Hamilton's equations for the teleparallel equivalent of general relativity

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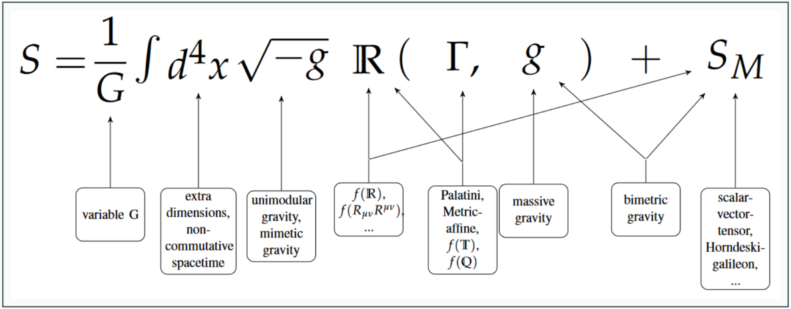
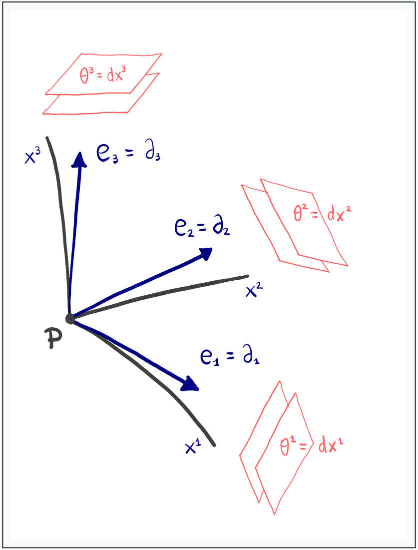
Modifying gravity

The simplest modification of the Einstein-Hilbert action is...



[A. Golovnev, M.J. Guzman (2022) arXiv:2203.16610].

Mathematical framework: (co)frames



- Local coordinates $\{x^\mu\}$ at P.
- They naturally define a basis vectors $e_\mu = \partial/\partial x^\mu = \partial_\mu$
- The basis 1-forms $\theta^\mu = dx^\mu$ are dual to the e_μ.
- In 4D, a linear combination of the θ^μ gives us an arbitrary frame, tetrad, or vierbein $\theta^a = \theta^a_\mu dx^\mu$
- Completeness relation $\theta^a(e_b) = \delta^a_b$, orthonormality condition $\eta_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b$ that defines the metric tensor $g_{\mu\nu} = \eta_{ab}\theta^a_\mu \theta^b_\nu$.

Mathematical framework: linear connection

The connection $\Gamma^{\alpha}{}_{\mu\nu}$ defines the parallel transport of a vector along a curve in a manifold. Generically it has three parts:

$$\Gamma^{\alpha}{}_{\mu\nu} = \underbrace{\bar{\Gamma}^{\alpha}{}_{\mu\nu}}_{\text{(CHRISTOFFEL'S)}} + K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}$$

CONTORTION:

$$K^{\alpha}{}_{\mu\nu} = \frac{1}{2}\left(T^{\alpha}{}_{\mu\nu} + T_{\mu}{}^{\alpha}{}_{\nu} + T_{\nu}{}^{\alpha}{}_{\mu}\right)$$

DISTORTION:

$$L^{\alpha}{}_{\mu\nu} = \frac{1}{2}\left(Q^{\alpha}{}_{\mu\nu} - Q_{\mu}{}^{\alpha}{}_{\nu} - Q_{\nu}{}^{\alpha}{}_{\mu}\right)$$

TORSION:

$$\boxed{T^{\rho}{}_{\mu\nu} = 2\Gamma^{\rho}{}_{[\mu\nu]}}$$

NON-METRICITY:

$$\boxed{Q_{\rho\mu\nu} = \nabla_{\rho} g_{\mu\nu}}$$

It is related with the spin connection $\omega^{a}{}_{b\mu}$ by the tetrad postulate

$$\partial_{\mu}\theta^{a}{}_{\nu} + \omega^{a}{}_{b\mu}\theta^{b}{}_{\nu} - \Gamma^{\rho}{}_{\mu\nu}\theta^{a}{}_{\rho} = 0 \tag{1}$$

