Collinear Parton Dynamics Beyond DGLAP

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in collaboration with

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Track Functions

Motivation

- Track-based measurements offer:
  - superior angular resolution
  - pileup mitigation.
- One problem: Track-based calculations are not IR safe in perturbation theory.

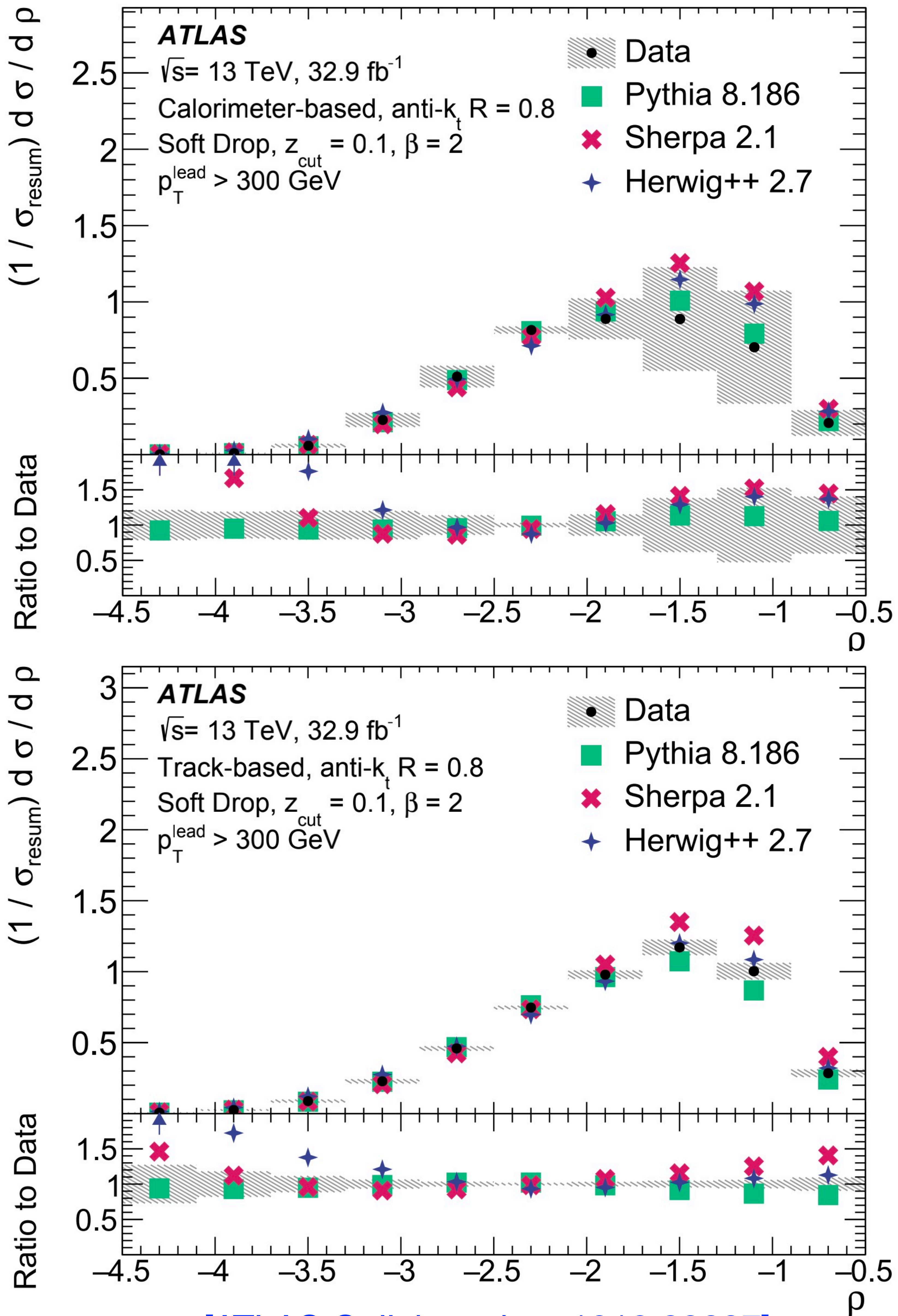
Track Functions

- IR divergences are absorbed into these **universal** non-perturbative functions.

(like the case of parton distribution functions and fragmentation functions)

calorimeter-based (all-particle)

track-based (charged-particle)



[ATLAS Collaboration, 1912.09837]

✓ Track functions introduced and studied at $\mathcal{O}(\alpha_s)$.

[H. Chang, M. Procura, J. Thaler, W. Waalewijn, arXiv:1303.6637, arXiv:1306.6630]

- Complicated:

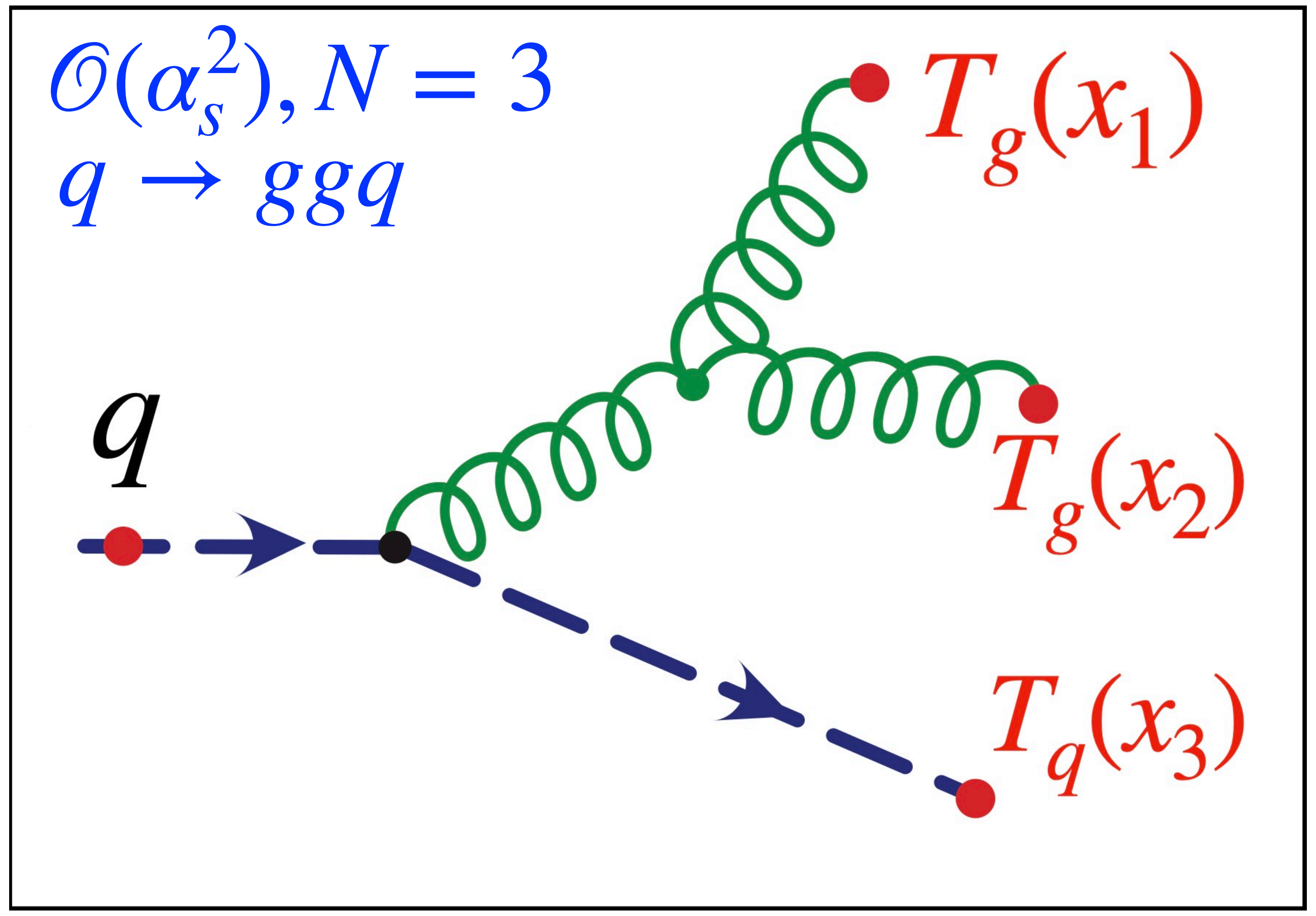
observables. For all of these observables, the uncertainties for the track-based observables are significantly smaller than those for the calorimeter-based observables, particularly for higher values of β, where more soft radiation is included within the jet. However, since no track-based calculations exist at the present time, calorimeter-based measurements are still useful for precision QCD studies.

[ATLAS Collaboration, 1912.09837]

the selection of charged particle jets. Note that track-based observables are IRC-unsafe. In general, nonperturbative track functions can be used to directly compare track-based measurements to analytical calculations [67–69]; however, such an approach has not yet been developed for jet angularities. Two

[ALICE Collaboration, 2107.11303]

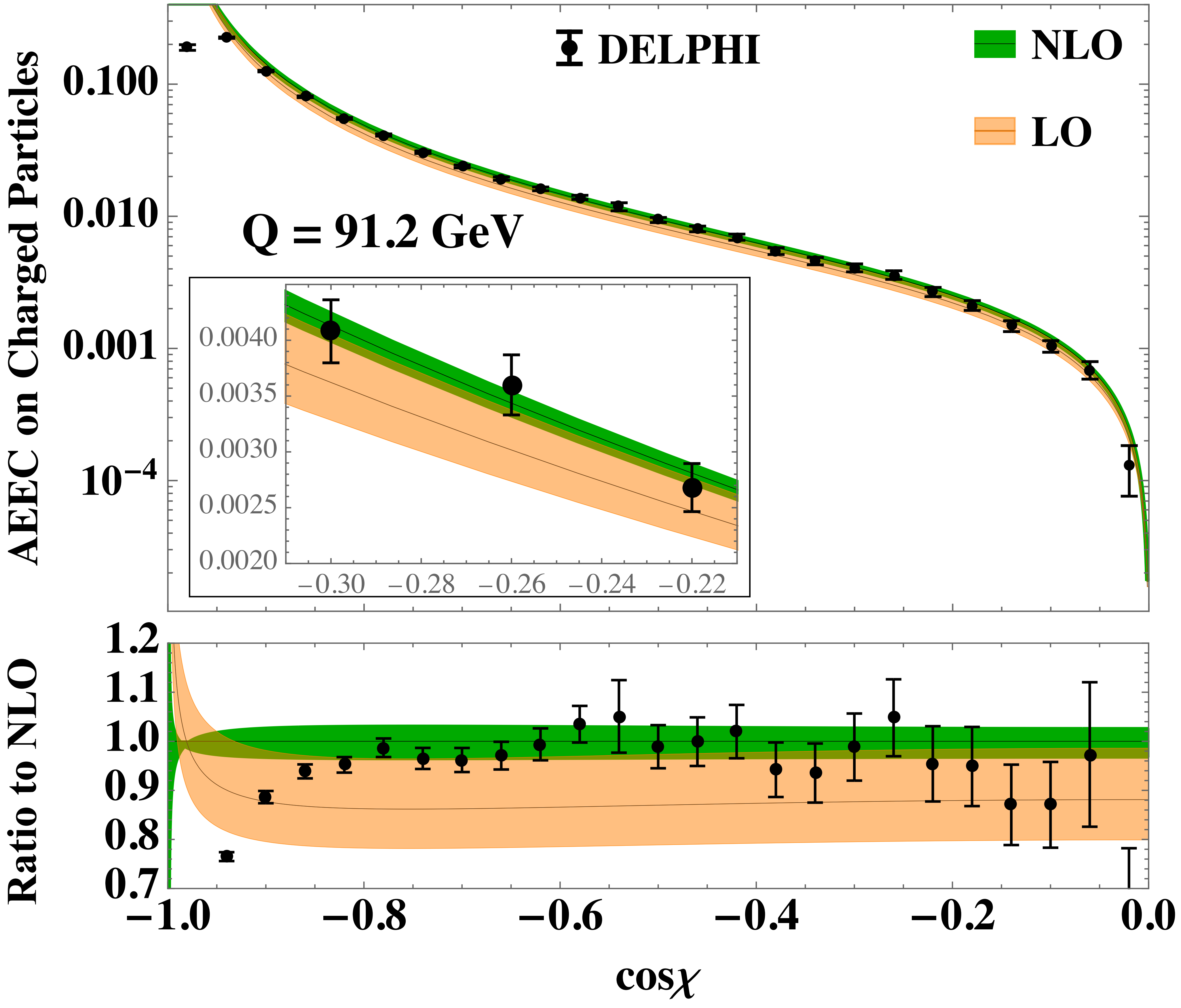
- The track function evolution encodes correlations in the hadronization process and thus involves the full collinear $1 \to N$ ($N \geq 3$) splittings in its kernel beyond LO.



✓ Our work: Track function formalism beyond leading order.

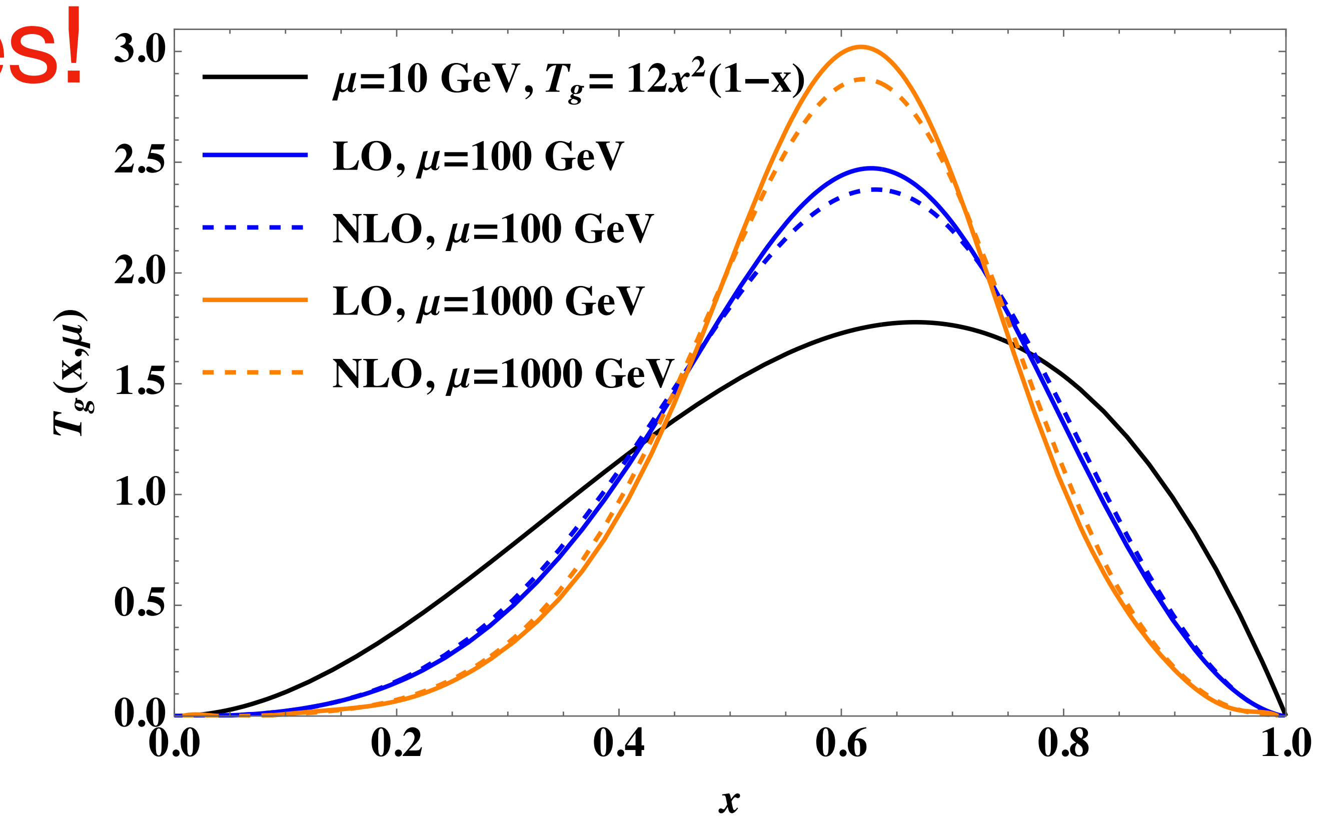
✓ Evolution of track functions in moment space and track energy correlators on tracks at $\mathcal{O}(\alpha_s^2)$.

