On conformal geometry and new invariants in absolute parallelism spaces

Ebtsam H. Taha

Department of Mathematics
Faculty of Science
Cairo University

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Conformal Geometry has been studied in different geometries e.g.

- In Riemannian geometry,it is angle-preserving.
- In the Lorentzian context, it preserves causality.
- In Finsler framework, one can consider a much more general concept of conformal geometry which is call **anisotropic conformal** geometry.
- Two pseudo-Finsler metrics are anisotropically conformally equivalent if and only if they have the same lightcone.[1]
- In parallelizable manifolds, it is angle-preserving and causality.

[1] M. A. Javaloyes and B. L. Soares, Anisotropic conformal invariance of lightlike geodesics in pseudo-Finsler manifolds, Class. Quantum Grav. 38 (2021) 025002.
N. Voicu, Conformal maps between pseudo-Finsler spaces, Int. J. Geom. Methods Mod. Phys. (2018) 15 1850003.

Definition

A **parallelizable manifold** is an n-dimensional smooth manifold **M** which admits n independent vector fields $\underset{i}{\lambda}$ $(i = 1, 2, ..., n)$ defined globally on M.

Let $\underset{i}{\lambda}^{\mu}$ $(\mu = 1, 2, ..., n)$ be the coordinate components of the *i-th* vector flied $\underset{i}{\lambda}$.

The covariant components of λ^{μ} are given via the relations

$$\underset{i}{\lambda}^{\mu}\underset{i}{\lambda}_{\nu} = \delta^{\mu}_{\nu}, \quad \underset{i}{\lambda}^{\mu}\underset{j}{\lambda}_{\mu} = \delta_{ij}. \tag{1}$$

The **metric structure** $g_{\mu\nu} := \underset{i}{\lambda}_{\mu}\underset{i}{\lambda}_{\nu}$ with inverse $g^{\mu\nu} := \underset{i}{\lambda}^{\mu}\underset{i}{\lambda}^{\nu}$.

The (built-in) natural connections

The **canonical** connection

$$\Gamma^{\alpha}_{\mu\nu} := \underset{i}{\lambda}^{\alpha} \underset{i}{\lambda}_{\mu,\nu}. \quad (2)$$

The **symmetric connection**

$$\widehat{\Gamma}^{\alpha}_{\mu\nu} := \frac{1}{2}(\Gamma^{\alpha}_{\nu\mu} + \Gamma^{\alpha}_{\mu\nu}) = \Gamma^{\alpha}_{(\mu\nu)}. \quad (3)$$

The **dual** connection

$$\widetilde{\Gamma}^{\alpha}_{\mu\nu} := \Gamma^{\alpha}_{\nu\mu}. \quad (4)$$

The **Riemannian connection**

$$\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} := \frac{1}{2} g^{\alpha\epsilon}(g_{\epsilon\nu,\mu} + g_{\epsilon\mu,\nu} - g_{\mu\nu,\epsilon}). \quad (5)$$

Definition

Two AP-spaces $(M, \underset{i}{\lambda})$ *and* $(M, \underset{i}{\overline{\lambda}})$ *are said to be conformal (or conformally related) if there exists a positive smooth function* $\rho(x)$ *such that*

$$\underset{i}{\overline{\lambda}} = e^{-\rho(x)} \underset{i}{\lambda} .$$

Locally,

$$\underset{i}{\overline{\lambda}}^{\mu} = e^{-\rho(x)} \underset{i}{\lambda}^{\mu} \;\; (or\; \underset{i}{\overline{\lambda}}_{\mu} = e^{\rho(x)} \underset{i}{\lambda}_{\mu}), \tag{6}$$

or, equivalently,

$$\overline{g}_{\mu\nu} = e^{2\rho(x)} g_{\mu\nu}.$$

Under the conformal change (6), *we have:*

- The Weitzenböck connections $\Gamma^{\alpha}_{\mu\nu}$ and $\overline{\Gamma}^{\alpha}_{\mu\nu}$ are related by

$$\overline{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + \delta^{\alpha}_{\mu}\rho_{\nu}.$$