Metric teleparallel framework

- A teleparallel framework is the one for which the linear connection has vanishing Riemann curvature $R^{\mu}{}_{\nu\alpha\beta} = 0$.
- In this setup we can still fix either the non-metricity tensor, or the torsion tensor, to zero. In the former case, the connection can be written in term of Lorentz matrices as

$$\omega^{a}{}_{b\mu} = -\left(\Lambda^{-1}\right)^{c}{}_{b}\partial_{\mu}\Lambda_{c}{}^{a}. \tag{2}$$

- Consequently, the torsion tensor is

$$T^{a}{}_{\mu\nu} = \partial_{\mu}\theta^{a}{}_{\nu} - \partial_{\nu}\theta^{a}{}_{\mu} + \omega^{a}{}_{b\mu}\theta^{b}{}_{\nu} - \omega^{a}{}_{b\nu}\theta^{b}{}_{\mu}. \tag{3}$$

- A useful object for later is the torsion scalar $\mathbb{T}$:

$$\mathbb{T} = -\frac{1}{4}T_{\rho\mu\nu}T^{\rho\mu\nu} - \frac{1}{2}T_{\rho\mu\nu}T^{\mu\rho\nu} + T^{\rho}{}_{\mu\rho}T^{\sigma\mu}{}_{\sigma}.$$

The teleparallel equivalent of general relativity

The action for the teleparallel equivalent of general relativity is [Aldrovandi, Pereira (2013)]

$$S_{\mathrm{TEGR}} = \frac{1}{2\kappa} \int d^4x \; \theta \mathbb{T},$$

contains $\mathbb{T}$, which satisfies the identity

$$\theta \mathbb{T} = -\theta \mathbb{R} + \partial_\mu (\theta T^{\nu}{}_{\nu}{}^{\mu}).$$

Here $\mathbb{R}$ depends exclusively on the metric, which is invariant under local Lorentz transformations of the tetrad

$$\theta^a \longrightarrow \theta^{a'} = \Lambda^{a'}{}_{a}(x) \theta^a.$$

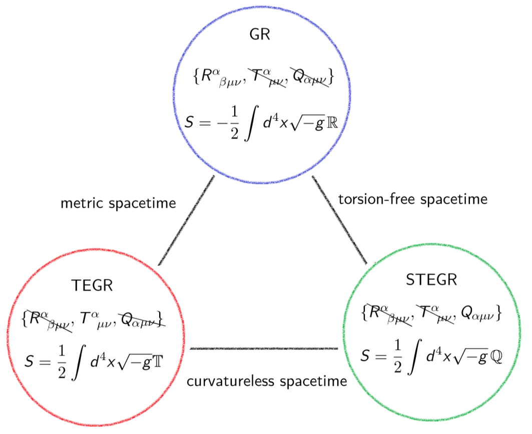
This is not true for $\partial_\mu (\theta T^{\nu}{}_{\nu}{}^{\mu})$, which is a harmless boundary term. Then, it is said that TEGR is a **Lorentz pseudo-invariant** theory. TEGR encompasses the same degrees of freedom than GR [Ferraro, Guzman (2016) arXiv: 1609.06766].

Teleparallel gravity (1928-1931)

- Einstein's attempts to build a unified theory of gravitational and electromagnetical fields on the mathematical framework of teleparallelism, is an episode that lasted for three years from summer 1928 until spring 1931.
- The tetrad field was introduced to get sixteen components instead of only ten for the symmetric metric tensor, and to exploit the additional degrees of freedom to accomodate the electromagnetic field.
- The mathematical structures had been developed before by Cartan and Weitzenboeck, purely in mathematical concepts.
- Einstein's pursuit of the approach triggered a more general discussion that involved a few other contemporary physicists and mathematicians, and it continued to be investigated further even when Einstein no longer took active part in the discussions.