[arXiv:2108.01674; arXiv:2201.05166]

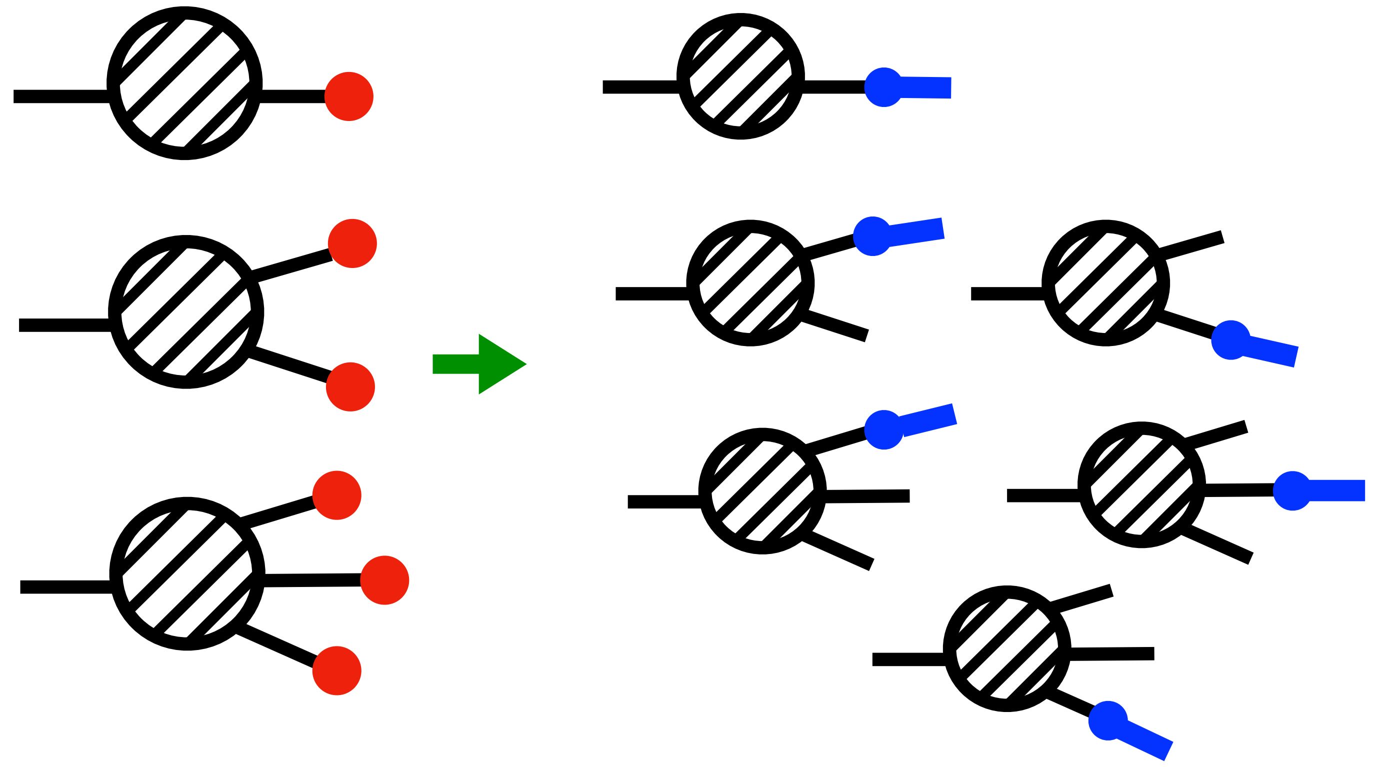


✦ **New** in this talk: [to appear soon]

- Results for the NLO non-linear x-space evolution enabling the use of tracks for generic substructure observables!

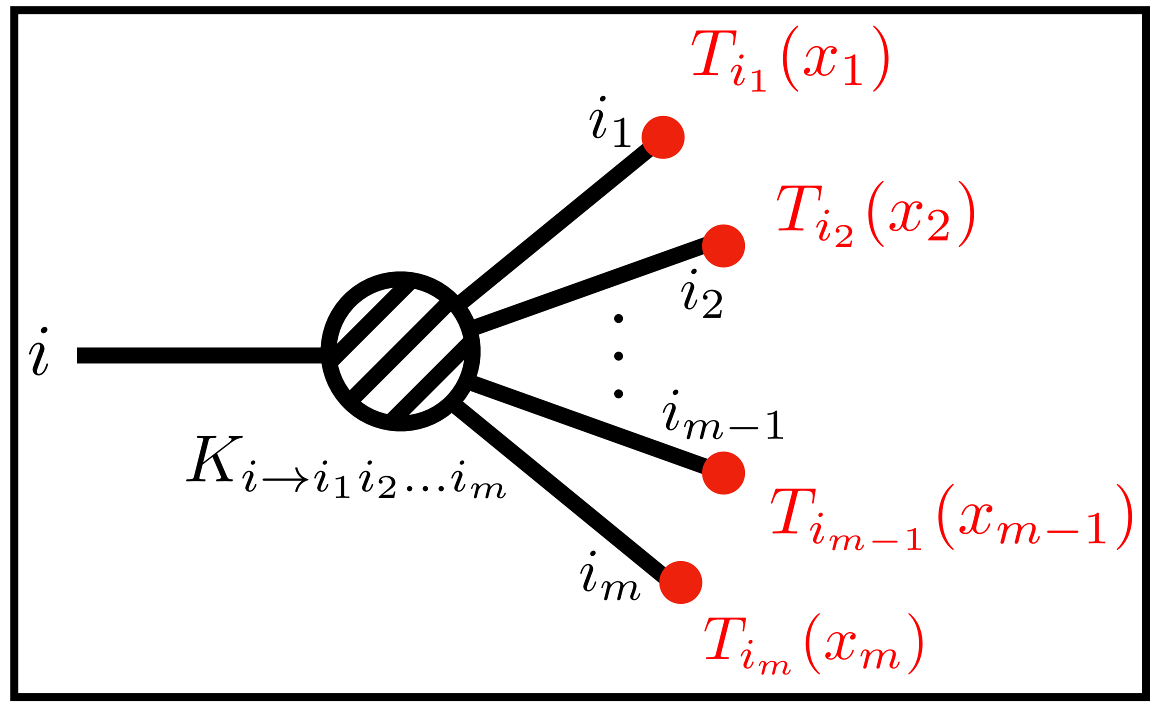


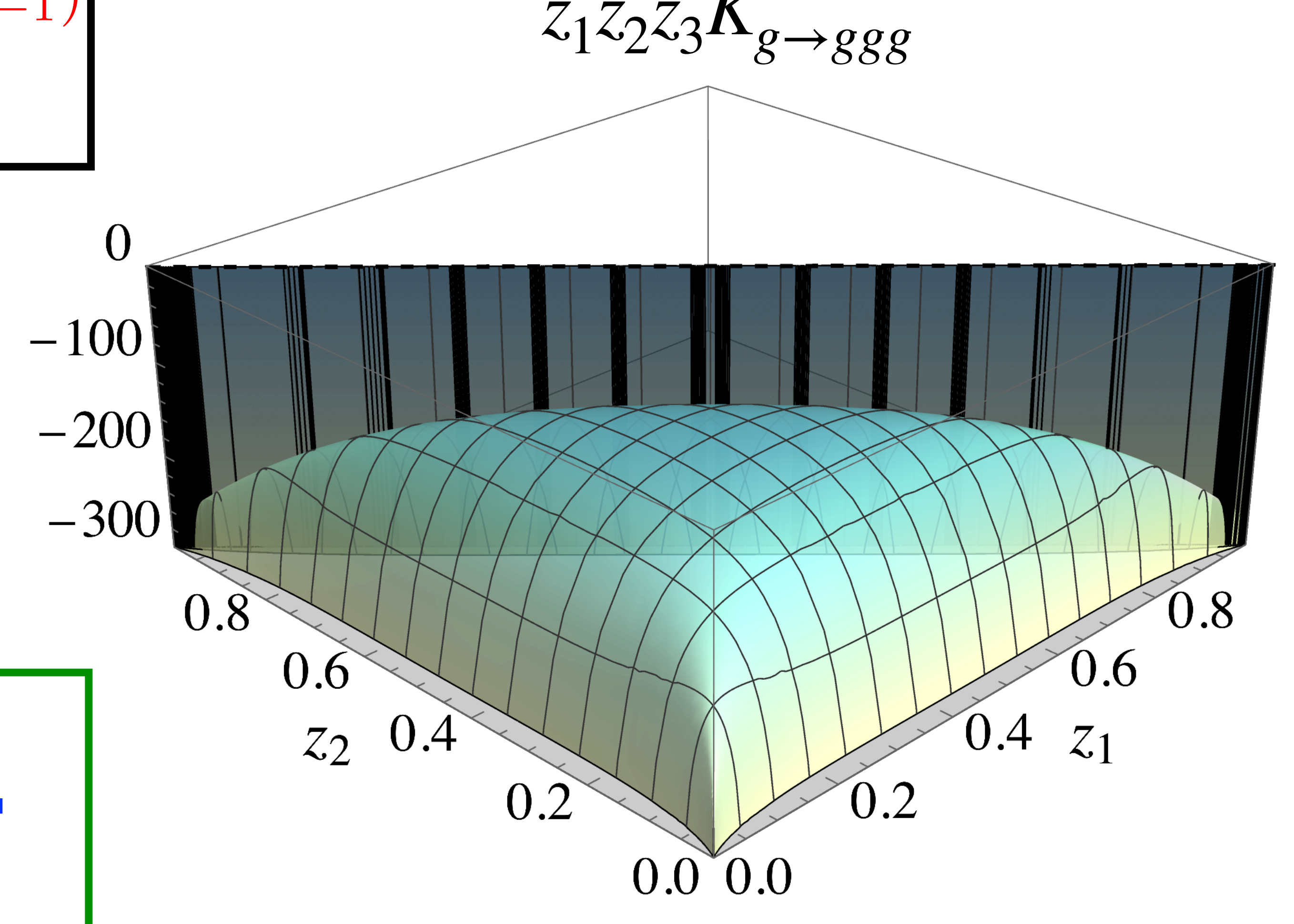
- Correspondence between the evolution of track functions and that of single- or multi-hadron fragmentation functions.

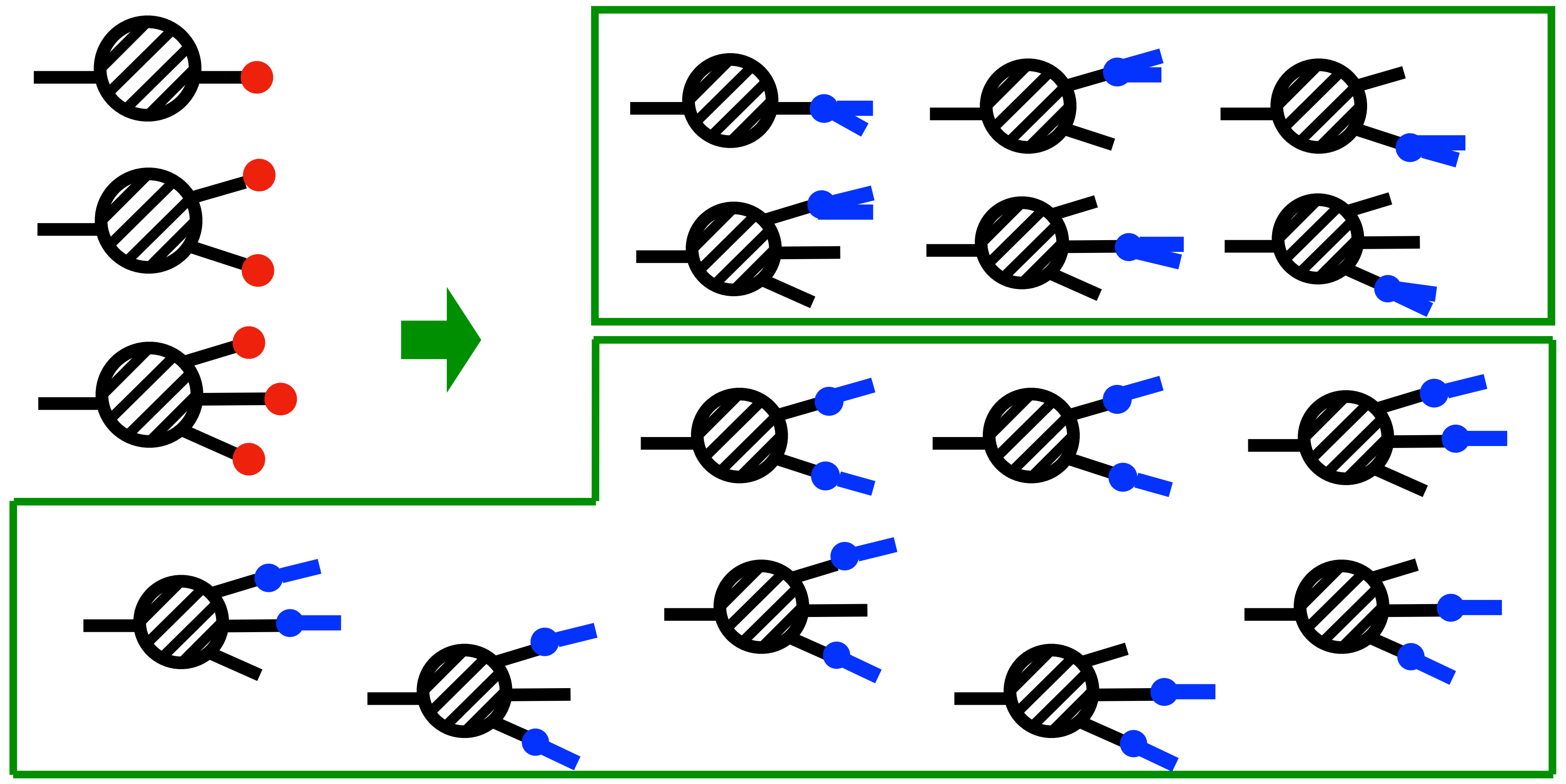


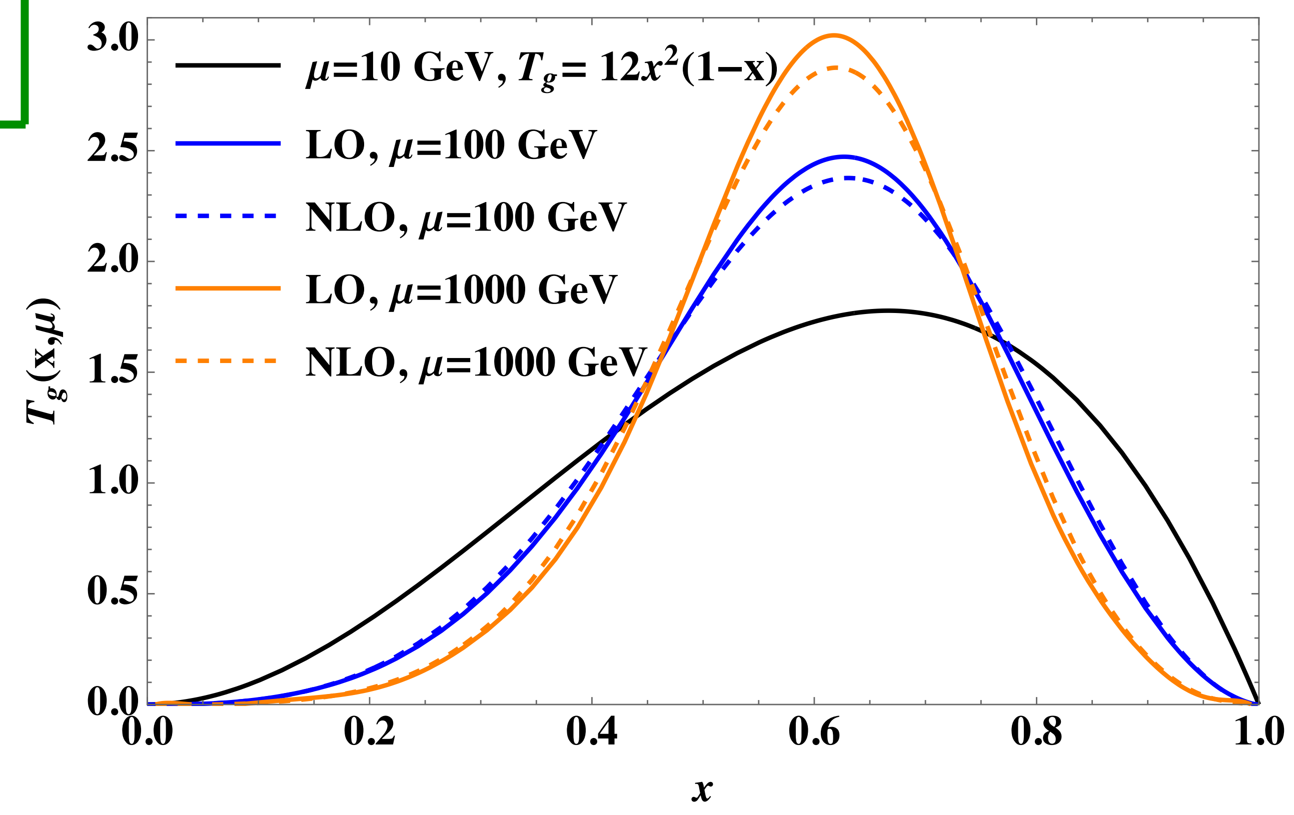
Outline

- Track Functions & Their Evolution
- Calculational Techniques & Results of the Nonlinear x-Space Evolution
- Reduction to Multi-hadron Fragmentation
- Numerical Implementation
- Summary

