A LAGRANGIAN GENERALISATION OF THE NOTION OF A STATIONARY LORENTZIAN METRIC

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Mathematics > Differential Geometry

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A variational setting for an indefinite Lagrangian with an affine Noether charge

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We introduce a variational setting for the action functional of an autonomous and indefinite Lagrangian on a finite dimensional manifold. Our basic assumption is the existence of an infinitesimal symmetry whose Noether charge is the sum of a one-form and a function. Our setting includes different types of Lorentz-Finsler Lagrangians admitting a timelike Killing vector field.

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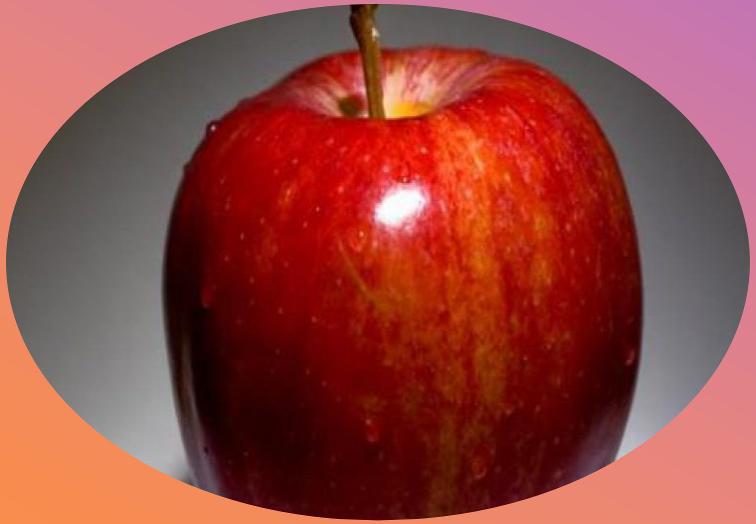

OUTLINE

Stationary Lorentzian metrics

Stationary Lagrangians with affine Noether charge

Examples

On the local expression of L

On the local injectivity of the Legendre transform

STATIONARY LORENTZIAN METRICS

(M,g) $\dim M = m+1$

g Lorentzian $(-,+,\dots,+)$

let $K \in \mathcal{X}(M)$ a Killing vector field

timelike (i.e. $g(K,K)<0$)

$L_K g = 0 \implies$ in some coordinate system adapted to K, $(U, t, x^1, \dots, x^m)$, $U = \underbrace{U_0}_{x} \times \underbrace{I}_{t}$

Riemannian on U_0 1-form on U_0 $\beta := -g(\partial_t, \partial_t)$, $\partial_t \equiv K$

$$\boxed{g = g_0 + 2\omega \otimes dt - \beta\, dt^2}$$

$Y \in \mathfrak{X}(M)$, $Y_p \neq 0 \quad \forall p \in M$

g_R Riemannian metric on M

$$g := g_R - 2\left(\frac{Y}{|Y|_R}\right)^{\flat} \otimes \left(\frac{Y}{|Y|_R}\right)^{\flat} \quad \text{is Lorentzian}$$

For $Y = K$, (recall that K is timelike $\Rightarrow K_p \neq 0$)

$$g_R := g + 2\left(\frac{K}{|K|}\right)^{\flat} \otimes \left(\frac{K}{|K|}\right)^{\flat} \quad \text{is Riemannian}$$

$$g = g_R + 2\left(\frac{k}{|k|_R}\right)^\flat \otimes \left(\frac{k}{|k|_R}\right)^\flat$$

$$g = g_R + \frac{2\, Q \otimes Q}{Q(k)}$$

The Noether charge associated with, is

$$\boxed{\frac{\partial g}{\partial v}(\cdot)[k] = 2g(\cdot, k)} =: Q$$

Notice that it is a 1-form on M

STATIONARY LAGRANGIANS WITH AFFINE NOETHER CHARGE

$L : TM \to \mathbb{R}$, $L = L(q,\dot q)$, $q \in M$, $\dot q \in T_qM$

L invariant by the action of a 1-parameter group of local diffeos with infinitesimal generator $K \in \mathcal{X}(M)$:

$$\mathcal{X}(TM) \ni K^c := K^i \frac{\partial}{\partial q^i} + \dot q^i \frac{\partial K^j}{\partial q^i} \frac{\partial}{\partial \dot q^j}$$