See Sauer (2004) arXiv:physics/0405142

The geometrical trinity of gravity



[adapted from Beltran-Jimenez, Heisenberg, Koivisto (2019)]

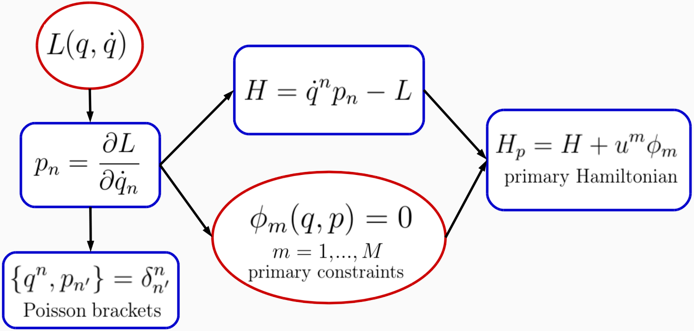
Hamiltonian formalism

Some motivations for studying the Hamiltonian formalism of the theory are:

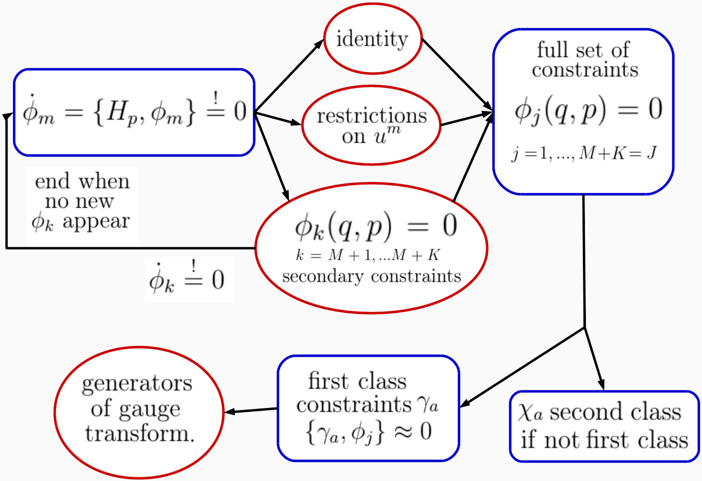
- allows a non-ambiguous identification of gauge symmetries and counting of physical degrees of freedom
- crucial in approaches to canonical quantum gravity
- assessing the well-posedness of the Cauchy problem, therefore the viability of any theory
- theoretical basis for numerical modified general relativity.

In particular, Hamilton's equations correspond 1-to-1 to the 3+1 Lagrangian decomposition. We make partial use of Dirac's algorithm.

Dirac algorithm part 1



Dirac algorithm part 2



ADM split in the tetrad

- We would like to perform a suitable 3+1 decomposition in the tetrad, reproducing the typical ADM split in the metric

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^i \beta^j \gamma_{ij} & \beta^i \\ \beta^j & \gamma_{ij} \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{pmatrix}. \tag{4}$$

- However, there are infinite possible representations. We begin by splitting the components ${\theta^A}_\mu$ as

$$ {\theta^A}_0 = \alpha \xi^A + \beta^i {\theta^A}_i, \quad {\theta^A}_i {\theta^B}_j \eta_{AB} = \gamma_{ij}, \tag{5}$$

where ξ^A is a vector satisfying the following properties

$$\eta_{AB} \xi^A \xi^B = \xi_A \xi^A = -1, \quad \eta_{AB} \xi^B {e^A}_i = \xi_A {\theta^A}_i = 0. \tag{6}$$

ADM split in the tetrad

- The requirement that $\xi_A \theta^A{}_i = 0$ indicates that ξ_A should be proportional to purely spatial components of the tetrad, i.e.

$$\xi_A \propto \epsilon_{ABCD} e^B{}_1 e^C{}_2 e^D{}_3 \tag{7}$$

- For the inverse components, the decomposition reads

$$e_A{}^0 = -\frac{1}{\alpha}\xi_A, \quad e_A{}^i = \theta_A{}^i + \xi_A \frac{\beta^i}{\alpha} \tag{8}$$

- The object $\theta_A{}^i$ is the shorthand notation for $\theta_A{}^i = \eta_{AB}\gamma^{ij}\theta^B{}_j$
- This ADM decomposition in the tetrad, although not unique, recovers unambiguously the ADM metric.

