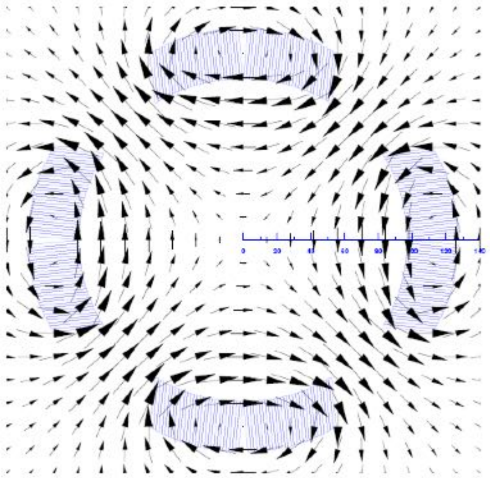
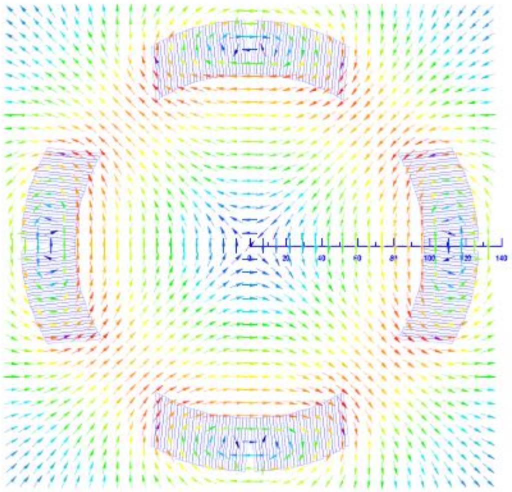
Electromagnetic Design of Accelerator Magnets and ROXIE User's Course