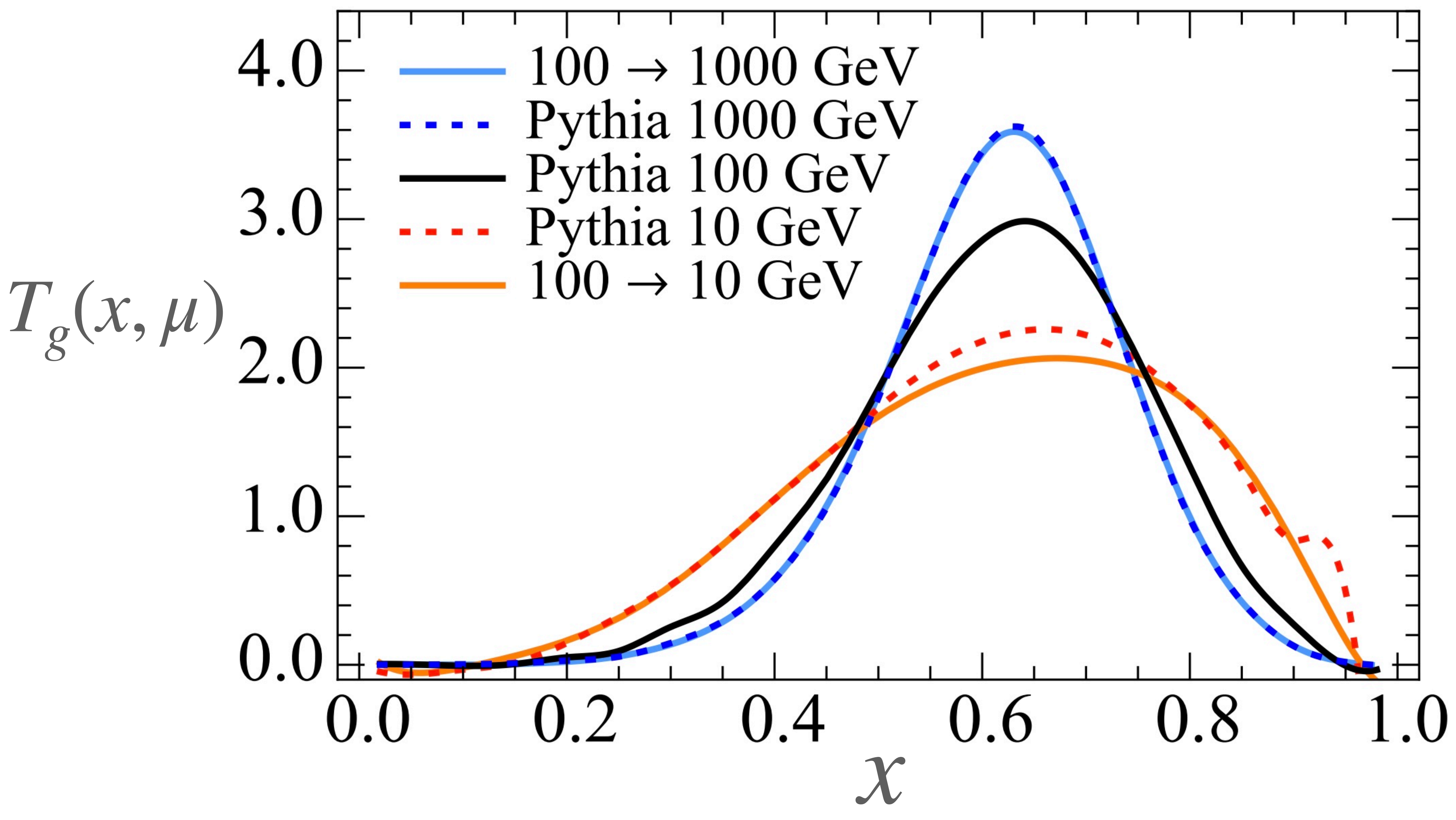


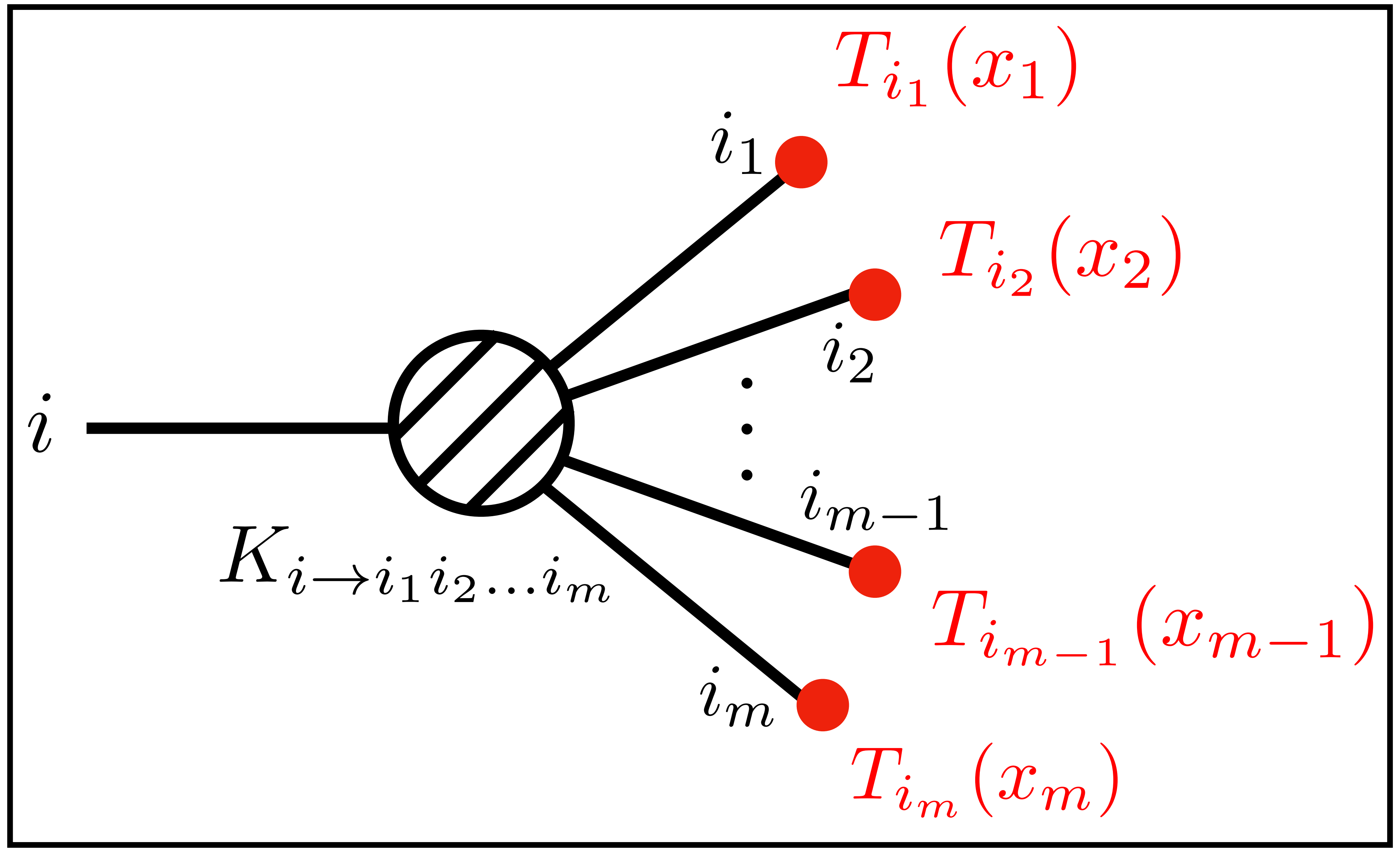






Track Functions and Their Evolution



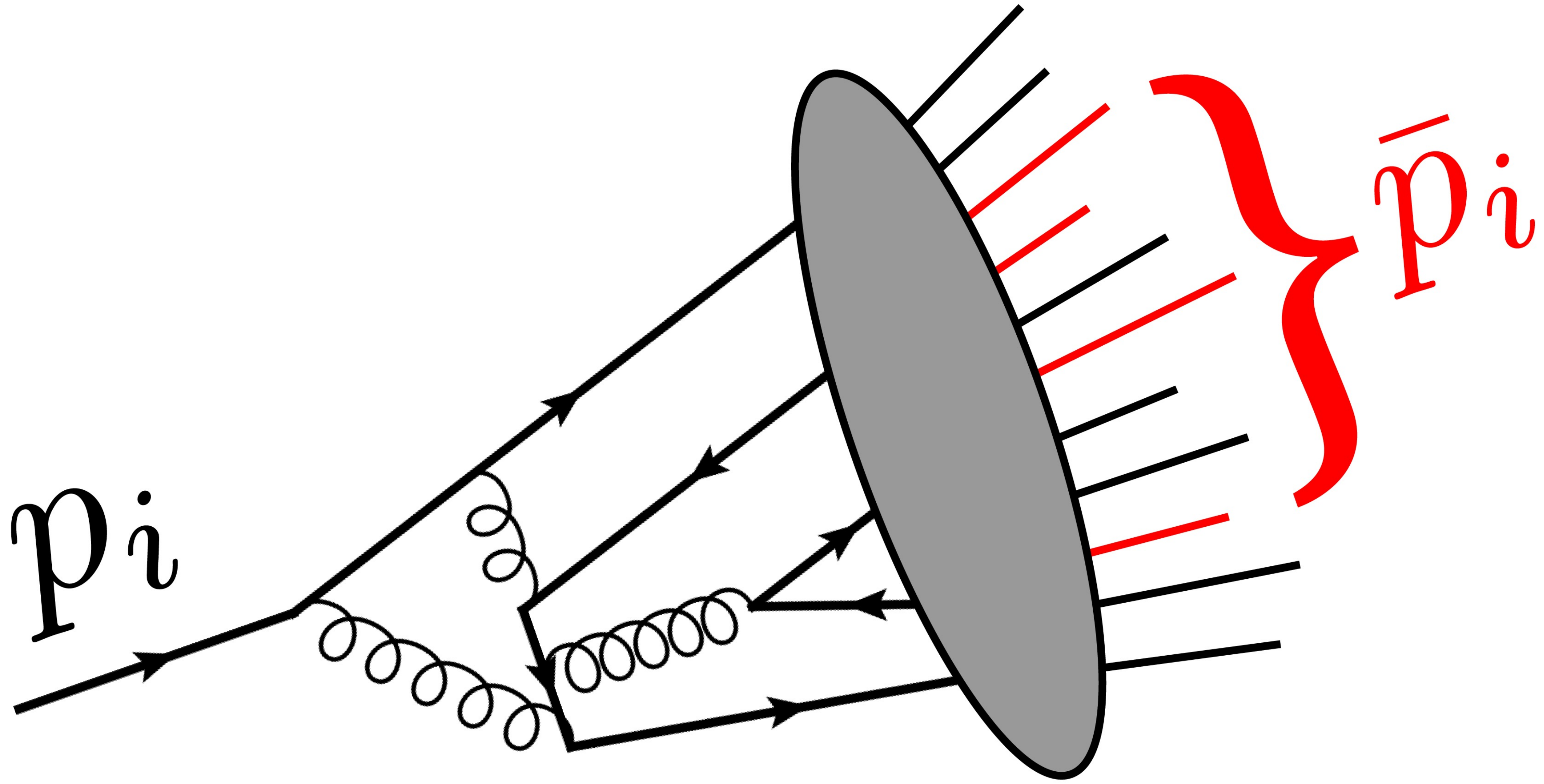


Track Functions $T_i(x, \mu)$ [H. Chang, M. Procura, J. Thaler, W. Waalewijn, 1303.6637, 1306.6630]

Definition

- The track function $T_i(x, \mu)$ describes the total momentum fraction x of *all* charged particles (tracks) in a jet initiated by a hard parton i.

$$\bar{p}_i^\mu = x p_i^\mu + O(\Lambda_{\mathrm{QCD}}) \,, (0 \leq x \leq 1) \,.$$

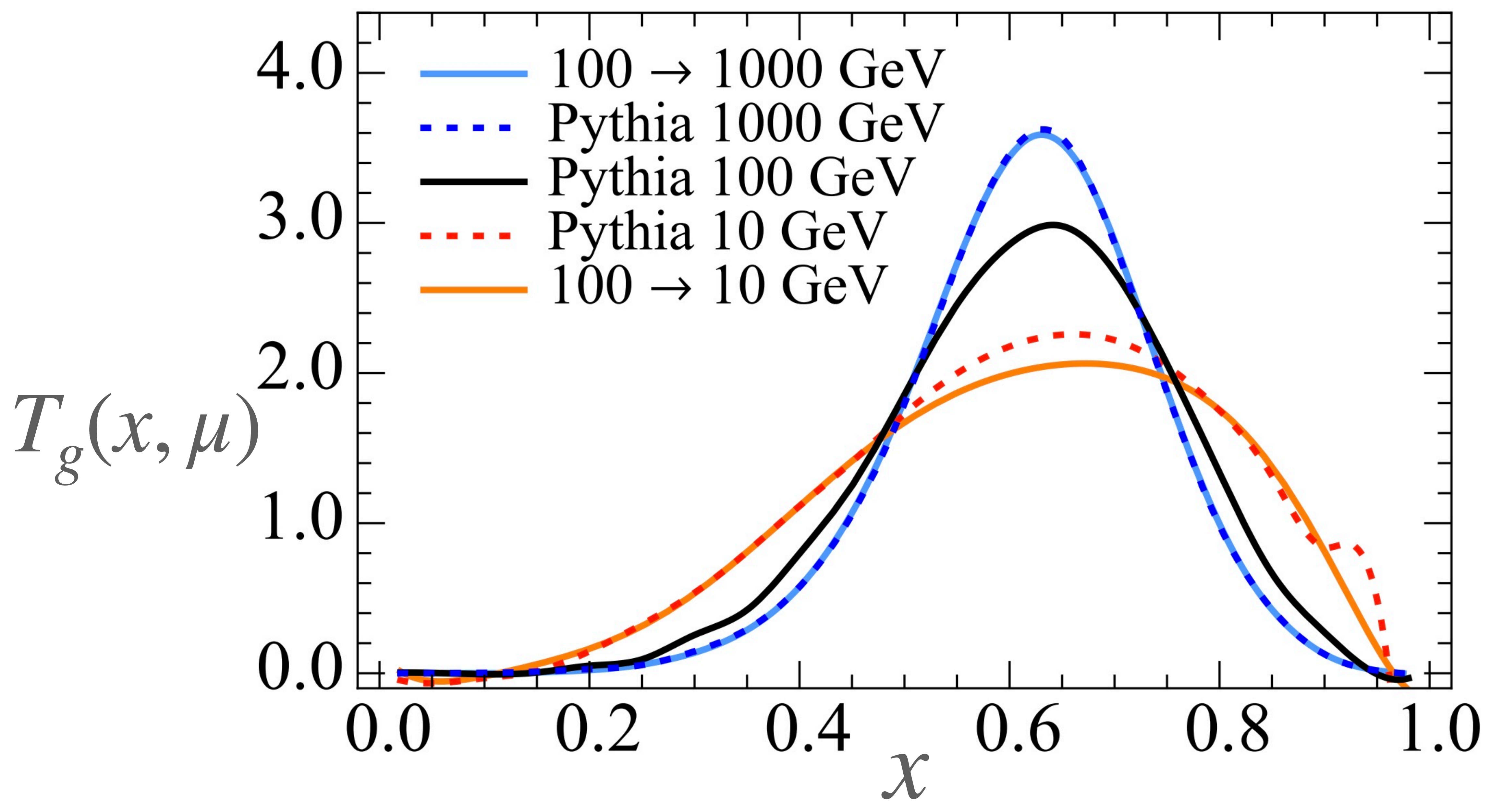


- This formalism applies to other subsets of particles (positively-charged, strange, etc).

Track Functions

Features [H. Chang, M. Procura, J. Thaler, W. Waalewijn, 1303.6637, 1306.6630]

- A generalization of fragmentation functions (FFs).
  - Independent of hard process.
  - Fundamentally non-perturbative, with a calculable scale (μ) dependence.
  - Incorporating correlations between final-state hadrons, like multi-hadron FFs.
  - Sum rule: $\int_0^1 dx\ T_i(x,\mu) = 1\ .$



- The single-hadron fragmentation function:
  - The probability of a parton to produce a single-hadron state considered.
  - The momentum sum rule:

$$\sum_h \int_0^1 dz\ z\ D_{i\to h}(z,\mu) = 1\ .$$

Incorporating Tracks

[1303.6637]

- For a δ-function type observable e measured using partons:

$$\frac{d\sigma}{de} = \sum_N \int d\Pi_N \frac{d\sigma_N}{d\Pi_N} \delta\left[e - \hat{e}(p_i^\mu)\right]$$

↓ tracks

full functional form of T

$$\frac{d\sigma}{d\bar{e}} = \sum_N \int d\Pi_N \frac{d\bar{\sigma}_N}{d\Pi_N} \int \prod_{i=1}^{N} dx_i T_i(x_i) \delta\left[\bar{e} - \hat{e}(x_i p_i^\mu)\right]$$

- For correlations of energy flow: k-point correlation functions

$$\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\cdots\mathcal{E}(\vec{n}_k)\rangle$$

- An energy flow operator that measures energy flow on a restricted set R of final states: $\mathcal{E}_R$ e.g. charged hadrons
- Then, the k-point correlator is

$$\langle \mathcal{E}_R(\vec{n}_1)\mathcal{E}_R(\vec{n}_2)\cdots\mathcal{E}_R(\vec{n}_k)\rangle$$

  - This can be related to the partonic-level correlation functions by a factorization formula:

$$\langle \mathcal{E}_R(\vec{n}_1)\mathcal{E}_R(\vec{n}_2)\cdots\mathcal{E}_R(\vec{n}_k)\rangle$$
$$= \sum_{i_1, i_2, \cdots i_k} T_{i_1}(1)\cdots T_{i_k}(1) \langle \mathcal{E}_{i_1}(\vec{n}_1)\mathcal{E}_{i_2}(\vec{n}_2)\cdots\mathcal{E}_{i_k}(\vec{n}_k)\rangle$$
$$+ \text{contact terms}$$