- The torsion tensors $\Lambda^{\alpha}_{\mu\nu}$ and $\overline{\Lambda}^{\alpha}_{\mu\nu}$ of $\Gamma^{\alpha}_{\mu\nu}$ and $\overline{\Gamma}^{\alpha}_{\mu\nu}$ are related by

$$\overline{\Lambda}^{\alpha}_{\mu\nu} = \Lambda^{\alpha}_{\mu\nu} + \left(\delta^{\alpha}_{\mu}\rho_{\nu} - \delta^{\alpha}_{\nu}\rho_{\mu}\right).$$

- The symmetric connections $\widehat{\Gamma}^{\alpha}_{\mu\nu}$ and $\overline{\widehat{\Gamma}}^{\alpha}_{\mu\nu}$ are related by:

$$\overline{\widehat{\Gamma}}^{\alpha}_{\mu\nu} = \widehat{\Gamma}^{\alpha}_{\mu\nu} + \frac{1}{2}(\delta^{\alpha}_{\mu}\rho_{\nu} + \delta^{\alpha}_{\nu}\rho_{\mu}),$$

- The curvature tensors $\widehat{R}^{\alpha}_{\mu\nu\sigma}$ and $\overline{\widehat{R}}^{\alpha}_{\mu\nu\sigma}$ of $\widehat{\Gamma}^{\alpha}_{\mu\nu}$ and $\overline{\widehat{\Gamma}}^{\alpha}_{\mu\nu}$ are related by:

$$\overline{\widehat{R}}^{\alpha}_{\mu\nu\sigma} = \widehat{R}^{\alpha}_{\mu\nu\sigma} + \frac{1}{2}\mathfrak{U}_{\nu\sigma}\{\delta^{\alpha}_{\sigma}\rho_{\mu\widehat{|}\nu} + \frac{1}{2}\delta^{\alpha}_{\nu}\rho_{\sigma}\rho_{\mu}\}. \qquad (7)$$

Remark

The tensor field $\widehat{r}_{\nu\sigma} := \widehat{R}^{\alpha}_{\ \alpha\nu\sigma}$ is **conformally invariant**:

$$\overline{\widehat{r}}_{\nu\sigma} = \widehat{R}^{\alpha}_{\alpha\nu\sigma} + \frac{1}{2}\mathfrak{U}_{\nu\sigma}\{\delta^{\alpha}_{\sigma}\rho_{\alpha\widehat{|}\nu} + \frac{1}{2}\delta^{\alpha}_{\nu}\rho_{\sigma}\rho_{\alpha}\} = \widehat{r}_{\nu\sigma}.$$

The Ricci-like tensor defined by $\widehat{R}_{\mu\nu} := \widehat{R}^{\alpha}_{\ \mu\nu\alpha}$ is not conformally invariant:

$$\overline{\widehat{R}}_{\mu\nu} = \widehat{R}_{\mu\nu} + \frac{(n-1)}{2}\left(\rho_{\mu\widehat{|}\nu} - \frac{1}{2}\rho_{\mu}\rho_{\nu}\right).$$

Under the conformal change (6), *we have:*

- The dual connections $\widetilde{\Gamma}^{\alpha}_{\mu\nu}$ and $\overline{\overline{\widetilde{\Gamma}}}^{\alpha}_{\mu\nu}$ are related by

$$\overline{\overline{\widetilde{\Gamma}}}^{\alpha}_{\mu\nu} = \widetilde{\Gamma}^{\alpha}_{\mu\nu} + \delta^{\alpha}_{\nu}\rho_{\mu}.$$

- The torsion tensors $\widetilde{\Lambda}^{\alpha}_{\mu\nu}$ and $\overline{\overline{\widetilde{\Lambda}}}^{\alpha}_{\mu\nu}$ of $\widetilde{\Gamma}^{\alpha}_{\mu\nu}$ and $\overline{\overline{\widetilde{\Gamma}}}^{\alpha}_{\mu\nu}$ are related by

$$\overline{\overline{\widetilde{\Lambda}}}^{\alpha}_{\mu\nu} = \widetilde{\Lambda}^{\alpha}_{\mu\nu} - \left(\delta^{\alpha}_{\mu}\rho_{\nu} - \delta^{\alpha}_{\nu}\rho_{\mu}\right).$$