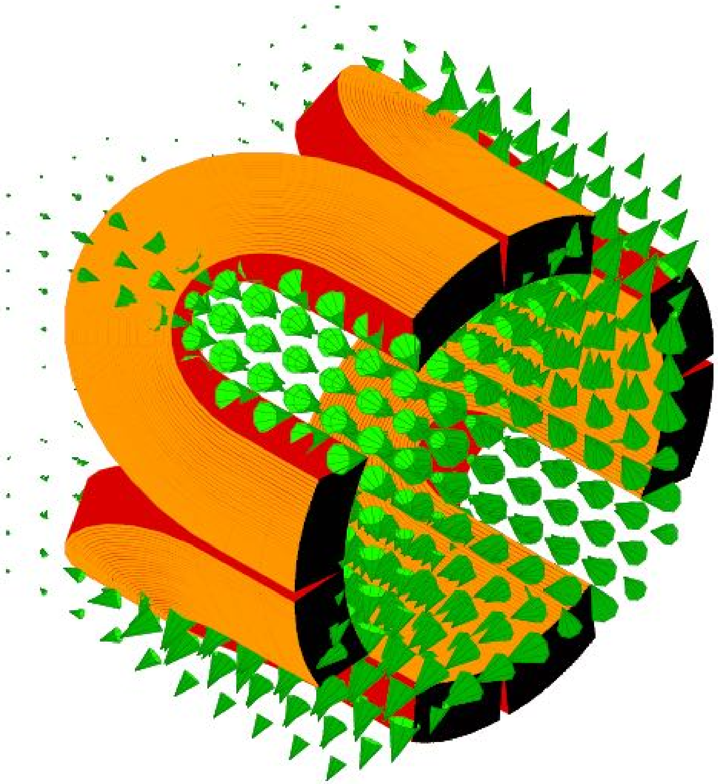
Theory 2

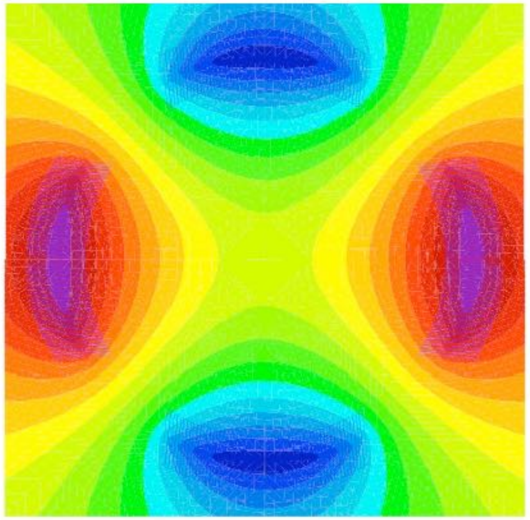
Stephan Russenschuck, CERN, 2022

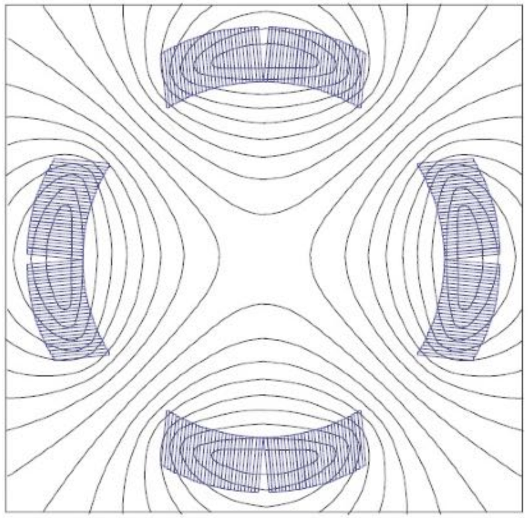
Different Renderings of the Same Vector Field