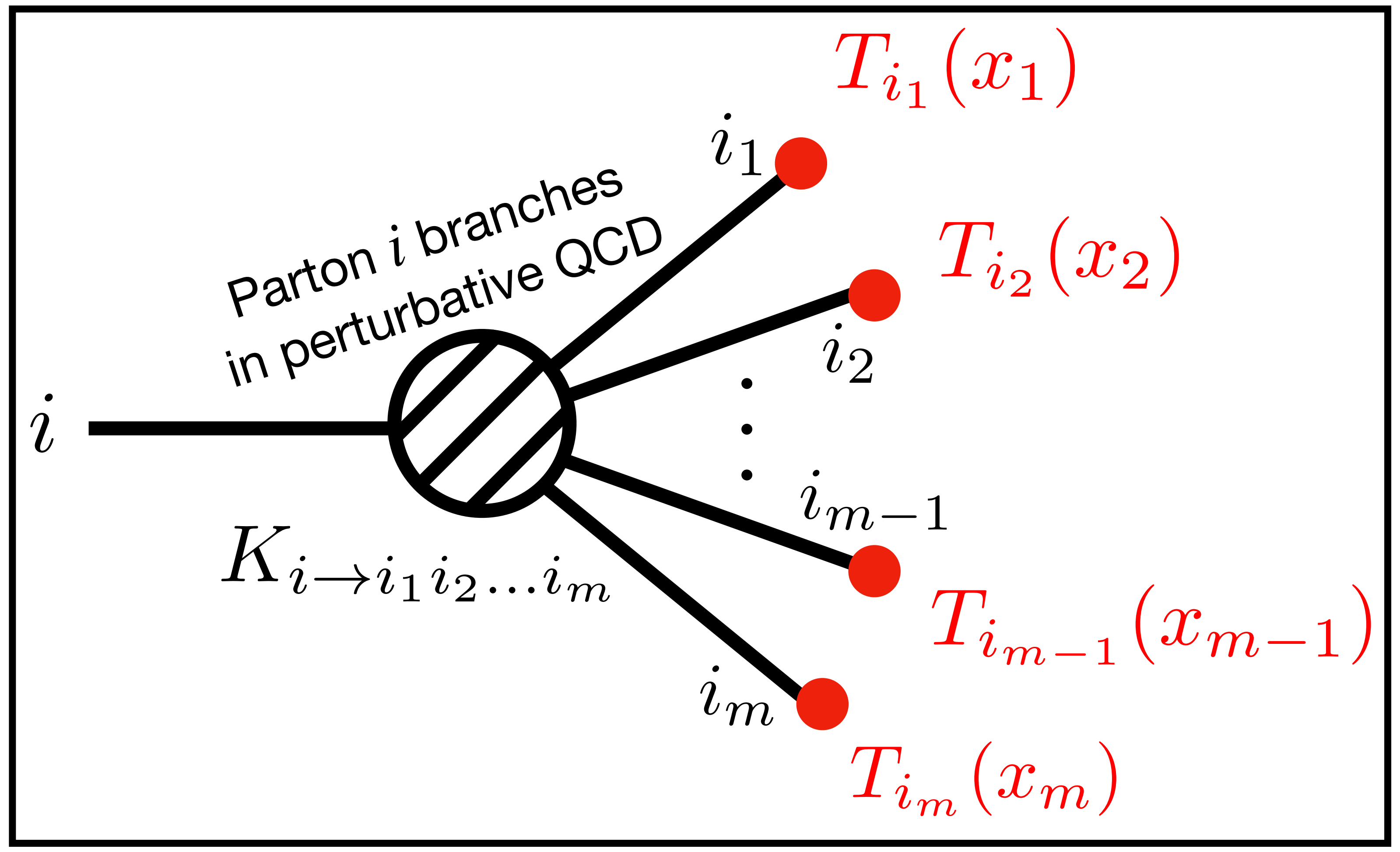
the first moments of T

with dependence on higher moments of T

Track Function Evolution

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x) = \sum_M \sum_{\{i_f\}} \left[\prod_{m=1}^{M}\int_0^1 \mathrm{d}z_m\right]\delta\Big(1-\sum_{m=1}^{M} z_m\Big) K_{i\to\{i_f\}}(\{z_f\})$$

$$\times \left[\prod_{m=1}^{M}\int_0^1 \mathrm{d}x_m\, T_{i_m}(x_m)\right]\delta\Big(x-\sum_{m=1}^{M} z_m x_m\Big)$$



$(i, i_f = g, u, \bar{u}, d, \cdots)$

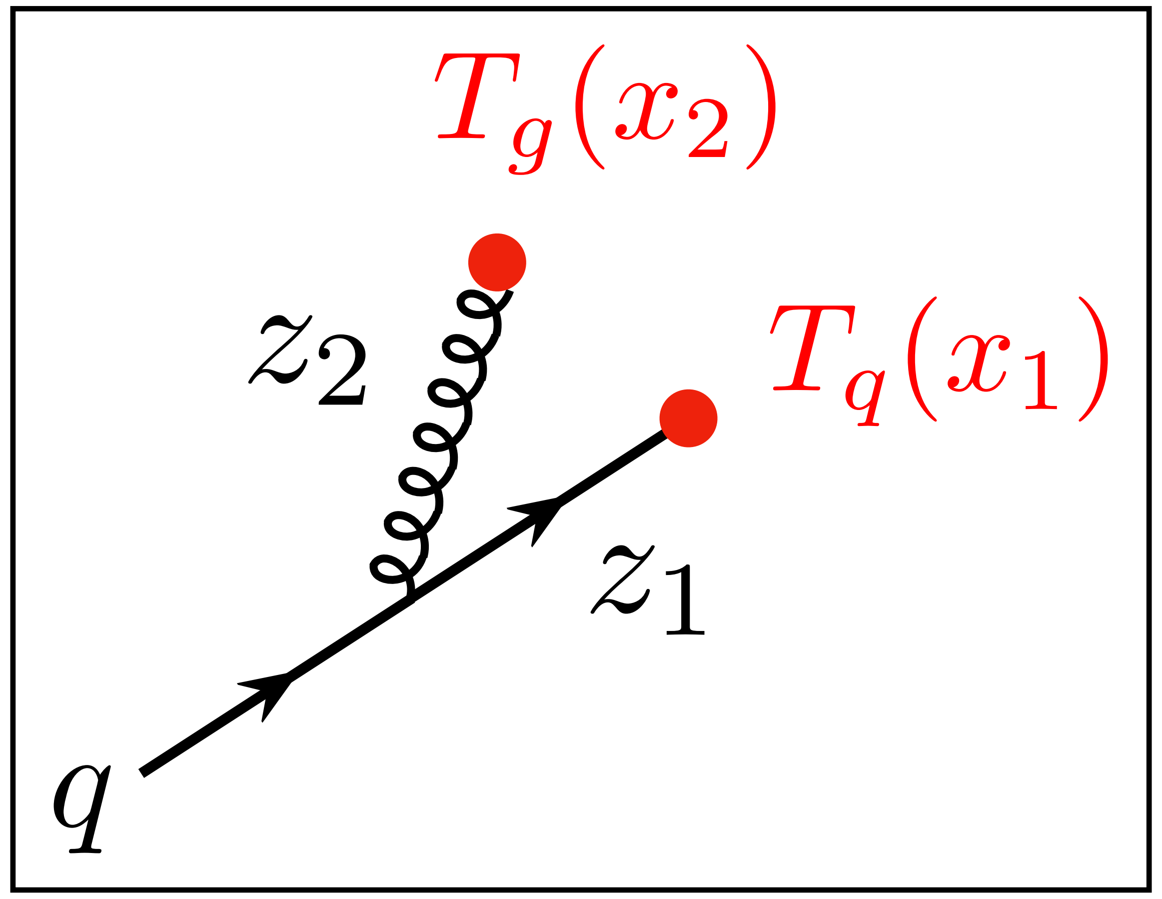
- **Nonlinear**, involving contributions from all branches of splittings.
- E.g., LO evolution:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x,\mu) = a_s(\mu)\sum_{\{jk\}}\int \mathrm{d}z_1\mathrm{d}z_2\, K^{(0)}_{i\to jk}(z_1,z_2)\delta(1-z_1-z_2)$$

$$\times \int \mathrm{d}x_1\mathrm{d}x_2 T_j(x_1,\mu)T_k(x_2,\mu)\delta[x-z_1x_1-z_2x_2]\ .$$

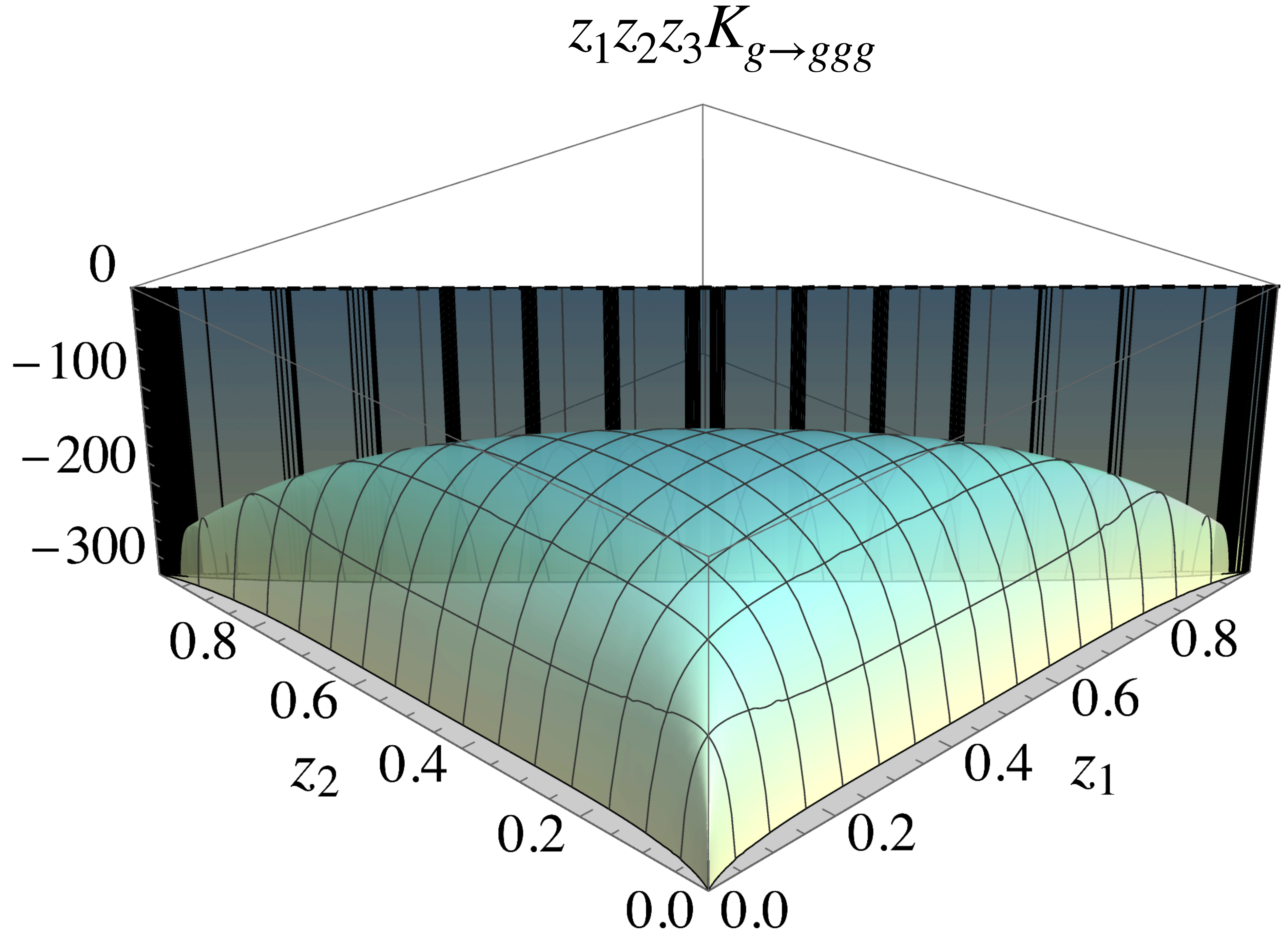
- For single-hadron FFs: Only one branch observed → **Linearity**

$$\frac{d}{d\ln\mu^2}D_{i\to h}(x) = \sum_j D_{j\to h}\otimes P^T_{ji}(x)$$



Involving contributions from both the branches of the splitting.

Calculational Techniques & Results



Track Jet Functions

In DR: $T_i^{(0)} = T_i^{\text{bare}}$

LO track jet function: $J_i^{(0)} = \delta(s) T_i^{(0)}$

- We use the jet function to extract the track function evolution.
- The definition for the jet function on tracks is

$$J_{\mathrm{tr},i}^{\mathrm{bare}}(s,x) = \sum_N \sum_{\{i_f\}} \int \mathrm{d}\Phi_N^c \delta(s-s')\sigma^c_{i\to\{i_f\}}(\{i_f\},\{s_{ff'}\},s') \int \left[\prod_{m=1}^{N} \mathrm{d}x_m T_{i_m}^{(0)}(x_m)\right] \delta\left(x - \sum_{m=1}^{N} x_m z_m\right)$$

- After integration over angular variables,

$$J_{\mathrm{tr},i}^{\mathrm{bare}}(s,x) \supset \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \int_0^1 \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \delta(1 - z_1 - z_2 - z_3) P_{i\to i_1 i_2 i_3}(z_1, z_2, z_3)$$

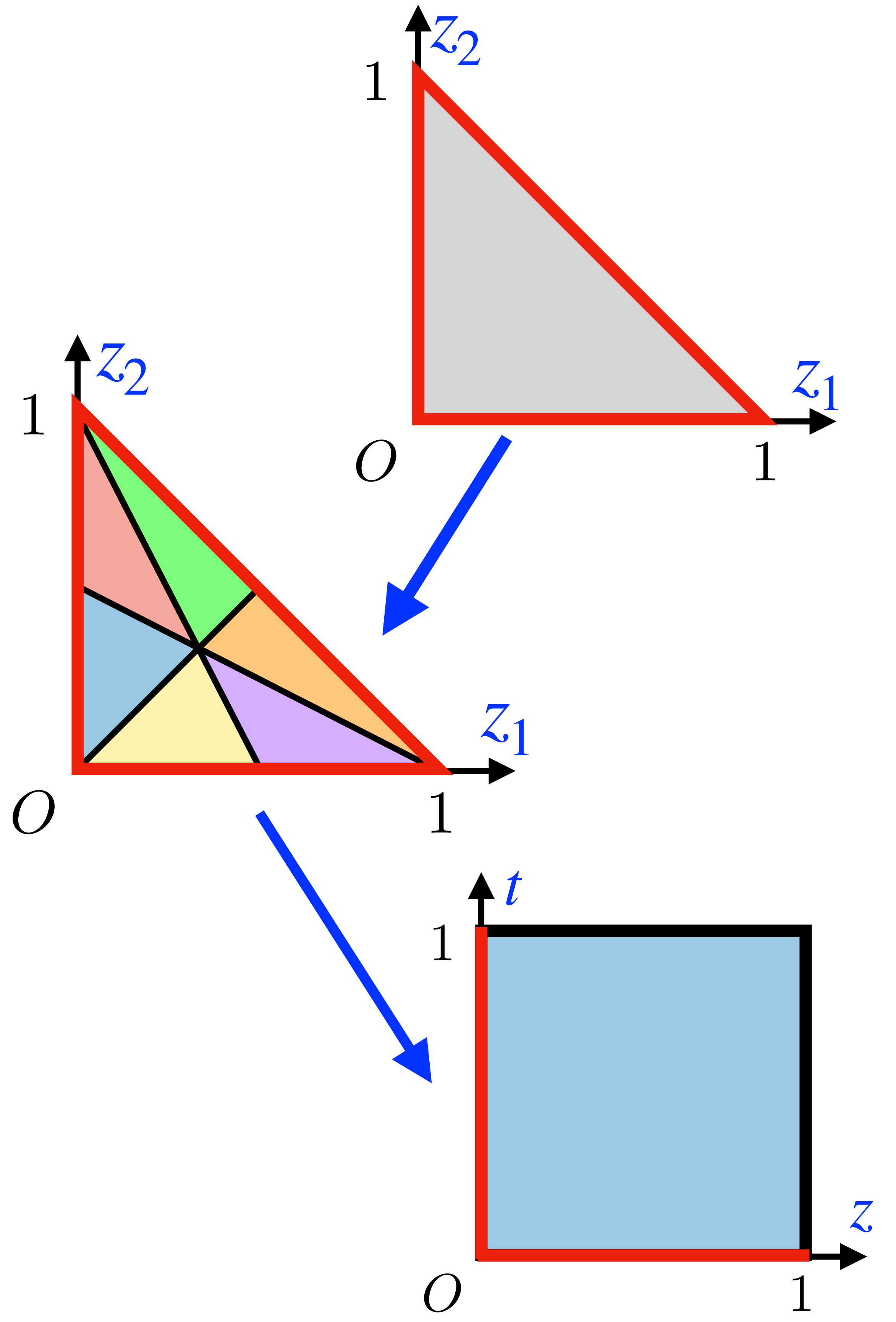
$$\times T_{i_1}^{(0)}(x_1) T_{i_2}^{(0)}(x_2) T_{i_3}^{(0)}(x_3) \delta(x - z_1 x_1 - z_2 x_2 - z_3 x_3)$$

have not been expanded in ϵ

- For $z_{i_1} < z_{i_2} < z_{i_3}$ $(i_1, i_2, i_3 = 1,2,3)$, do the coordinate transformation

[Sector decomposition (Heinrich, arXiv:0803.4177)]

$$t = \frac{z_{i_1}}{z_{i_2}}, z = \frac{z_{i_2}}{z_{i_3}} \quad \text{, i.e.,} \quad z_{i_1} \to \frac{zt}{1+z+zt}, z_{i_2} \to \frac{z}{1+z+zt}, z_{i_3} \to \frac{1}{1+z+zt}$$



Results in $\mathcal{N} = 4$ SYM

a: t' Hooft coupling constant

$$\frac{d}{d\ln\mu^2}T(x) = a^2 \Bigg\{ K^{(1)}_{1\to 1}\, T(x) + \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}z\; K^{(1)}_{1\to 2}(z)\; T(x_1)T(x_2)\; \delta\left(x - x_1\frac{1}{1+z} - x_2\frac{z}{1+z}\right)$$

$$+ \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}x_3 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}t\; K^{(1)}_{1\to 3}(z,t)\; T(x_1)T(x_2)T(x_3)$$

$$\times\, \delta\left(x - x_1\frac{1}{1+z+zt} - x_2\frac{z}{1+z+zt} - x_3\frac{zt}{1+z+zt}\right)\Bigg\}$$

where

$$K^{(1)}_{1\to 1} = -25\zeta_3$$

$$K^{(1)}_{1\to 2}(z) = \frac{8}{3}\pi^2\left[\frac{1}{z}\right]_+ + \frac{32\ln^2(z+1)}{z} - \frac{16\ln(z)\ln(z+1)}{z}$$

$$K^{(1)}_{1\to 3}(z,t) = 8\Bigg\{\frac{4\ln(1+z)}{z}\left[\frac{1}{t}\right]_+ + \left[\frac{1}{z}\right]_+\left(4\left[\frac{\ln t}{t}\right]_+ - \frac{\ln t}{1+t} - \frac{7\ln(1+t)}{t}\right)$$

$$+\frac{2\left[\ln(1+tz) - \ln(1+z+tz)\right]}{(1+t)(1+z)(1+tz)} + \frac{10\left[\ln(1+z+tz) - \ln(1+z)\right]}{tz} + \frac{\ln(1+tz)}{(1+t)z(1+z)}$$

$$-\frac{7\ln(1+tz)}{tz} + \frac{\ln(1+t) - \ln t}{(1+t)(1+tz)} + \frac{\ln(1+z) + \ln(1+t)}{(1+t)(1+z)} - \frac{\ln(1+z)}{(1+t)z} - \frac{z\ln(1+z)}{(1+z)(1+tz)}\Bigg\}$$