The flow of K^c is $\varphi^c(t,q,\dot q) = \left(\varphi(t,q), \partial_q \varphi(t,q)[\dot q] \right)$

$$K^c(L) = 0 \iff \frac{d}{dt} L(\varphi^c(t,q,\dot q)) = 0$$

We call K an infinitesimal symmetry for L

$N : TM \to \mathbb{R}$, $N(q,\dot q) := \frac{\partial L}{\partial \dot q}(q,\dot q)[K]$

Assume that:

$$\boxed{N(q,\dot q) = Q(\dot q) + \alpha(q)}$$

Q 1-form on M

$\alpha : M \to \mathbb{R}$

$$\boxed{Q(K) < 0}$$

Let us define

$$L_c := L - \frac{Q^2}{Q(K)}$$

and assume that L_c is fiberwise strongly convex:

$$\left(\frac{\partial L_c}{\partial \dot q}(q,\dot q_2) - \frac{\partial L_c}{\partial \dot q}(q,\dot q_1)\right)[\dot q_2 - \dot q_1] \geq$$

$$\geq \lambda(q)\,|\dot q_2 - \dot q_1|^2, \quad \lambda : M \to \mathbb{R}, \ \lambda > 0$$

SOME CLASSES OF EXAMPLES

1) Let $M = \overset{x}{S} \times \overset{t}{\mathbb{R}}$

$L_0 : TS \to \mathbb{R}$, $w \in \Lambda_1(S)$, $\alpha : S \to \mathbb{R}$, $\beta : S \to \mathbb{R}$, $\beta > 0$

$$L((x,t),(v,\tau)) = L_0(x,v) + 2\left(w(v) + \frac{\alpha(x)}{2}\right)\tau - \beta(x)\tau^2$$

with L_0 fiberwise strongly convex

$\frac{\partial}{\partial t} \equiv K$ is an infinitesimal symmetry for L

$$N((x,t),(v,\tau)) = \underbrace{2\left(w(v) - \beta\, dt\right)}_{\Omega} + \underbrace{\alpha(x)}_{\alpha}$$

$Q(K) = -2\beta < 0$

Here $$L_c((x,t),(\nu,\tau)) = L_0(x,\nu) + \left(\frac{1}{\sqrt{\beta(x)}}\omega(\nu) - \sqrt{\beta(x)}\tau\right)^2 + \frac{1}{\beta(x)}\omega^2(\nu) + \frac{d(x)}{2}\tau.$$

which is strongly convex

1-b) A special case is $L_0 = F^2 + \omega_0 + V$
with F Finsler metric, $\omega_0 \in \Lambda_1(S)$
and $V : S \to \mathbb{R}$

This case generalises static Lorentz-Finsler metric corresponding to $\omega \equiv 0$, $d \equiv 0$, $\omega_0 \equiv 0$, $V \equiv 0$

Introduced in

C. Lämmerzahl, V. Perlick and W. Hasse: Observable effects in a class of spherically symmetric static Finsler spacetimes. Phys. Rev. D, 86 (2012), 104042.

and studied also in

E. Caponio and G. Stancarone: Standard static Finsler spacetimes. Int. J. Geom. Methods Mod. Phys., 13 (2016), 1650040.

They also generalises a class of stationary Lorentz-Finsler spacetimes studied in

E. Caponio and G. Stancarone: On Finsler spacetimes with a timelike Killing vector field. Classical Quantum Gravity, 35 (2018), 085007.

that corresponds to the case $d \equiv 0$, $\omega_0 \equiv 0$, $V \equiv 0$

2) Interchanging the role of L and L_c

Let L_b be a strongly convex Lagrangian admitting an infinitesimal symmetry K such that $N_b := \frac{\partial L_b}{\partial \dot q}(\cdot)[K]$ is pointwise affine $N_b = Q_b + d$ and $Q_b(K) > 0$

Then $L := L_b - Q_b^2 / Q(K)$ has K as an infinitesimal symmetry,

$$N = -Q_b + d \quad \text{and} \quad L_c = L_b$$

2-b)

Let $F : TM \to \mathbb{R}$ be a Finsler metric on M admitting K as an infinitesimal symmetry s.t.