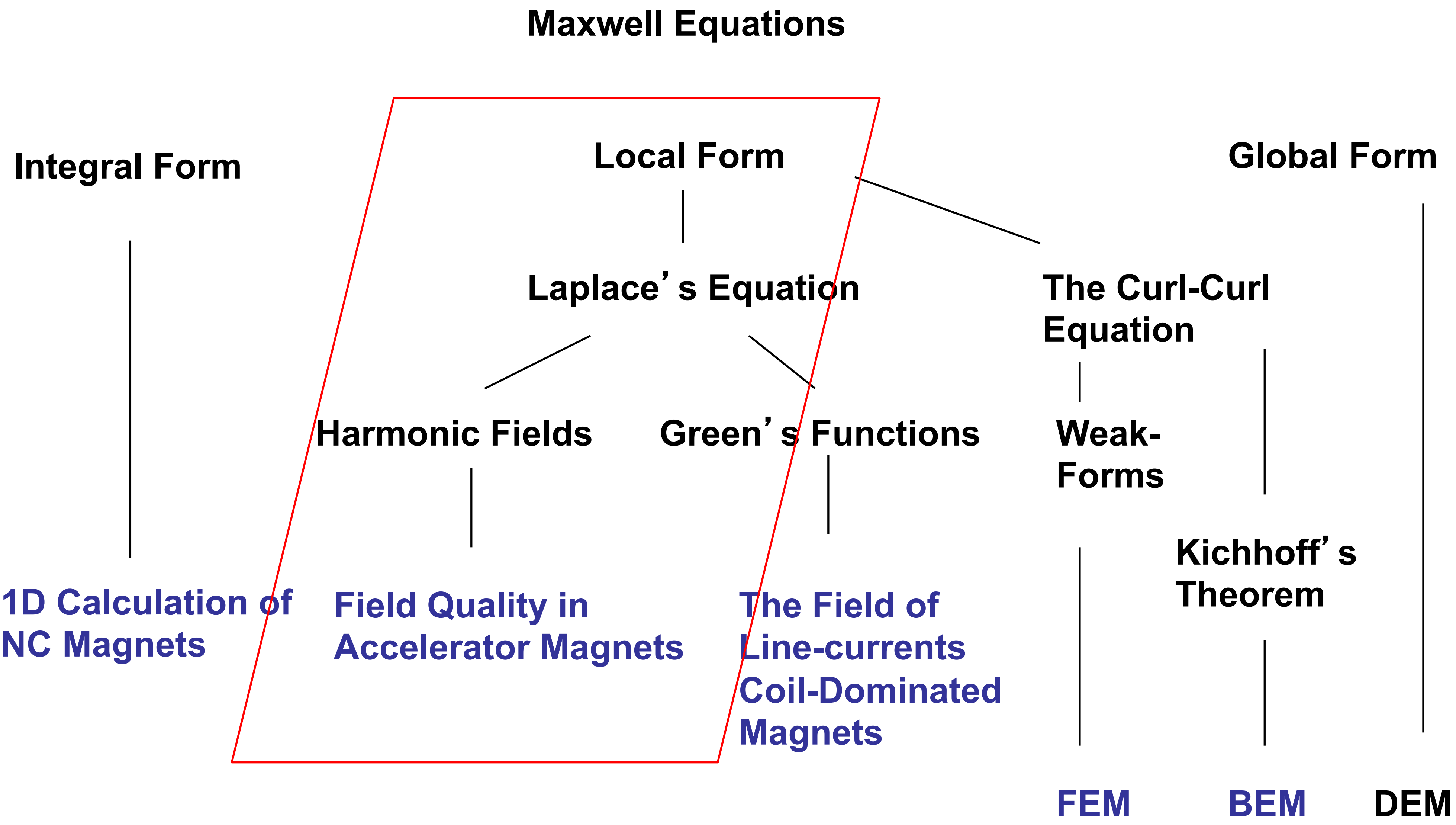








Mathematical Foundations of Magnet Design



Maxwell Equations

Integral Form

Local Form

Global Form

Laplace’s Equation

The Curl-Curl Equation

Harmonic Fields

Green’s Functions

Weak-Forms

Kichhoff’s Theorem

1D Calculation of NC Magnets

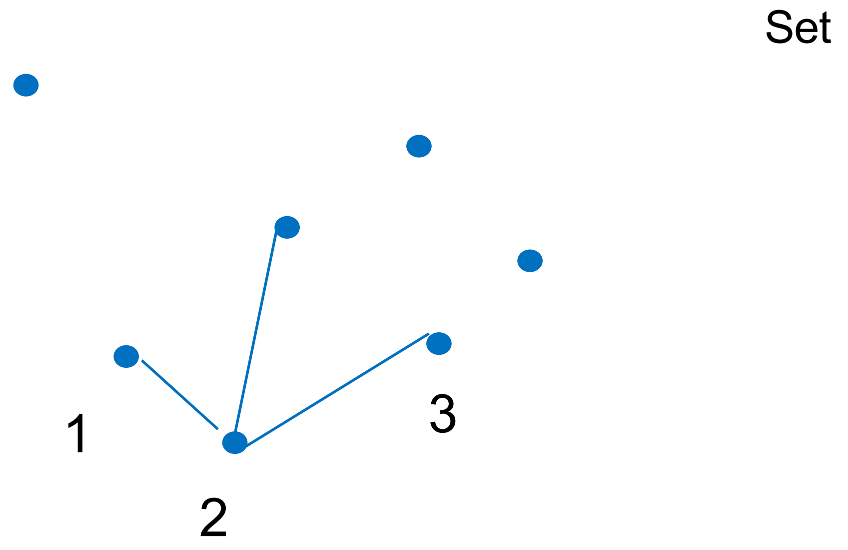
Field Quality in Accelerator Magnets

The Field of Line-currents Coil-Dominated Magnets

FEM

BEM

DEM



Sequence {1,2,3}
Connections (Cell Complexes)

Greenhoe, D. J.: Structure and Analysis of Mathematical Spaces, E-book, 2017
Tonti, E.: The Mathematical Structure of Classical and Relativistic Physics, Birkhauser, 2013
Russenschuck, S.: Chapters 2 and 3

Vector space

For any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V : (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

There is a zero vector $\mathbf{0}$ for which $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for any vector $\mathbf{a}$.

For each vector $\mathbf{a} \in V$ there is a vector $-\mathbf{a}$ in V for which $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$.