Results in QCD

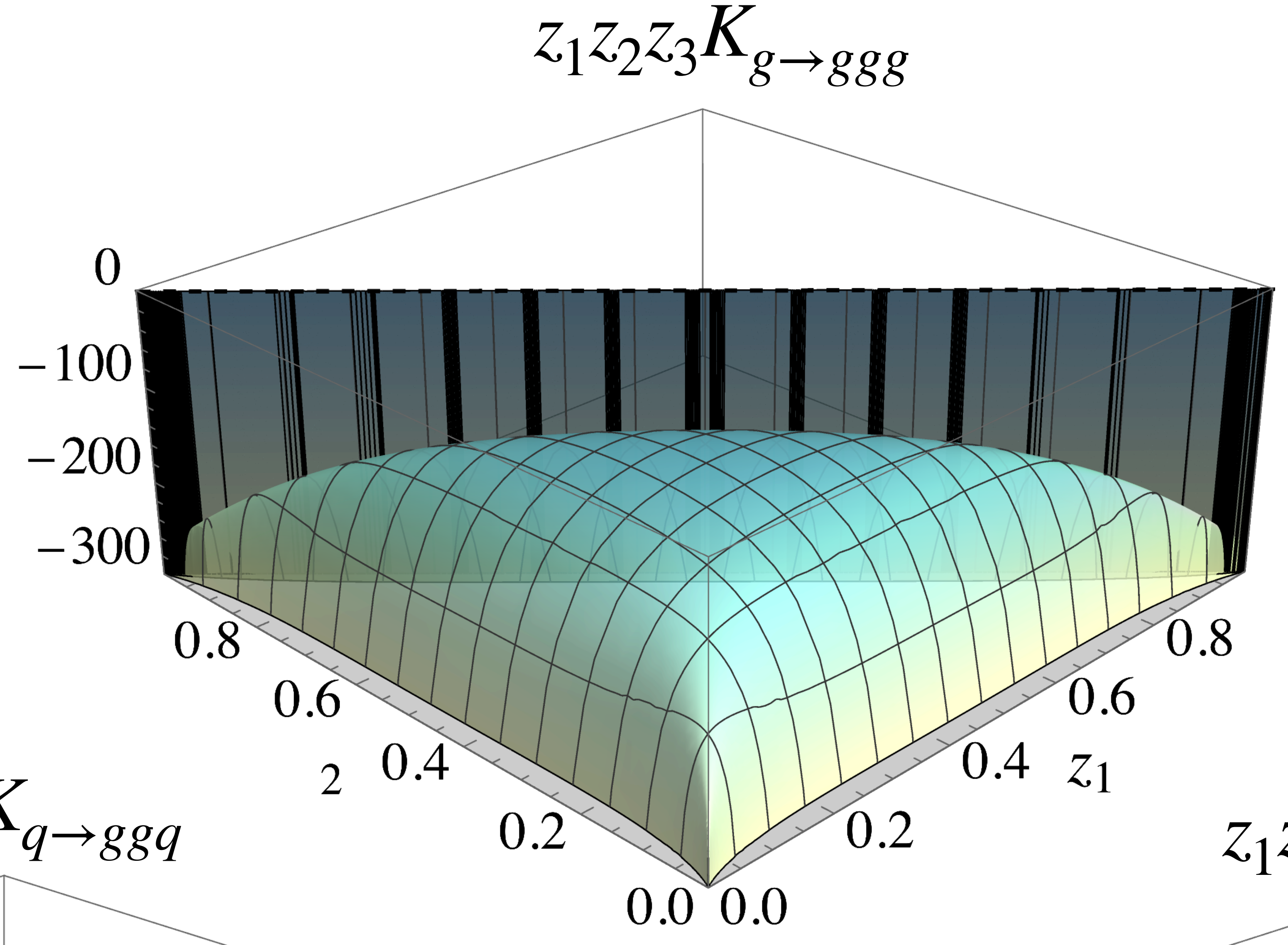
E.g. Gluon case:

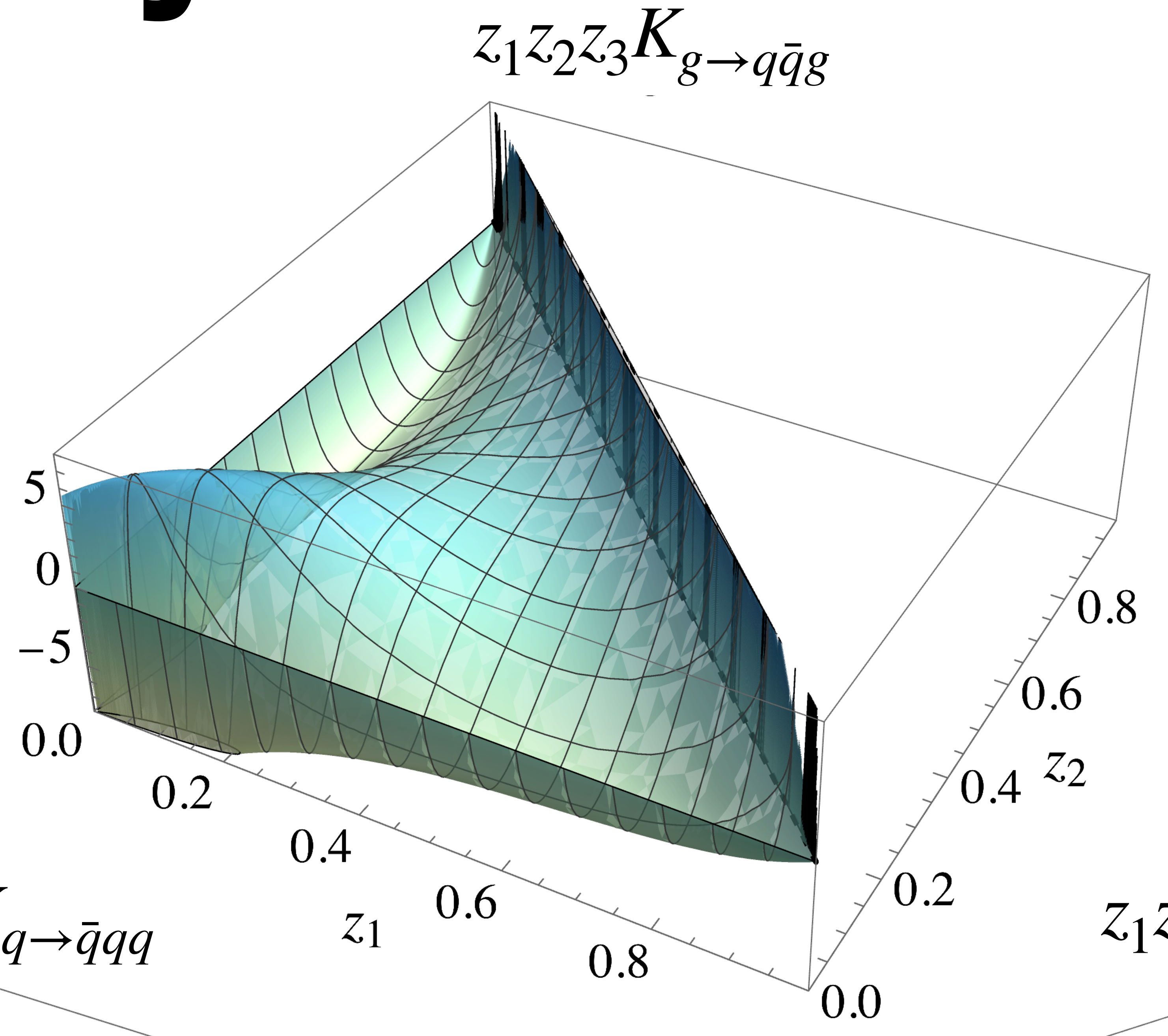
For brevity, $a_s^2 = [\alpha_s(\mu)/(4\pi)]^2$ is suppressed.

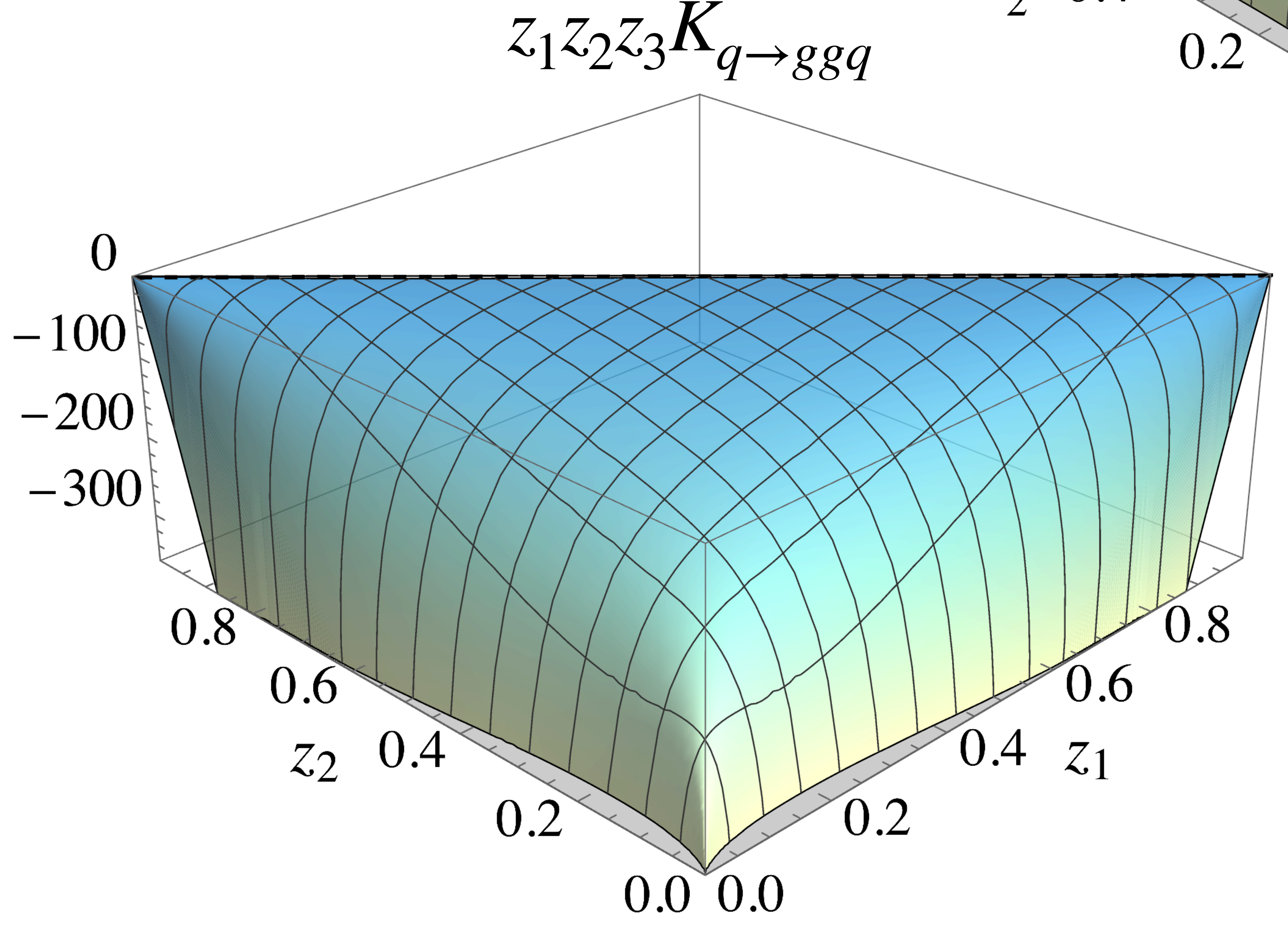
$$
\begin{aligned}
\frac{d}{d\ln\mu^2} T_g(x) = {} & T_g(x)\; K_g^{(1)} \\
& + \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}z \delta\left(x - x_1 \frac{1}{1+z} - x_2 \frac{z}{1+z}\right) \Big[T_g(x_1) T_g(x_2)\; K^{(1)}_{gg,1}(z) \\
& + \sum_q \big(T_q(x_1) T_{\bar{q}}(x_2) + T_q(x_2) T_{\bar{q}}(x_1)\big)\; K^{(1)}_{q\bar{q},1}(z) \Big] \\
& + \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}x_3 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}t\; \delta\left(x - x_1 \frac{1}{1+z+zt} - x_2 \frac{z}{1+z+zt} - x_3 \frac{zt}{1+z+zt}\right) \\
& \times \Big\{ 6\; T_g(x_1) T_g(x_2) T_g(x_3)\; K^{(1)}_{ggg,1}(z,t) \\
& \quad + \sum_q \Big[T_g(x_3) \big(T_q(x_2) T_{\bar{q}}(x_1) + T_q(x_1) T_{\bar{q}}(x_2)\big)\; K^{(1)}_{gq\bar{q},1}(z,t) \\
& \quad + T_g(x_2) \big(T_q(x_3) T_{\bar{q}}(x_1) + T_q(x_1) T_{\bar{q}}(x_3)\big)\; K^{(1)}_{gq\bar{q},2}(z,t) \\
& \quad + T_g(x_1) \big(T_q(x_3) T_{\bar{q}}(x_2) + T_q(x_2) T_{\bar{q}}(x_3)\big)\; K^{(1)}_{gq\bar{q},3}(z,t) \Big] \Big\} .
\end{aligned}
$$