$$K_x \neq 0 \ , \ \forall x \in M \quad \text{and} \quad N_F = Q_F$$

Let $w_0 \in \Lambda_1(M)$ and $V : M \to \mathbb{R}$ invariant by the flow of K

let

$$L := F^2 + w_0 + V - \cancel{\frac{Q_F^2}{Q(K)}}$$

Notice that
$Q_F(K) = 2F^2(K) > 0$

In this case $L_b = F^2 + w_0 + V$ and $N_b = N_F + w_0(K)$;

hence $Q_b(K) = Q_F(K) > 0$

3)
Beem's Lorentz-Finsler metrics

J. K. Beem: Indefinite Finsler spaces and timelike spaces. Canad. J. Math., 22 (1970), pp. 1035–1039.

- $L_F : TM \to \mathbb{R}$, $L_F \in C^2(TM \setminus 0)$, $L_F(x,\lambda v) = \lambda^2 L_F(x,v)$ $\forall v \in TM$, $\forall \lambda > 0$
- $\forall (x,v) \in TM \setminus 0$:

$$\frac{\partial^2 L_F}{\partial v^\alpha \partial v^\beta}(x,v), \quad \alpha,\beta = 0,\dots,m, \text{ is non-degenerate of index } 1$$

- K is an infinitesimal symmetry for L_F, $L_F(K) < 0$ and $N_{L_F} = Q_{L_F} + d_{L_F}$

$L_C := L_F - \mathfrak{a}_{L_F}^2 / \mathfrak{a}_{L_F}(k)$ is then strongly convex

This example can be generalized by replacing

$L_F \longrightarrow L_F + \omega_1 + V$, ω_1, V being k-invariant

3)-b) In particular if $F : TM \to \mathbb{R}$ is a standard Finsler metric on M and $\omega \in \Lambda_1(M)$; if K is an infinitesimal symmetry for F with $N_F = \mathfrak{a}_F$ and ω is K-invariant then

$L_F := F^2 - \omega^2$ is **Beem** provided $F^2(k) - \omega^2(k) < 0$

ON THE LOCAL EXPRESSION OF L

We want to show that L can be always expressed locally as a Lagrangian in Example 1)

i.e. for each $p \in M$ there exist

- $U_p \subset M$, open neighborhood of p
- $\phi : S_p \times I \to U_p$ diffeo.
- $L_0 : TS_p \to \mathbb{R}$, $w \in \Lambda_1(S_p)$, $d : S_p \to \mathbb{R}$, $\beta : S_p \to \mathbb{R}$, $\beta > 0$ s.t.

$$L \circ \phi\big((x,t),(v,\tau)\big) = L_0(x,v) + 2\left(w(v) + \frac{d(x)}{2}\right)\tau - \beta(x)\tau^2$$

Let then $\mathcal{Q} \subset TM$ be the rank m distribution generated by $\operatorname{Ker} Q$ (Recall that $Q(K)<0$).

Let $p \in M$ and S_p be a hypersurface s.t. $p \in S_p$, $T_p S_p = \mathcal{Q}_p$ and K_p is transversal to S_p for all $q \in S_p$ (i.e. $T_q M = T_q S_p \oplus [K_q]$)

Recall that $L_c = L - Q^2/Q(K)$ thus

$$\frac{\partial L}{\partial \dot q}(q,\dot q) = \frac{\partial L_c}{\partial \dot q}(q,\dot q) + \frac{2}{Q(K)} Q(\dot q)\, Q$$ and

$$\left(\frac{\partial L}{\partial \dot q}(q,\dot q_2) - \frac{\partial L}{\partial \dot q}(q,\dot q_1)\right)[\dot q_2 - \dot q_1] = \left(\frac{\partial L_c}{\partial \dot q}(q,\dot q_2) - \frac{\partial L_c}{\partial \dot q}(q,\dot q_1)\right)[\dot q_2 - \dot q_1] + 2Q^2(\dot q_2 - \dot q_1)/Q(K)$$

$$\left(\frac{\partial L}{\partial \dot q}(q,\dot q_2)-\frac{\partial L}{\partial \dot q}(q,\dot q_1)\right)[\dot q_2-\dot q_1]=\left(\frac{\partial L_c}{\partial \dot q}(q,\dot q_2)-\frac{\partial L_c}{\partial \dot q}(q,\dot q_1)\right)[\dot q_2-\dot q_1]+2Q^2(\dot q_2-\dot q_1)/Q(K)$$