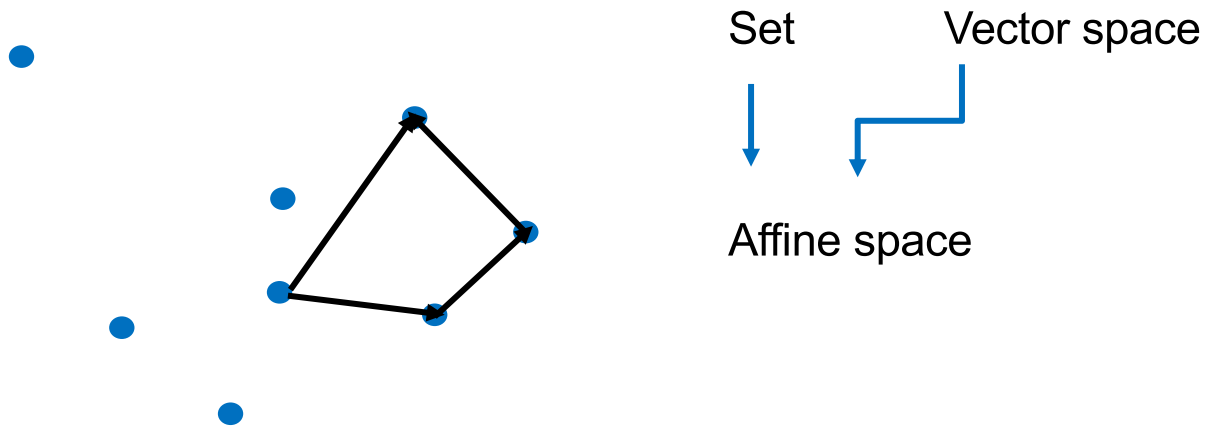
For any vectors $\mathbf{a}, \mathbf{b} \in V : \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

For any scalar $\lambda \in \mathbb{F}$ and any vectors $\mathbf{a}, \mathbf{b} \in V$: $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$.

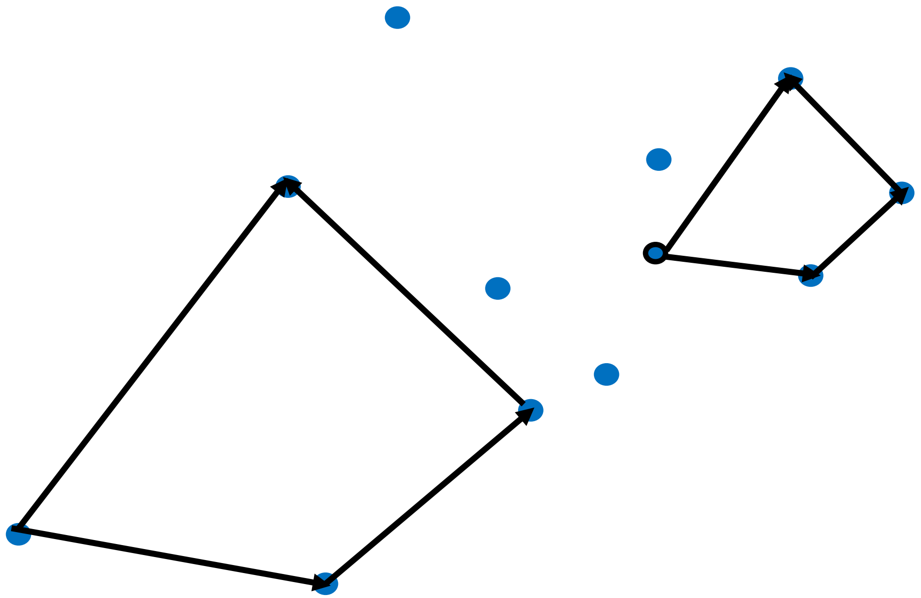
For any scalars $\lambda, \mu \in \mathbb{F}$ and any vector $\mathbf{a} \in V$: $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$.