ADM Lagrangian

After performing the ADM split in the tetrad, we obtain the following form of the TEGR Lagrangian

$$\begin{aligned} L_{\mathrm{TEGR}} = {} & \frac{\sqrt{\gamma}}{2\alpha} M^{i}{}_{A}{}^{j}{}_{B} T^{A}{}_{0i} T^{B}{}_{0j} \\ & - \frac{\sqrt{\gamma}}{\alpha} T^{A}{}_{0i} T^{B}{}_{kl} \cdot \left[M^{i}{}_{A}{}^{l}{}_{B} \beta^{k} + \frac{\alpha}{\kappa} \gamma^{il} \left(\frac{1}{2} \xi_{B} \theta_{A}{}^{k} - \xi_{A} \theta_{B}{}^{k} \right) \right] + L_{S}, \end{aligned} \tag{9}$$

where the Hessian is

$$M^{i}{}_{A}{}^{j}{}_{B} = -\frac{1}{\kappa} \left(\frac{1}{2} \gamma^{ij} \eta_{AB} + \frac{1}{2} \xi_{A} \xi_{B} \gamma^{ij} + \frac{1}{2} \theta_{A}{}^{j} \theta_{B}{}^{i} - \theta_{A}{}^{i} \theta_{B}{}^{j} \right), \tag{10}$$

and a term independent from time derivatives of the tetrad is

$$L_{S} = \frac{\sqrt{\gamma}}{\alpha} T^{A}{}_{ij} T^{B}{}_{kl} \beta^{i} \left[\frac{1}{2} M^{j}{}_{A}{}^{l}{}_{B} \beta^{k} + \frac{\alpha}{\kappa} \gamma^{jl} \left(\frac{1}{2} \xi_{B} \theta_{A}{}^{k} - \xi_{A} \theta_{B}{}^{k} \right) \right] + \frac{\alpha \sqrt{\gamma}}{2\kappa} {}^{3}\mathbb{T}. \tag{11}$$

Canonical momenta in TEGR

- Given that $T^A{}_{0i} = \partial_0 e^A{}_i + \ldots$, the canonical momenta can be obtained through variation wrt $T^A{}_{0i}$, that is,

$$\pi_A{}^i = \frac{\partial L}{\partial T^A{}_{0i}} = \frac{\sqrt{\gamma}}{\alpha}\left[M^{ij}_{AB} T^B{}_{0j} - T^B{}_{kl}(M^{il}_{AB}\beta^k + \frac{\alpha}{\kappa}\left(\frac{1}{2}\xi_B\theta_A{}^k - \xi_A\theta_B{}^k\right))\right]. \tag{12}$$

- Since the TEGR Lagrangian does not depend on $\partial_0 e^A{}_0$, the primary constraint $\pi_A{}^0 = 0$ appears.
- Additional primary constraints are obtained from the contraction $e_{i[B}\pi_{A]}{}^i + \text{boundary term} = 0$