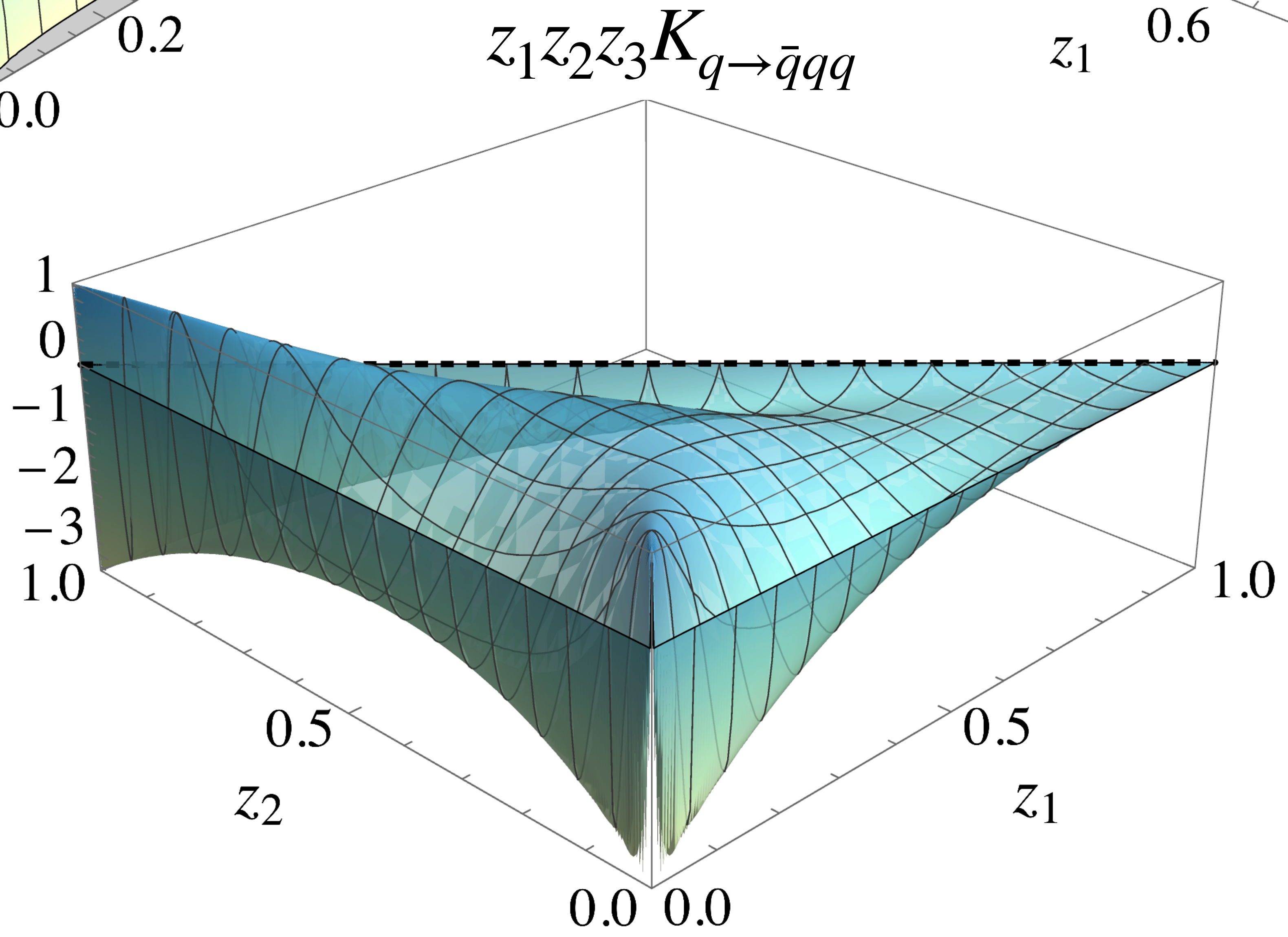
Results in QCD, Pictorially

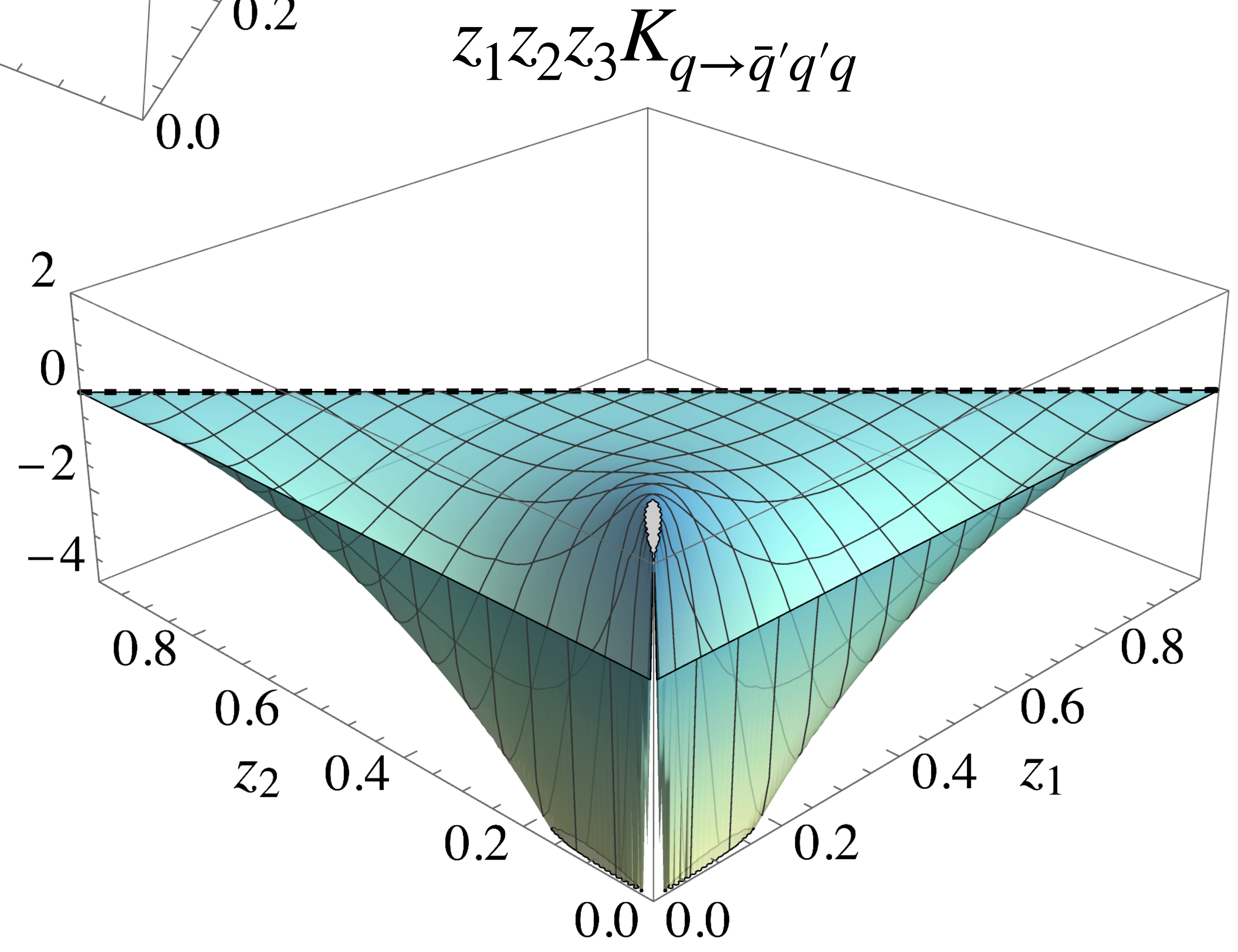
the 1→3 Kernels



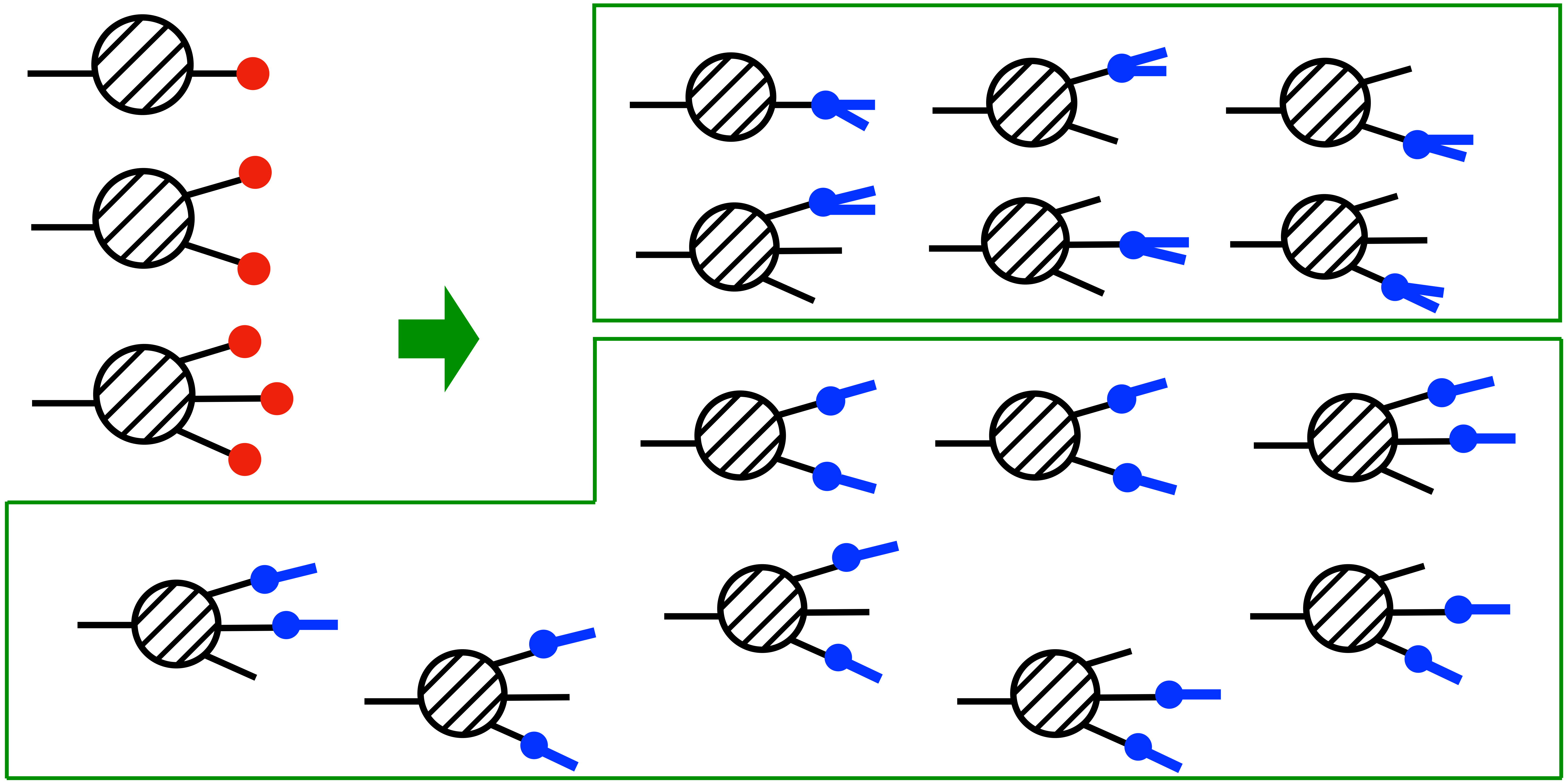








Reduction to Multi-hadron Fragmentation



Fragmentation Functions

Single- and Multi-hadron cases

[U. P. Sukhatme and K. E. Lassila, *Phys.Rev.D* 22 (1980) 1184]
[D. de Florian, L. Vanni: arXiv:0310196]

- **The single-hadron fragmentation function** $D_{i\to h}(y)$ gives the probability of finding in a jet a single hadron h with momentum fraction y of that possessed by the jet-initiating parton i (a quark, antiquark or gluon).

- **The** N**-hadron fragmentation function** $D_{i\to h_1h_2\cdots h_N}(y_1, y_2, \cdots, y_N)$ for fragmentation of parton i into N hadrons which carry fractions $y_1, y_2, \cdots, y_N$ of the momentum carried by the initial parton.

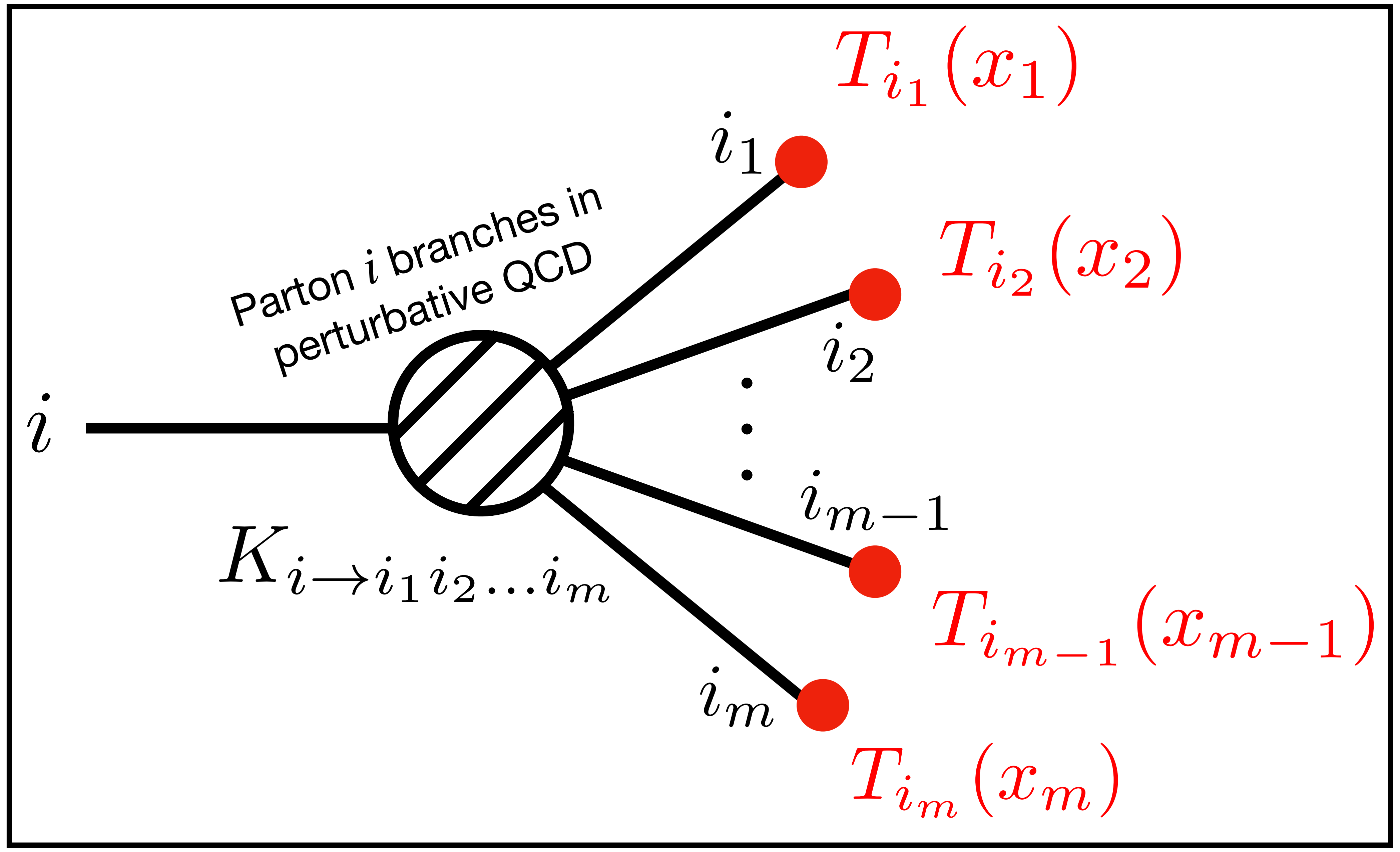
- $N=2$: **Di-hadron fragmentation function** $D_{i\to h_1h_2}(y_1, y_2)$.

Notation

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x) = \sum_M \sum_{\{i_f\}} \left[\prod_{m=1}^{M}\int_0^1 \mathrm{d}z_m\right]\delta\Big(1-\sum_{m=1}^{M} z_m\Big) K_{i\to\{i_f\}}(\{z_f\})$$

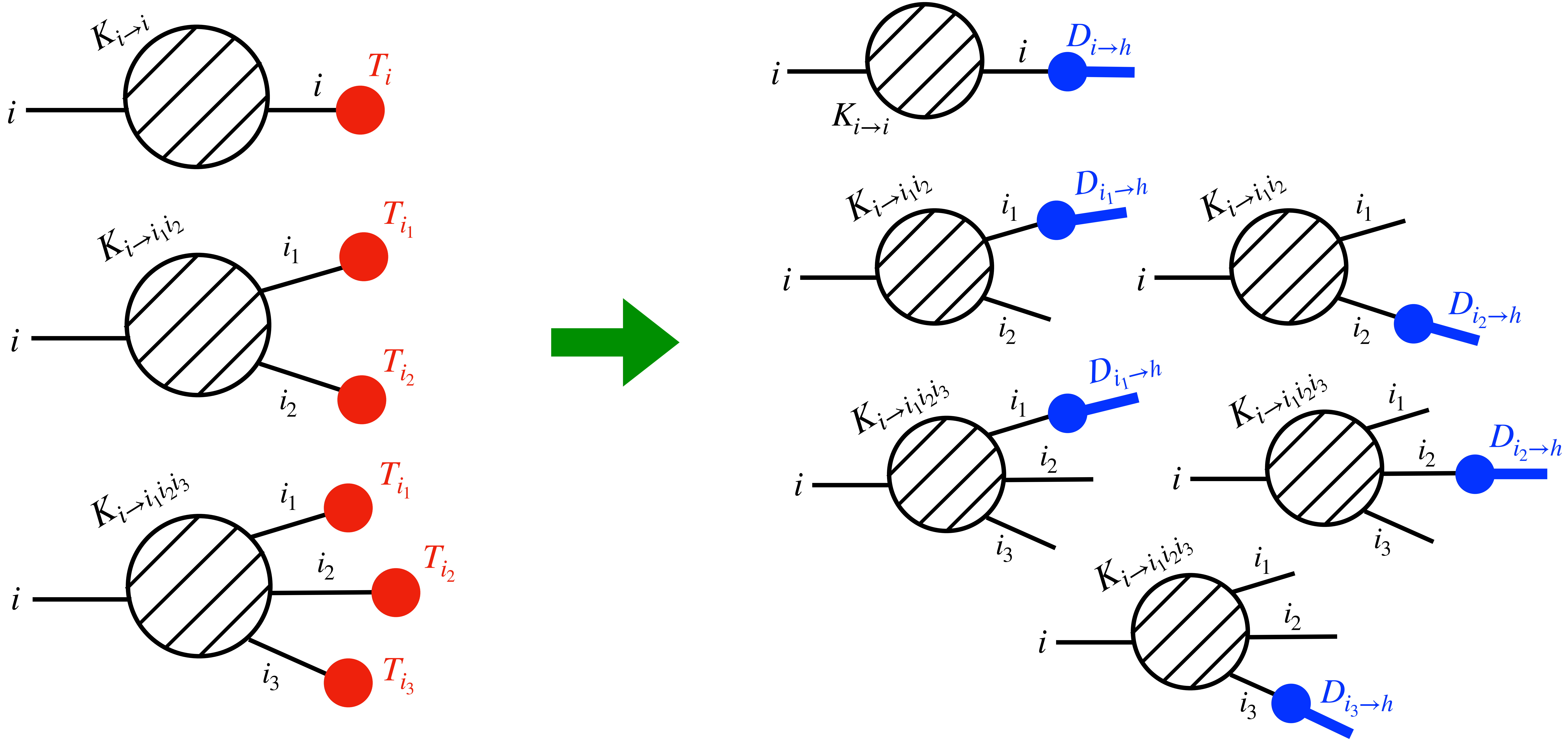
$$\times \left[\prod_{m=1}^{M}\int_0^1 \mathrm{d}x_m\, T_{i_m}(x_m)\right]\delta\Big(x-\sum_{m=1}^{M} z_m x_m\Big)$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x) = \sum_M \sum_{\{i_f\}} K_{i\to i_1 i_2\cdots i_M} \otimes T_{i_1}T_{i_2}\cdots T_{i_M}(x)$$



- For notational simplicity, set $M \leq 3$.