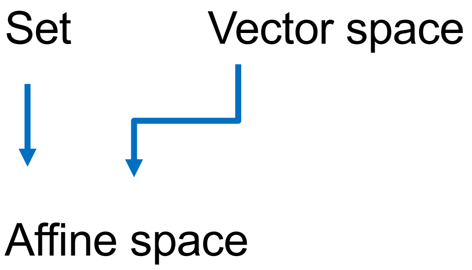
For any scalars $\lambda, \mu \in \mathbb{F}$ and any vector $\mathbf{a} \in V$: $(\lambda\mu)\mathbf{a} = \lambda(\mu\mathbf{a})$.

For the unit scalar $1 \in \mathbb{F}$ and any vector $\mathbf{a} \in V$: $1\mathbf{a} = \mathbf{a}$.

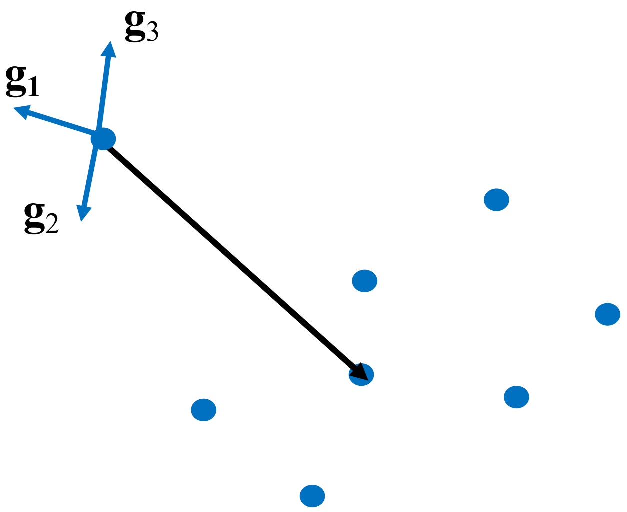


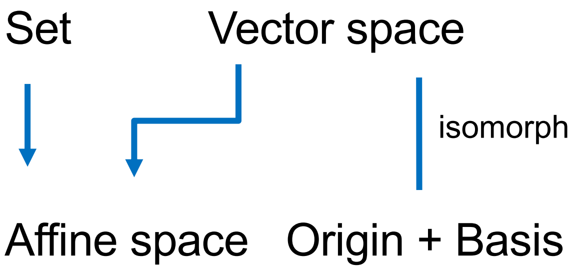
1. $\mathscr{P} + \mathbf{x} \in A$ if $\mathscr{P} \in A$ and $\mathbf{x} \in V$.
2. $(\mathscr{P} + \mathbf{x}) + \mathbf{y} = \mathscr{P} + (\mathbf{x} + \mathbf{y})$ for $\mathscr{P} \in A$ and $\mathbf{x}, \mathbf{y} \in V$.
3. There is a unique $\mathbf{x} \in V$ such that $\mathscr{P}_1 = \mathscr{P}_2 + \mathbf{x}$ for $\mathscr{P}_1, \mathscr{P}_2 \in A$.





Affine transformations (preserve barycenters)
but the translation is nonlinear!




Set

Vector space

isomorph

Affine space

Origin + Basis

$$\mathscr{P} \in A_n \xrightarrow{\text{Origin}} \mathbf{r} \in V_n \xrightarrow{\text{Basis}} (x^1, \dots, x^n) \in \mathbb{R}^n .$$