Irreducible decomposition

We consider useful to decompose the momenta and velocity under irreducible parts under the rotation group $\mathscr{O}(3)$

$$\dot{\theta}^A{}_i = {}^{\mathbb{V}}\dot{\theta}_i \xi^A + {}^{\mathbb{A}}\dot{\theta}_{ji}\gamma^{kj}\theta^A{}_k + {}^{\mathbb{S}}\dot{\theta}_{ji}\gamma^{kj}\theta^A{}_k + {}^{\mathbb{T}}\dot{\theta}\theta^A{}_i, \tag{13}$$

$$\pi_A{}^i = {}^{\mathbb{V}}\pi^i \xi_A + {}^{\mathbb{A}}\pi^{ji}\gamma_{kj}\theta_A{}^k + {}^{\mathbb{S}}\pi^{ji}\gamma_{kj}\theta_A{}^k + {}^{\mathbb{T}}\pi\theta_A{}^i. \tag{14}$$

The superscripts are for $\mathbb{V}$ectorial, $\mathbb{A}$ntisymmetric, $\mathbb{S}$ymmetric trace-free and $\mathbb{T}$race parts [Blixt, Hohmann, Pfeifer 1811.11137] .

The decomposition allows to invert the Hessian by blocks, obtain the velocities in term of the momenta, and write the Hamiltonian $H = \dot{\theta}^A{}_i \pi_A{}^i - L$.

Hamiltonian for TEGR

We obtain that the Hamiltonian for TEGR has the following form

$$
\begin{aligned}
H_{\mathrm{TEGR}} = \alpha \Bigg[-\frac{\kappa}{16\sqrt{\gamma}} \left[\pi_A{}^i \pi_B{}^l \theta^A{}_k \theta^B{}_j \gamma^{jk} \gamma_{li} + \pi_A{}^i \pi_B{}^j \theta^A{}_j \theta^B{}_i \right. \\
\left. - \pi_A{}^i \pi_B{}^j \theta^A{}_i \theta^B{}_j \right] - \frac{\sqrt{\gamma}}{2\kappa} {}^3\mathbb{T} - \xi^A \partial_i \pi_A{}^i \Bigg] \\
+ \beta^j \left[-\theta^A{}_j \partial_i \pi_A{}^i - \pi_A{}^i T^A{}_{ij} \right] + \text{primary constraints}
\end{aligned}
\tag{15}
$$

Let us compare with the Hamiltonian for GR

$$
H = \sqrt{\gamma} \left\{ \alpha \left[-{}^{(3)}R + \frac{\pi^{ij}\pi_{ij}}{\gamma} - \frac{1}{2}\frac{\pi^2}{\gamma} \right] - 2\beta_j D_i \left(\frac{\pi^{ij}}{\sqrt{\gamma}} \right) + 2 D_i \left(\beta_j \frac{\pi^{ij}}{\sqrt{\gamma}} \right) \right\}.
\tag{16}
$$

Derivation of Hamilton's equations

We recall that the momenta for GR and TEGR are defined as

$$\pi^{ij} = \frac{\partial L}{\partial \dot{\gamma}_{ij}}, \quad {\pi^i}_A = \frac{\partial L}{\partial {\dot{\theta}^A}_i}, \tag{17}$$

therefore it is easy to see that they are dependent from each other, since

$$\dot{\gamma}_{ij} = \eta_{AB}({\dot{\theta}^A}_i {\theta^B}_j + {\theta^A}_i {\dot{\theta}^B}_j). \tag{18}$$

In particular,

$${\pi_A}^i = 2\eta_{AB} {\theta^B}_j \pi^{ij} \tag{19}$$

Hamilton's equations for TEGR are then obtained as

$${\dot{\theta}^A}_i = \{{\theta^A}_i, H\} = \frac{\delta H}{\delta {\pi_A}^i}, \qquad {\dot{\pi}_A}{}^i = \{{\pi_A}^i, H\} = -\frac{\delta H}{\delta {\theta^A}_i}. \tag{20}$$

