniadis in arXiv:1410.8845v2)

VI. Gauss-Bonnet Black Hole

Black Hole with α < 0



- Geodesics: around $r = 2(m\alpha)^{1/4}$

$$\dot{r} = -C_1 + C_2\sqrt{r - 2\,(m\alpha)^{1/4}}$$

$$f(r) = (1 - \sqrt{m/\alpha}) + \sqrt{2}\left(\frac{m}{\alpha}\right)^{3/8}\alpha^{-1/4}\sqrt{r - 2\,(m\alpha)^{1/4}}$$

- *This fn is continues but not differentiable at $r=2(m\alpha)^{1/4}$, which is saying that one can extend geodesics beyond this point, but the extension is not unique.*
- *Another difference is that geodesic extensions are not going to be C^2 as in the case for cosmology but at most C^1.*

IV. Extending spacetime and singularity crossing: geodesic extension

- For these first-order eqn.'s Picard-Lindelof theorem states that if f^i and g are continuous in λ and Lipshitz continuous in t, there exist a unique soln. for the first-order eqn.'s
- Now joining the two branches (soln's for t ≥ 0 and t < 0) together leads to following scale factor

$$a(t) = a_0\,[1 + |H_0 t| - \frac{2}{3}|H_0 t|^{3/2}] + O(t^2).$$

- Notice: First, the above theorem shows the possibility of geodesic extensions for sudden singularities in general.
- Furthermore, the invariance of Raychaudhuri eqn. under t → -t, and H → -H leads to existence of time-reflected soln. which works as a natural extension for FLRW for t<0.

II. Cosmology with Weyl anomaly

- In the special case of conformally flat background, $g_{\mu\nu} = \chi\, \eta_{\mu\nu}$, stress tensor is

$$\langle T(g)^{(ren)}{}_{\mu\nu} \rangle = T^{(m)}_{\mu\nu} + \alpha H_{\mu\nu}{}^{(1)} + \beta H_{\mu\nu}{}^{(3)} \quad *$$

- We are interested in this tensor since it modifies Einstein FE;

$$\frac{2}{\sqrt{-g}} \frac{\delta s_{eff}}{\delta g_{\mu\nu}} = G^{\mu\nu} - \langle T^{\mu\nu} \rangle = 0,$$

where $T^{(m)}{}_{\mu\nu}$ is a local (not geometric) conserved traceless tensor and $H^{(1)}$ and $H^{(3)}$ are given by

$$H_{\mu\nu}{}^{(1)} = 2R_{;\mu\nu} - 2g_{\mu\nu}\Box R - \frac{1}{2} g_{\mu\nu} R^2 + 2RR_{\mu\nu}$$

$$H_{\mu\nu}{}^{(3)} = \frac{1}{12} R^2 g_{\mu\nu} - R^{\rho\sigma} R_{\rho\mu\sigma\nu}$$

- Then the trace is

$$\langle T^{(ren)\mu}{}_{\mu} \rangle = -6\alpha \Box R - \beta \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right),$$

- α is a regularization-scheme dependent (and gauge dependent too). Later we will set $\alpha = 0$.

* J. Brown and N. Cassidy, Phys. Rev. D 15, 2810 (1977)