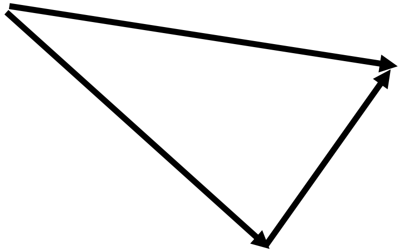
Euclidean space

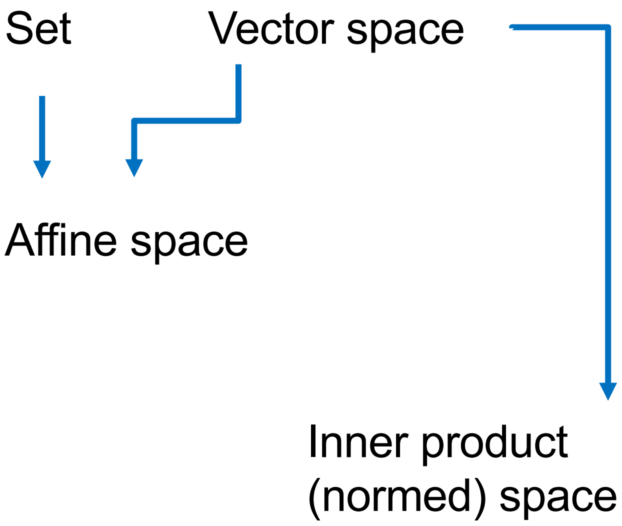
$$\mathbf{a} \cdot \mathbf{b} := a^1 b^1 + a^2 b^2 + a^3 b^3$$