Let $\mathbb{R}\ni\lambda_0=\min\limits_{q\in S_p}\left(\lambda(q)+2\max\limits_{\substack{|v|=1\\ v\in TS_p}} Q^2(v)/Q(K)\right)>0$

$$\left(\frac{\partial L}{\partial \dot q}(q,\dot q_2)-\frac{\partial L}{\partial \dot q}(q,\dot q_1)\right)[\dot q_2-\dot q_1]\geq\lambda|\dot q_2-\dot q_1|^2.$$

Hence $L_0:=L|TS_p$

$$\frac{d}{ds}L(q,\dot q+s\tau K_q)=\frac{\partial L}{\partial \dot q}(q,\dot q+s\tau K_q)[\tau K_q]=\tau N(q,\dot q+s\tau K_q)$$

$$=\tau\left(Q(\dot q)+d(q)\right)+\tau^2 sQ(K_q)$$

Thus $L(q,\dot q+\tau K_q)-L(q,\dot q)=\tau\left(Q(\dot q)+d(q)\right)+\frac{1}{2}\tau^2Q(K_q)$

and then for all $(q,\dot q)\in TS_p$

$$L(q,\dot q+\tau K_q)=L_0(q,\dot q)+\big(Q(\dot q)+d(q)\big)\tau+\tfrac{1}{2}Q(K_q)\tau^2$$

Thus $\forall\, w\in T_qM$, $q\in S_p$, $w=\dot q+\tau_w K_q$ for some $\dot q\in TS_p$,

and $L(q,w)=L_0(q,\dot q)+\big(Q(\dot q)+d(q)\big)\tau_w+\tfrac{1}{2}Q(K_q)\tau_w^2$

which is of the stated type, i.e. :

$$L_0(x,v)+2\left(W(v)+\frac{d(x)}{2}\right)\tau-\beta(x)\tau^2,$$

by letting $W:=\frac{Q}{2}\Big|_{TS_p}$, $\beta=-\frac{1}{2}Q(K|_{S_p})$

and then for all $(q,\dot{q}) \in TS_p$

$$L(q,\dot{q}+\tau K_q) = L_0(q,\dot{q}) + \big(Q(\dot{q}) + d(q)\big)\tau + \tfrac{1}{2}Q(K_q)\tau^2$$

Thus $\forall\, w \in T_qM$, $q \in S_p$, $w = \dot{q} + \tau_w K_q$ for some $\dot{q} \in TS_p$

and $L(q,w) = L_0(q,\dot{q}) + \big(Q(\dot{q}) + d(q)\big)\tau_w + \tfrac{1}{2}Q(K_q)\tau_w^2$

which is of the stated type

Finally by using the flow φ of K, we can construct the diffeo $\phi : S_p \times I \to U_p$ and obtaining the above expression for $L \circ \phi$ thanks to the K-invariance of L

ON THE LOCAL INJECTIVITY OF THE LEGENDRE TRANSFORM

We want to show that

$$(q, \dot q) \in TM \mapsto \left(q, \frac{\partial L}{\partial \dot q}(q,\dot q)\right) \in TM^*$$ is locally injective

We can then assume that locally L is given as

$$L\big((x,t),(v,\tau)\big) = L_0(x,v) + 2\left(\omega(v) + \frac{d(x)}{2}\right)\tau - \beta(x)\tau^2$$

$$(q,\dot q) \cong \big((x,t),(v,\tau)\big)$$

If L_0 admits second derivatives w.r.t. v outside the zero section then

$$\frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j} \cong \begin{pmatrix} \frac{\partial^2 L_0}{\partial v^i \partial v^j} & \omega_i \\ \omega_i & -\beta \end{pmatrix}$$

Since $W_0 \equiv 0 \Rightarrow \|W\| \ll 1$ on S_p (up to take a smaller S_p)

and then being $\beta > 0$ the above matrix is non-degenerate

This might be useful for modified dispersion relations, see e.g.

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Redshift and lateshift from homogeneous and isotropic modified dispersion relations

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THANKS FOR YOUR ATTENTION!

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