Reduction to DGLAP



Reduction to DGLAP

- At NLO,

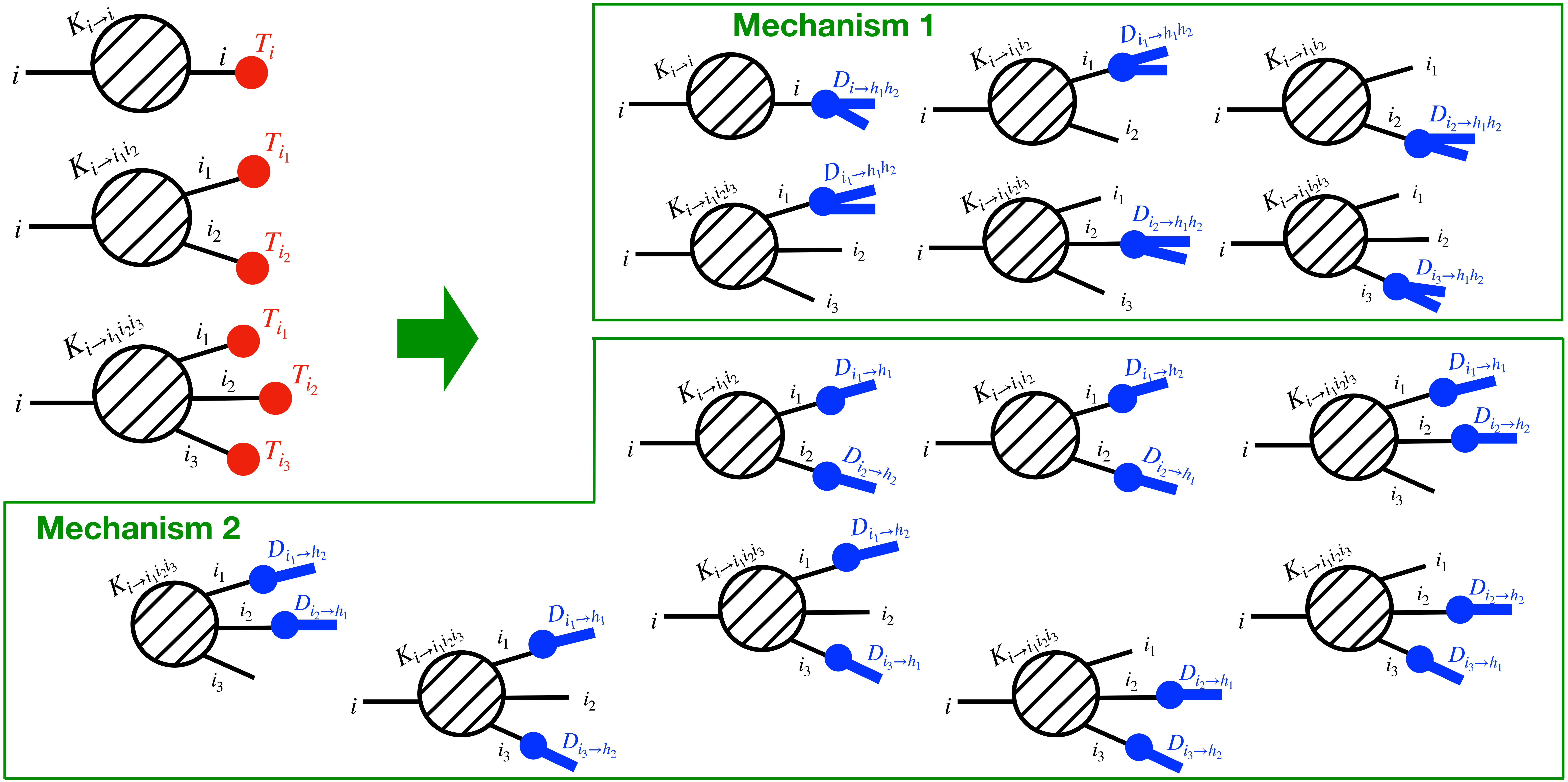
$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x) = K^{(1)}_{i\to i}T_i(x) + \sum_{\{i_f\}} K^{(1)}_{i\to i_1 i_2} \otimes T_{i_1}(x_1)T_{i_2}(x_2) + \sum_{\{i_f\}} K^{(1)}_{i\to i_1 i_2 i_3} \otimes T_{i_1}(x_1)T_{i_2}(x_2)T_{i_3}(x_3)$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}D_{i\to h}(x) = K^{(1)}_{i\to i}D_{i\to h}(x) + \sum_{\{i_f\}} K^{(1)}_{i\to i_1 i_2} \otimes [D_{i_1\to h}(x_1) + D_{i_2\to h}(x_2)] + \sum_{\{i_f\}} K^{(1)}_{i\to i_1 i_2 i_3} \otimes [D_{i_1\to h}(x_1) + D_{i_2\to h}(x_2) + D_{i_3\to h}(x_3)]$$

equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}D_{i\to h}(x) = \sum_j D_{j\to h} \otimes P^T_{ji}(x)$$

Reduction to Di-hadron Fragmentation



Reduction to Di-hadron Fragmentation

- First NLO ($\mathcal{O}(\alpha_s^2)$) equation for the di-hadron fragmentation function evolution:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2} D_{i\to h_1 h_2}(y_1, y_2) = \Bigg\{ K^{(1)}_{i\to i} D_{i\to h_1 h_2}(y_1, y_2) + \sum_{\{i_f\}} K^{(1)}_{i\to i_1 i_2} \otimes [D_{i_1\to h_1 h_2} + D_{i_2\to h_1 h_2}]$$

$$+ \sum_{\{i_f\}} K^{(1)}_{i\to i_1 i_2 i_3} \otimes [D_{i_1\to h_1 h_2} + D_{i_2\to h_1 h_2} + D_{i_3\to h_1 h_2}] \Bigg\}$$

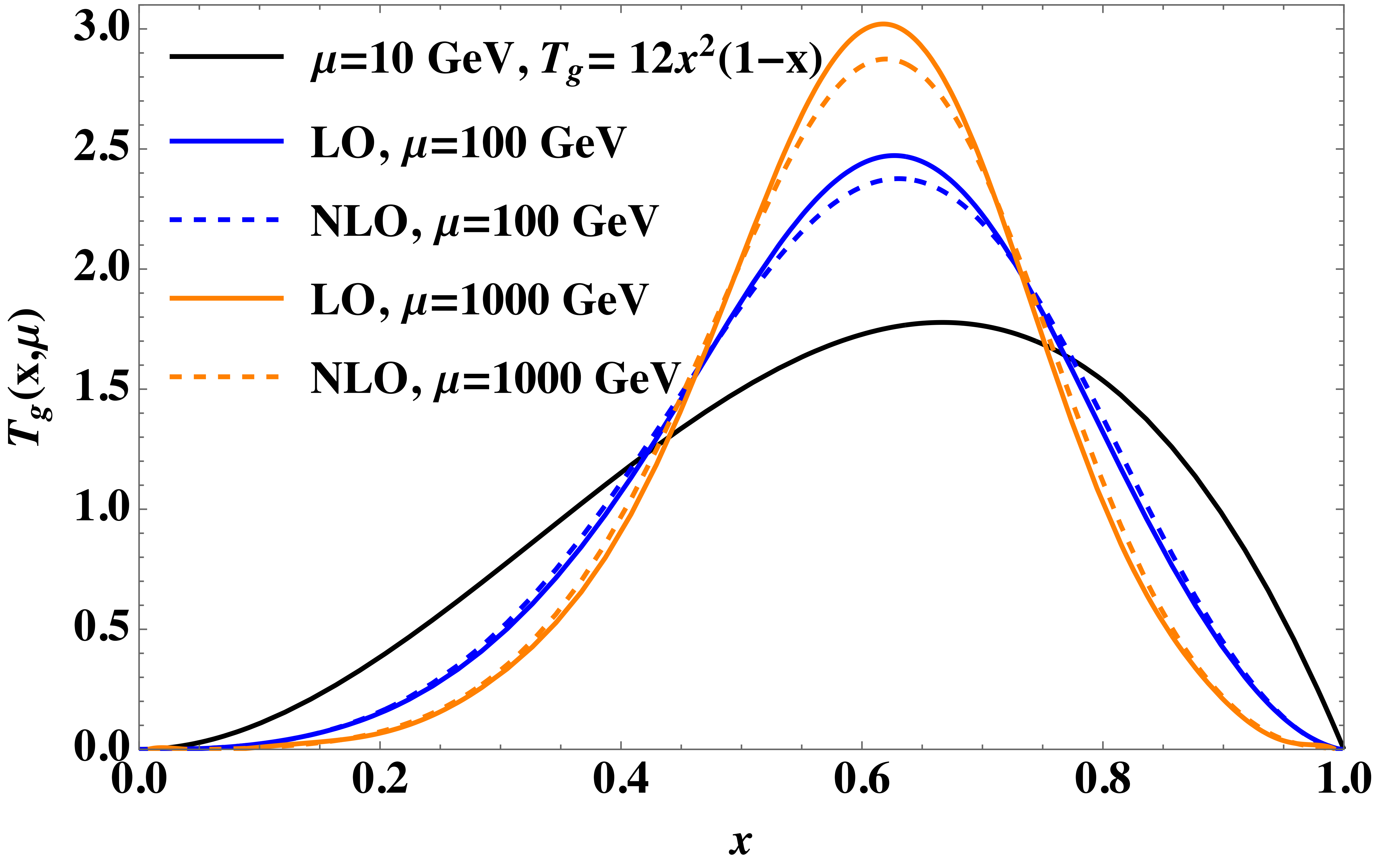
$$+ \Bigg\{ \sum_{\{i_f\}} K^{(1)}_{i\to i_1 i_2} \otimes [D_{i_1\to h_1} D_{i_2\to h_2} + D_{i_1\to h_2} D_{i_2\to h_1}]$$

$$+ \sum_{\{i_f\}} K^{(1)}_{i\to i_1 i_2 i_3} \otimes [D_{i_1\to h_1} D_{i_2\to h_2} + D_{i_1\to h_2} D_{i_2\to h_1}$$

$$+ D_{i_1\to h_1} D_{i_3\to h_2} + D_{i_1\to h_2} D_{i_3\to h_1}$$

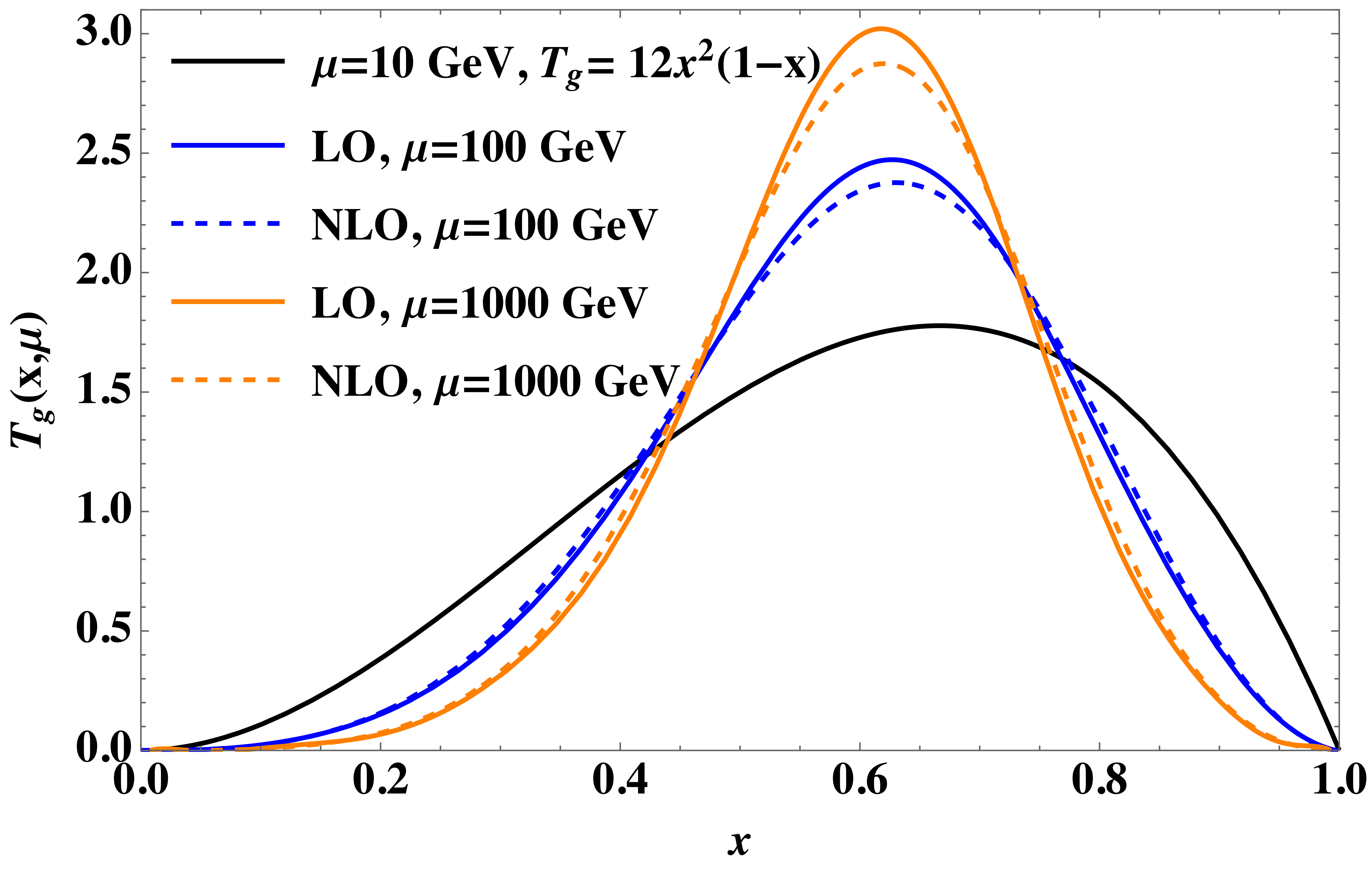
$$+ D_{i_2\to h_1} D_{i_3\to h_2} + D_{i_2\to h_2} D_{i_3\to h_1}] \Bigg\}$$

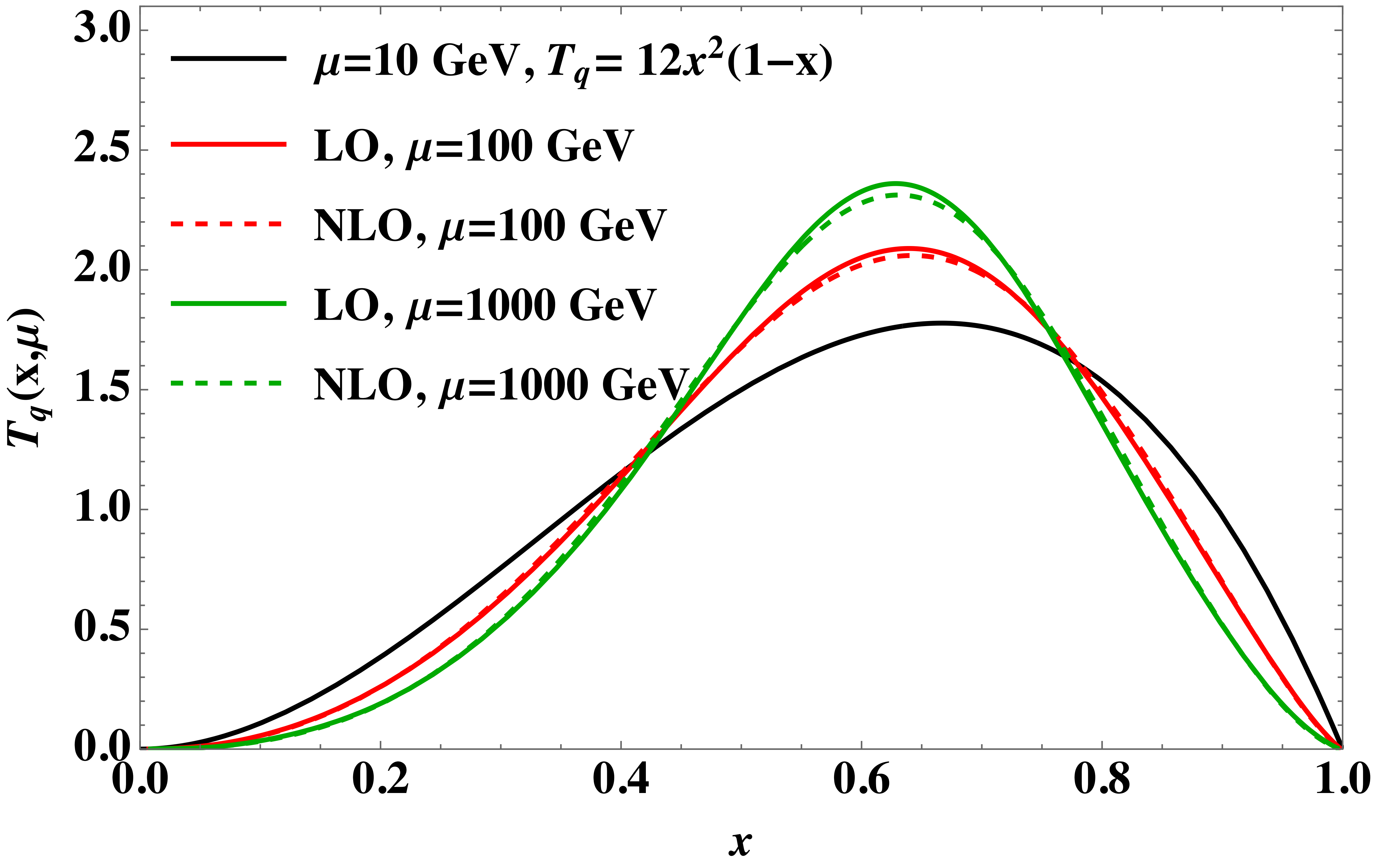
Numerical Implementation



- Full non-linear RG can be solved numerically (see back-up slides for details).

  - A toy model: at $\mu_0 = 10$ GeV, $T_i(x) = 12x^2(1-x)$ $(i = q, g)$ of which the first moment is 0.6 ~ that in real world QCD.