- The curvature tensors $\widetilde{R}^{\alpha}{}_{\mu\nu\sigma}$ and $\overline{\overline{\widetilde{R}}}^{\alpha}{}_{\mu\nu\sigma}$ are related by

$$\overline{\overline{\widetilde{R}}}^{\alpha}{}_{\mu\nu\sigma} = \widetilde{R}^{\alpha}{}_{\mu\nu\sigma} + \mathfrak{U}_{\nu\sigma}\{\delta^{\alpha}_{\sigma}\rho_{\mu\widetilde{|}\nu} + \delta^{\alpha}_{\nu}\,\rho_{\mu}\,\rho_{\sigma} + \frac{1}{2}\rho_{\mu}\Lambda^{\alpha}_{\nu\sigma}\},$$

where $\mathfrak{U}_{\mu\nu}\{A_{\mu\nu}\} := A_{\mu\nu} - A_{\nu\mu}$.

- The Levi-Civita connections $\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}$ and $\overline{\overset{\circ}{\Gamma}}{}^{\alpha}_{\mu\nu}$ are related by

$$\overline{\overset{\circ}{\Gamma}}{}^{\alpha}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} + \left(\delta^{\alpha}_{\mu}\rho_{\nu} + \delta^{\alpha}_{\nu}\rho_{\mu} - g_{\mu\nu}\rho^{\alpha}\right).$$

Under the conformal change (6), *we have:*

- The curvature tensor $\overset{\circ}{R}{}^{\alpha}_{\ \mu\nu\sigma}$ of $\overset{\circ}{\Gamma}{}^{\alpha}_{\ \mu\nu}$ is transformed as

$$\overline{\overset{\circ}{R}}{}^{\alpha}_{\ \mu\nu\sigma} = \overset{\circ}{R}{}^{\alpha}_{\ \mu\nu\sigma} + \mathfrak{U}_{\nu\sigma}\{\delta^{\alpha}_{\sigma} S_{\mu\nu} - g_{\mu\sigma} S^{\alpha}_{\nu}\},$$

where $S_{\mu\nu} := \rho_{\mu \overset{\circ}{|} \nu} - \rho_\mu \rho_\nu - \frac{1}{2} g_{\mu\nu}\rho^2$, $\rho^2 := \rho^\epsilon \rho_\epsilon$ and $S^{\alpha}_{\nu} := g^{\alpha\epsilon} S_{\epsilon\nu}$.

- The contortion tensor $\gamma^{\alpha}_{\mu\nu}$ is transformed as

$$\overline{\gamma}^{\alpha}_{\mu\nu} = \gamma^{\alpha}_{\mu\nu} - \delta^{\alpha}_{\nu}\rho_\mu + g_{\mu\nu}\rho^\alpha.$$

- The W-tensor $W^{\alpha}_{\mu\nu\sigma}$ is transformed as

$$\begin{aligned}\overline{W}^{\alpha}_{\mu\nu\sigma} \quad = \quad & W^{\alpha}_{\mu\nu\sigma} + \mathfrak{U}_{\nu\sigma}\Big\{\delta^{\alpha}_{\sigma}\rho_{\nu|\mu} - 2\delta^{\alpha}_{\sigma}\rho_\nu\rho_\mu - \frac{1}{2}\Lambda^{\alpha}_{\sigma\nu}\rho_\mu \\ & + \frac{1}{2}\delta^{\alpha}_{\mu}\Lambda^{\varepsilon}_{\sigma\nu}\rho_\epsilon - \Lambda^{\alpha}_{\sigma\mu}\rho_\nu\Big\}.\end{aligned}$$

It should be noted that

In the next three theorems, we assume that $(M, \underset{i}{\lambda})$ is an AP-space of dimension $n \geq 2$.