$$\| \mathbf{a} \| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle},$$

Functional space

$$\langle f, g \rangle := \int_{-1}^{1} f(x) g(x) \mathrm{d}x.$$





Set

Vector space

Affine space

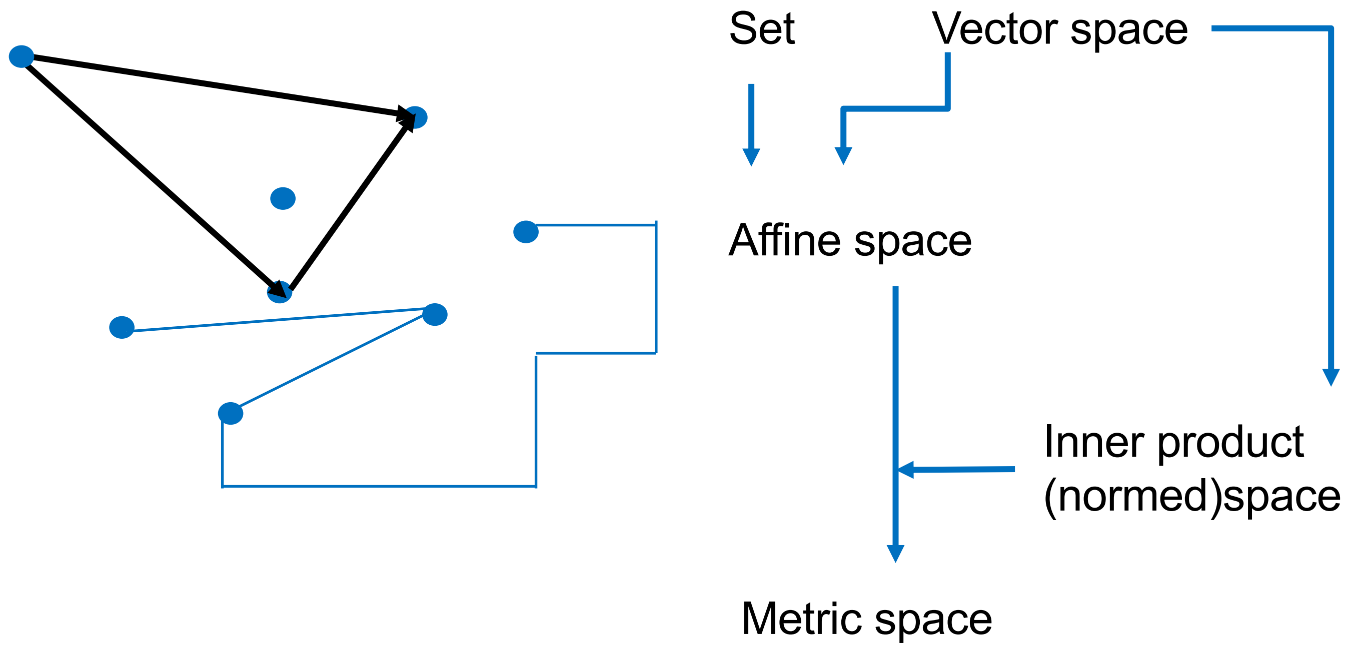
Inner product (normed) space

Length

1. $\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle + \langle \mathbf{b}, \mathbf{c} \rangle$ and $\langle \mathbf{a}, \lambda \mathbf{b} + \mu \mathbf{c} \rangle = \lambda \langle \mathbf{a}, \mathbf{b} \rangle + \mu \langle \mathbf{a}, \mathbf{c} \rangle$.
2. $\langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{a} \rangle$.
3. $\langle \mathbf{a}, \mathbf{a} \rangle > 0$ and $\langle \mathbf{a}, \mathbf{a} \rangle = 0$ if and only if $\mathbf{a} = \mathbf{0}$.

$$d(\mathscr{P}_1,\mathscr{P}_2) := \| \mathbf{r}_{\mathscr{P}_1} - \mathbf{r}_{\mathscr{P}_2} \|$$

Distance



Set

Vector space

Affine space

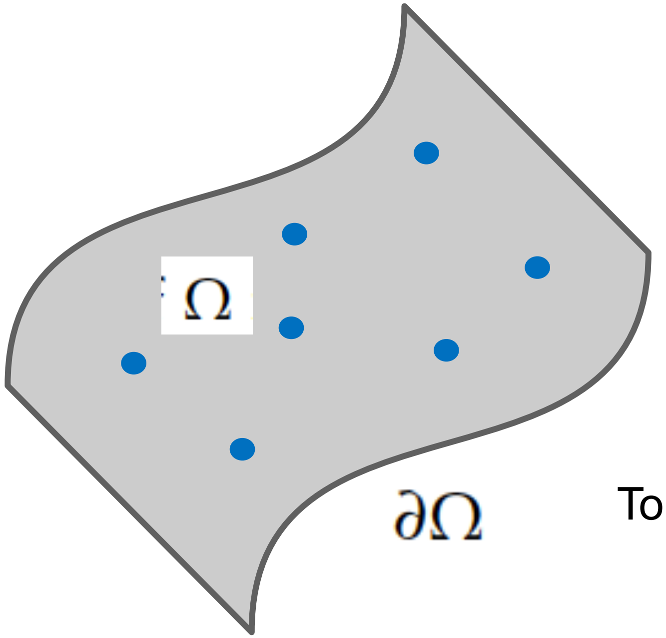
Inner product
(normed)space

Metric space

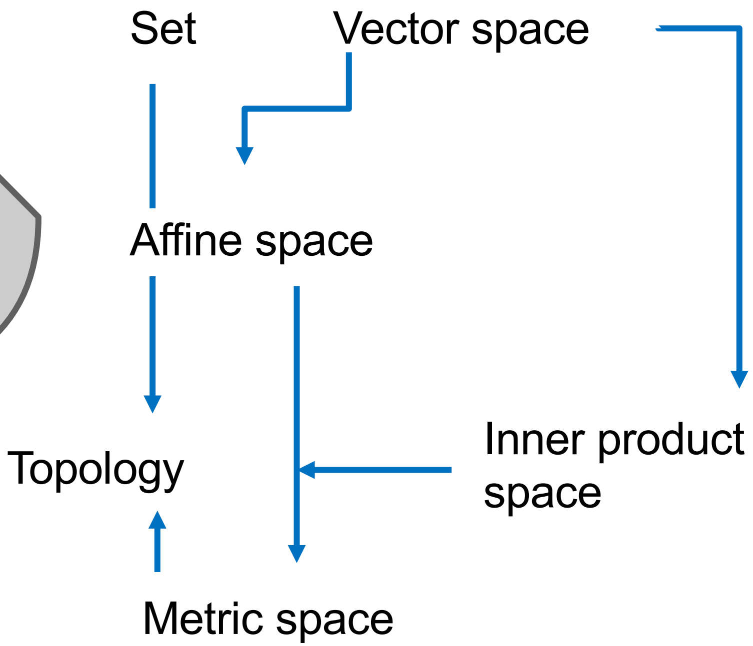
$$\cos\alpha(\mathbf{a},\mathbf{b}) := \frac{\langle \mathbf{a},\mathbf{b}\rangle}{\| \mathbf{a} \| \| \mathbf{b} \|}, \qquad 0 \leq \alpha \leq \pi.$$

$$| \langle \mathbf{a},\mathbf{b}\rangle | \leq \| \mathbf{a} \| \| \mathbf{b} \|,$$