Hamilton's equations for TEGR

Our computations give [Blixt, Guzman, Pati, in preparation]

$$
\begin{aligned}
-\dot{\pi}_A{}^i = \frac{\delta H}{\delta \theta^A{}_i} \\
&= \frac{\alpha\kappa}{8\sqrt{\gamma}}\left(\pi_A{}^i\pi_B{}^j\theta^B{}_j - \pi_A{}^j\pi_B{}^i\theta^B{}_j - \gamma^{il}\gamma_{jk}\pi_A{}^j\pi_B{}^k\theta^B{}_l - \beta^i\partial_j\pi_A{}^j + \alpha\xi_A\gamma^{ik}\theta^B{}_k\partial_j\pi_B{}^j\right) \\
&+ \frac{\alpha\kappa}{16\sqrt{\gamma}}\theta^A{}_i\pi_B{}^j\pi_D{}^k(\theta^B{}_k\theta^D{}_j - \theta^B{}_j\theta^D{}_k + \gamma_{jk}\gamma^{ln}\theta^B{}_l\theta^D{}_n) \\
&+ \frac{\alpha\kappa}{8\sqrt{\gamma}}\eta_{AC}\left(\gamma_{lm}\pi_B{}^l\pi_D{}^m\theta^B{}_n\theta^D{}_k\theta^C{}_j\gamma^{k[i}\gamma^{j]n} - \gamma^{mn}\theta^B{}_m\theta^C{}_j\theta^D{}_n\pi_D{}^{(i}\pi_B{}^{j)}\right) + \mathscr{O}(\partial_i\theta)
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\dot{\theta}^A{}_i = \frac{\delta H}{\delta \pi_A{}^i} &= \left[\alpha\xi^A + \beta^k\theta^A{}_k\right]\partial_i^x\delta(x-y) - \beta^k T^A{}_{ik}\delta(x-y) \\
&\frac{\kappa\alpha}{8\sqrt{\gamma}}\left[\pi_B{}^j\theta^A{}_{[i}\theta^B{}_{j]} - \gamma_{ij}\gamma^{kl}\pi_B{}^j\theta^A{}_{(k}\theta^B{}_{l)} + \pi_C{}^k\theta^A{}_{[i}\theta^C{}_{k]}\right]\delta(x-y)
\end{aligned}
\tag{22}
$$

Hamilton's equations for GR

In comparison, in general relativity it is known that

$$\dot{\gamma}_{ij} = \frac{\delta H}{\delta \pi^{ij}} = 2\gamma^{-1/2}\alpha\left(\pi_{ij} - \frac{1}{2}\gamma_{ij}\pi\right) + 2D_{(i}\beta_{j)}, \tag{23}$$

and

$$\begin{aligned}\dot{\pi}^{ij} = &- \frac{\delta H}{\delta \gamma_{ij}} = -\alpha\sqrt{\gamma}\left({}^{(3)}R^{ij} - \frac{1}{2}{}^{(3)}R\gamma^{ij}\right) + \frac{1}{2}\alpha\gamma^{-1/2}\gamma^{ij}\left(\pi^{kl}\pi_{kl} - \frac{1}{2}\pi^2\right)\\ &- 2\alpha\gamma^{-1/2}\left(\pi^{ik}\pi_k{}^j - \frac{1}{2}\pi\pi^{ij}\right)\\ &+ \sqrt{\gamma}(D^iD^j\alpha - \gamma^{ij}D^kD_k\alpha) + \sqrt{\gamma}D_k(\gamma^{-1/2}\beta^k\pi^{ij}) - 2\pi^{k(i}D_k\beta^{j)}.\end{aligned} \tag{24}$$

Discussion and conclusions

- Hamilton's equations in TEGR are generically more complicated than GR ones
- ADM decomposition of Einstein's equations aren't strongly hyperbolic, a reformulation called BSSN it is. It is widely used in numerical relativity for strong gravity regimes as merge of black holes.
- Such study in TEGR is not present in the literature. Some authors [Capozziello, Finch, Levi-Said, Magro 2108.03075] claim that in TEGR simulations should be computationally more efficient. We suspect it could be exactly the opposite.
- We have also obtained Hamilton's equations for the covariant version of TEGR.


gracias
спасибо
شكرا
danke
aitäh
merci
thanks
grazie
kiitos
obrigada