Theorem 1

The tensors

$$\begin{aligned} T^{\alpha}_{\mu\nu} &:= \Lambda^{\alpha}_{\mu\nu} - \frac{1}{(n-1)}\{\delta^{\alpha}_{\mu} C_{\nu} - \delta^{\alpha}_{\nu} C_{\mu}\}, \\ K^{\alpha}_{\mu\nu\sigma} &:= \frac{1}{(n-1)}\{\delta^{\alpha}_{\mu} C_{\nu,\sigma} - \delta^{\alpha}_{\mu} C_{\sigma,\nu}\}, \end{aligned}$$

are conformally invariant. Moreover, the tensors $T^{\alpha}_{\mu\nu}$ and $K^{\alpha}_{\mu\nu\sigma}$ are the torsion and curvature tensors of a conformal connection on M

$$\Gamma^{\alpha}_{\mu\nu} := \Gamma^{\alpha}_{\mu\nu} - \frac{1}{(n-1)}\delta^{\alpha}_{\mu} C_{\nu}. \tag{8}$$

New conformally invariant geometric objects

Theorem 2

The tensor

$$B^{\alpha}_{\mu\nu\sigma} := \frac{1}{4}\mathfrak{U}_{\nu\sigma}\{2\Lambda^{\alpha}_{\mu\nu|\sigma} + \Lambda^{\epsilon}_{\mu\nu}\Lambda^{\alpha}_{\sigma\epsilon} + \Lambda^{\epsilon}_{\sigma\nu}\Lambda^{\alpha}_{\epsilon\mu}\}$$
$$-\frac{1}{2(n-1)}\mathfrak{U}_{\nu\sigma}\{\delta^{\alpha}_{\mu}C_{\sigma,\nu} + \delta^{\alpha}_{\sigma}C_{\mu\widehat{|}\nu} - \frac{1}{2(n-1)}\delta^{\alpha}_{\nu}C_{\mu}C_{\sigma}\}$$

is conformally invariant. Moreover, $B^{\alpha}_{\mu\nu\sigma}$ is precisely the curvature tensor of the conformal connection on M

$$\widehat{\Gamma}^{\alpha}_{\mu\nu} := \widehat{\Gamma}^{\alpha}_{\mu\nu} - \frac{1}{2(n-1)}(\delta^{\alpha}_{\mu}C_{\nu} + \delta^{\alpha}_{\nu}C_{\mu}). \quad (9)$$

New conformally invariant geometric objects

Theorem 3

The tensor

$$\begin{aligned} Q^{\alpha}_{\mu\nu\sigma} \;&:=\; \mathfrak{U}_{\nu\sigma}\left\{\gamma^{\alpha}_{\mu\nu|\sigma}+\gamma^{\epsilon}_{\mu\sigma}\gamma^{\alpha}_{\epsilon\nu}+\frac{1}{2}\gamma^{\alpha}_{\mu\epsilon}\Lambda^{\epsilon}_{\nu\sigma}\right\} \\ &\quad -\frac{1}{(n-1)}\mathfrak{U}_{\nu\sigma}\left\{\delta^{\alpha}_{\mu}C_{\sigma,\nu}+\delta^{\alpha}_{\sigma}C_{\mu\overset{\circ}{|}\nu}+g_{\mu\sigma}C^{\alpha}{}_{\overset{\circ}{|}\nu}\right. \\ &\quad \left.-\frac{1}{(n-1)}\left(\delta^{\alpha}_{\nu}C_{\mu}C_{\sigma}-\delta^{\alpha}_{\nu}g_{\mu\sigma}C^{2}+g_{\mu\sigma}C_{\nu}C^{\alpha}\right)\right\} \qquad (10) \end{aligned}$$

is conformally invariant. Moreover, $Q^{\alpha}_{\mu\nu\sigma}$ is precisely the curvature tensor of the following conformal connection on M

$$\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} := \overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} - \frac{1}{(n-1)}\left(\delta^{\alpha}_{\mu}C_{\nu}+\delta^{\alpha}_{\nu}C_{\mu}-g_{\mu\nu}C^{\alpha}\right). \qquad (11)$$

Further properties of the new invariants:

- The invariant connection $\Gamma^{\alpha}_{\mu\nu}$ defined by (8) is non-metric and recurrent metric with recurrence form $\frac{2}{n-1}\, C_{\sigma}$. That is,

$$g_{\mu\nu\|\sigma} = \frac{2}{n-1}\, g_{\mu\nu}\, C_{\sigma}.$$

- The invariant connection $\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}$ defined by (11) is non-metric, symmetric and recurrent metric with recurrence form $\frac{2}{n-1}\, C_{\sigma}$. That is,

$$g_{\mu\nu\overset{\circ}{\|}\sigma} = \frac{2}{n-1}\, g_{\mu\nu}\, C_{\sigma}.$$

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Thank you