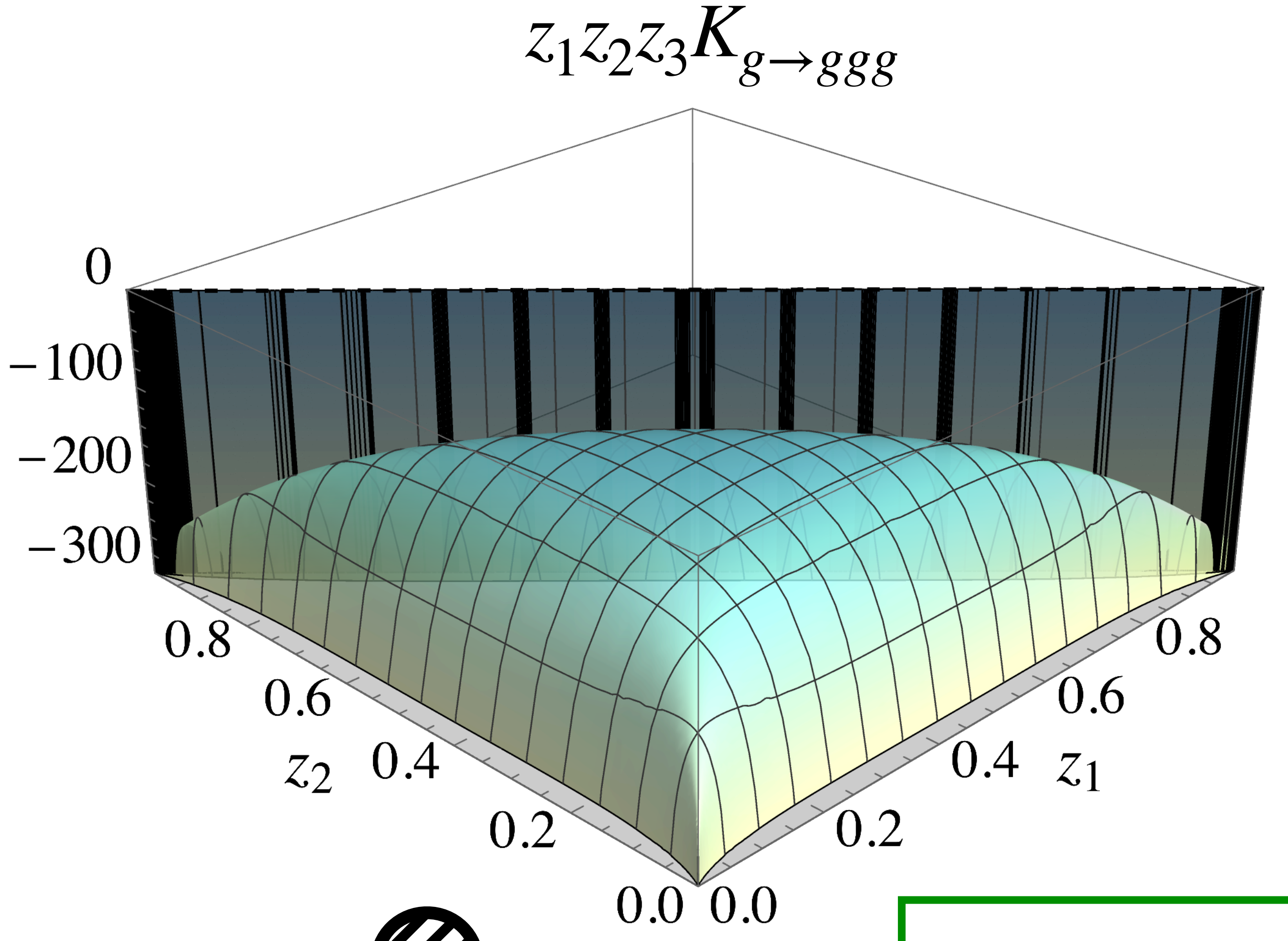


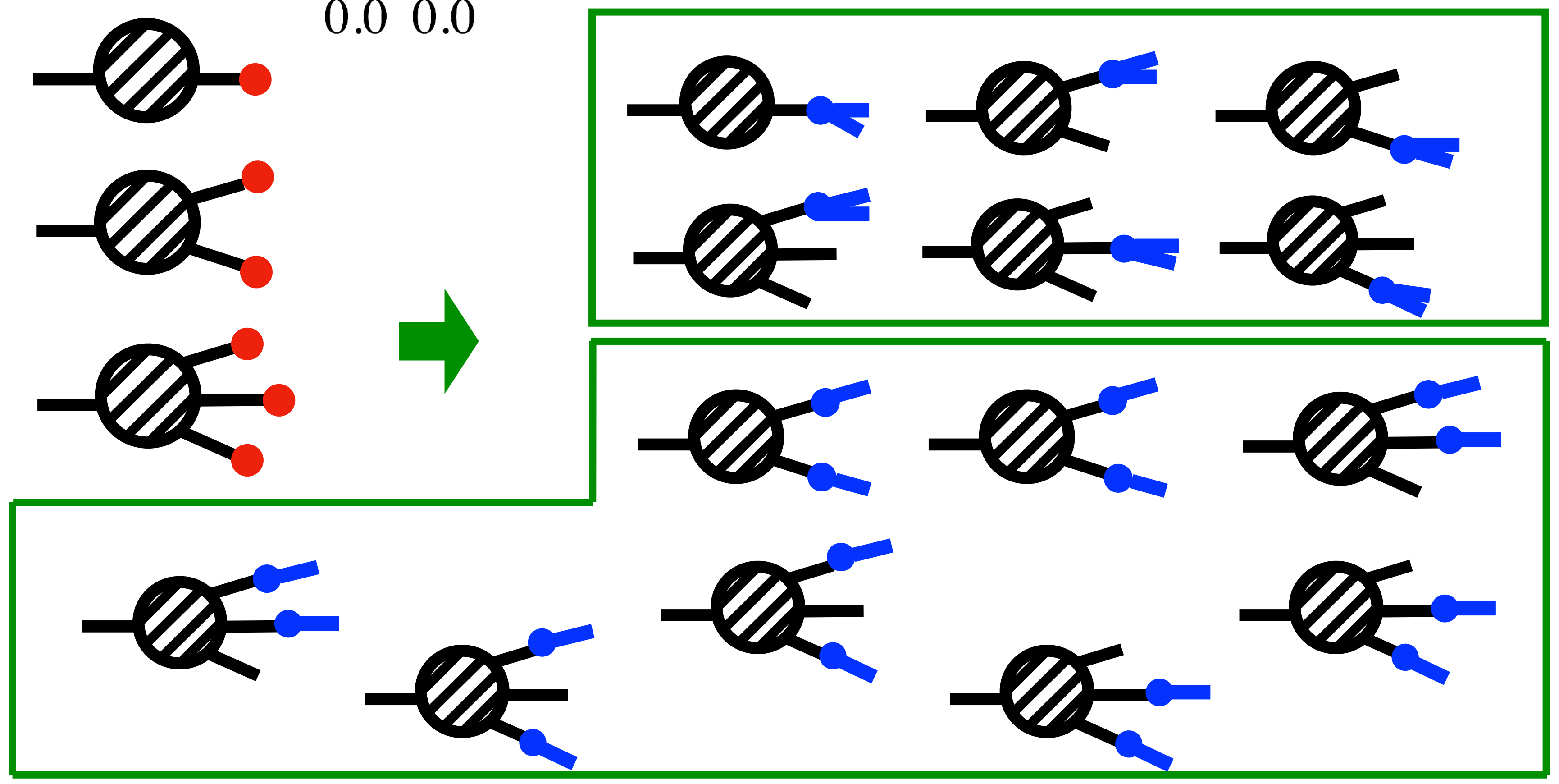
- **Ready for phenomenology!**

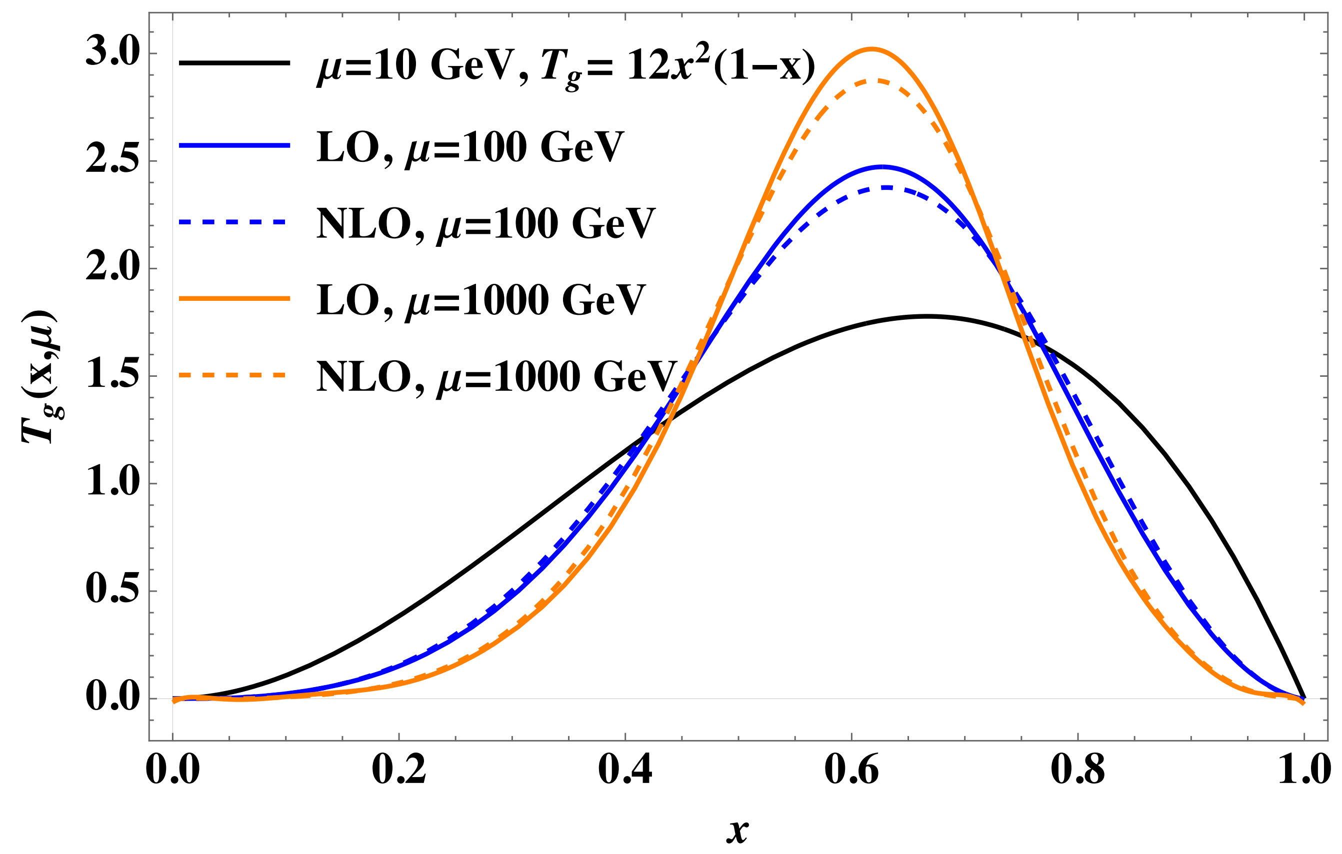
Summary

- Track functions offer a QFT approach to calculating track-based observables.
- We derived the full result of the nonlinear x-space evolution at $\mathcal{O}(\alpha_s^2)$:
  - The most general equation for collinear evolution at NLO;

    ➡ the NLO corrections to any N-hadron fragmentation functions evolution derived.
  - Numerical implementation.







Outlook

- A benchmark for triple collinear evolution in parton showers.
- Precision phenomenology with tracks.
- Applications of multi-hadron fragmentation functions.
- To describe the leading hadron/(sub)jet fragmentation.

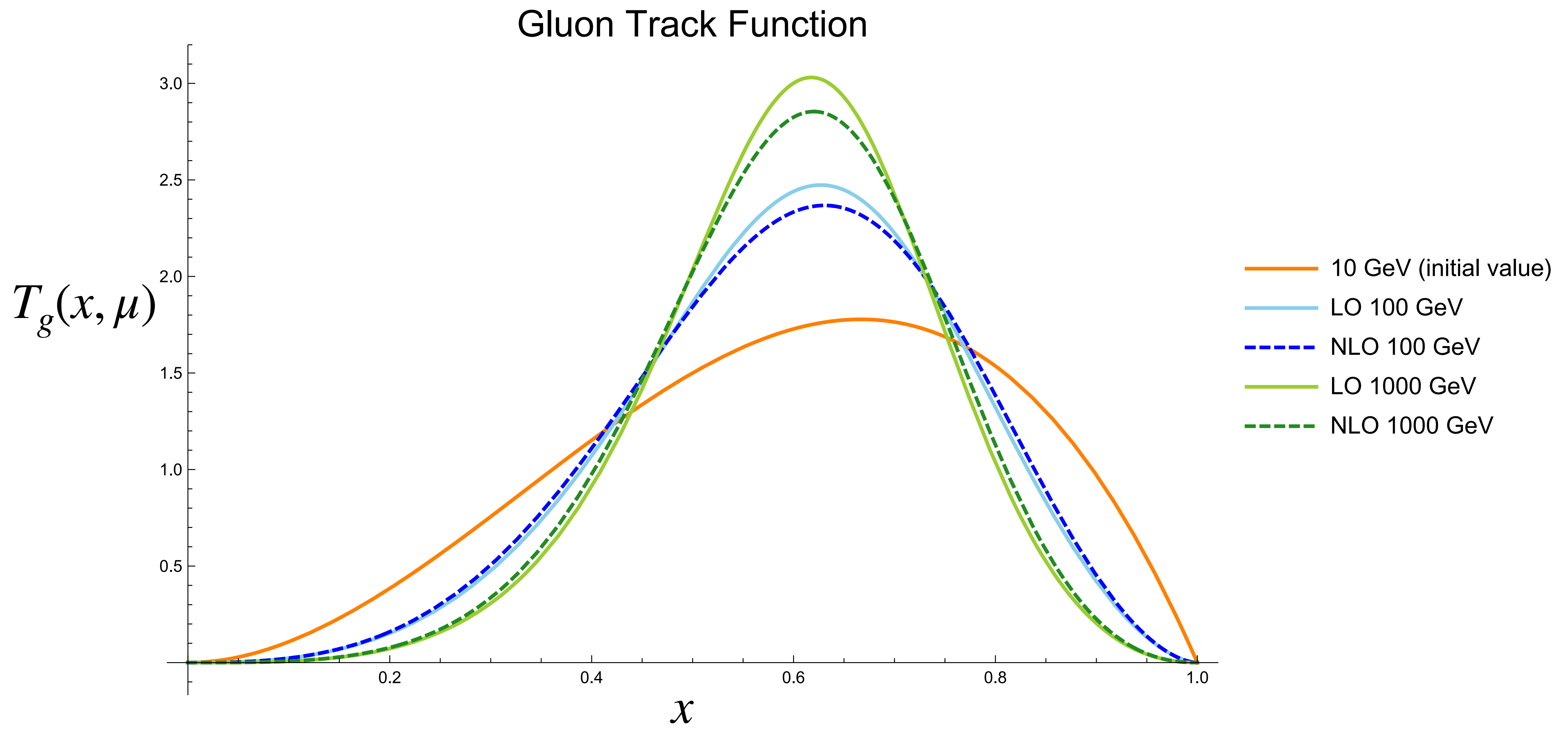
Backup

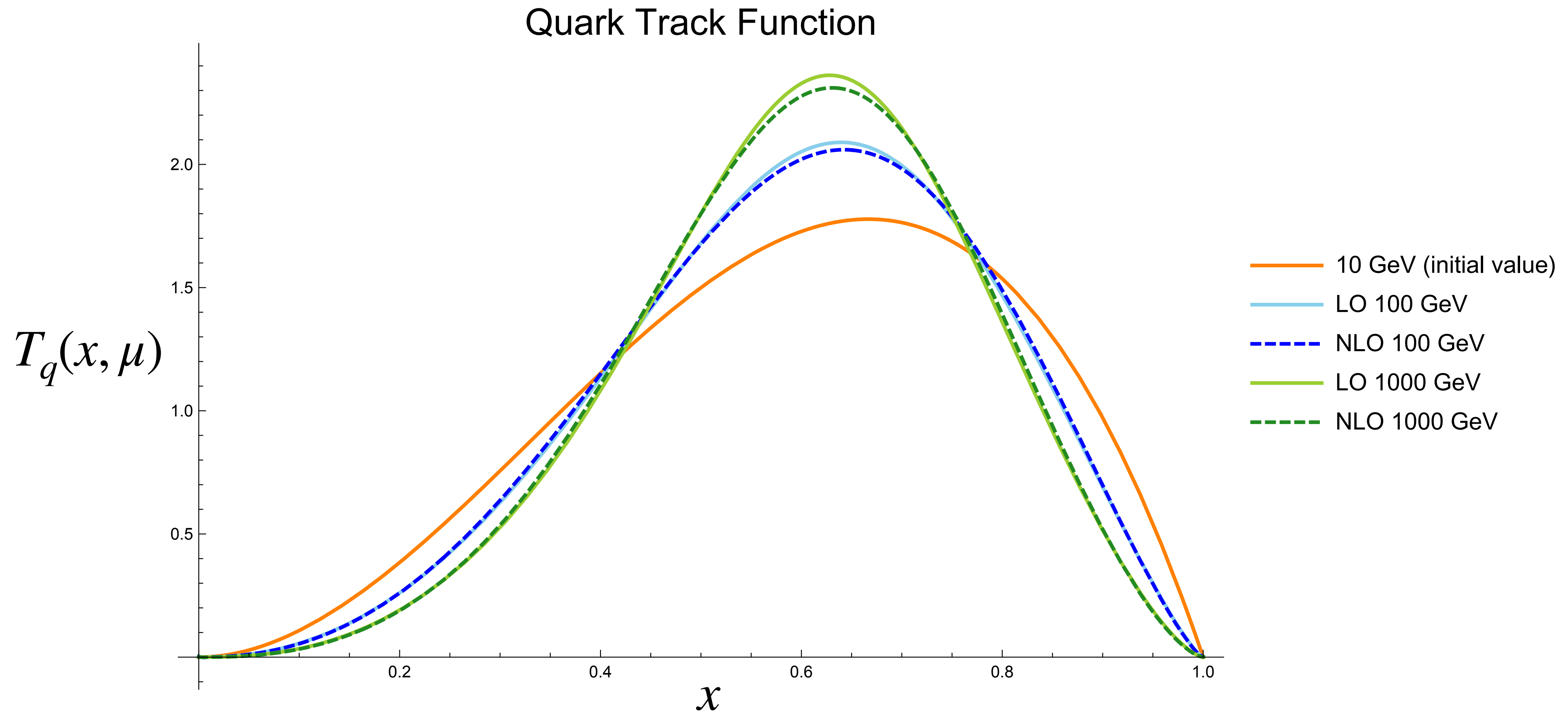
Solving the RGEs Numerically

➡ The approaches to the solution to the non-linear evolution equations, we used:

1. Moment method:
   - Suppose that the track function at any scale, $T(x, \mu_j)$, can be well described by a polynomial of some degree; $T(x, \mu_j)$ can then be restored from a finite number of its moments.
   - The curve is smooth but more likely to deviate from the normal at the endpoints.

2. Fourier series: Working better at the peak but leading to oscillations at the endpoints (especially at $x = 0$).





Solving the RGEs Numerically

➡ The approaches to the solution to the non-linear evolution equations, we used:

3. Discretization: Better at the endpoints.

4. Legendre wavelets: Piecewise but behaving well at the endpoints, like discretization.