Cauchy Schwarz inequality



Closed, simply connected



Set

Vector space

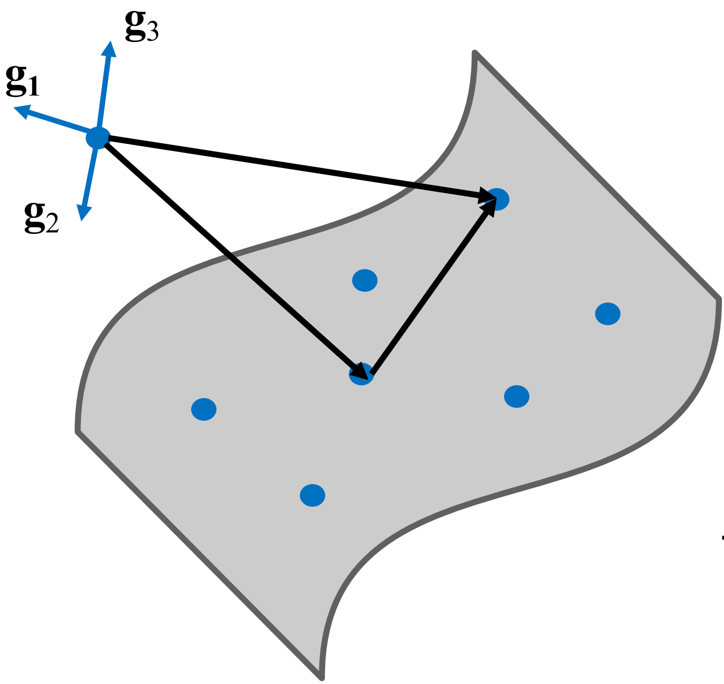
Affine space

Topology

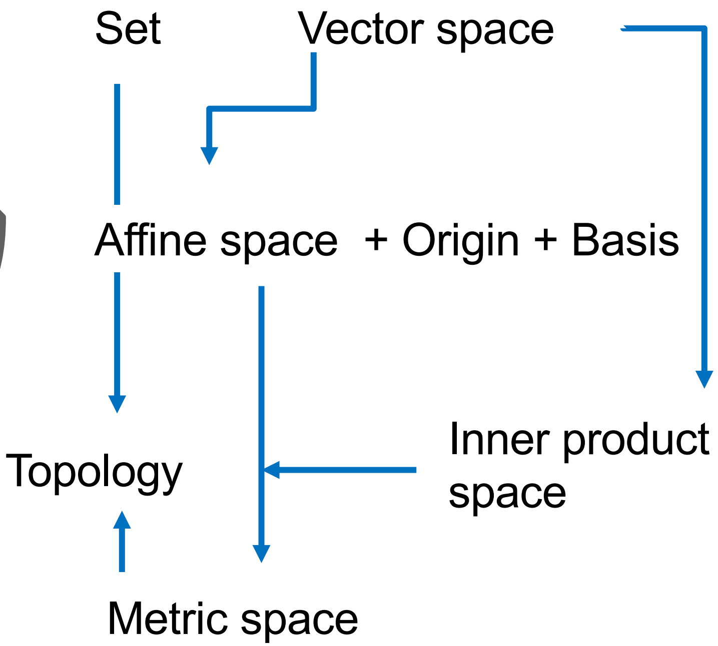
Inner product space

Metric space

The *boundary* $\partial\Omega$ of Ω is the set of all points for which the distances $d(\mathscr{P},\Omega) = 0$ and $d(\mathscr{P}, A \setminus \Omega) = 0$.



Embedded



Set

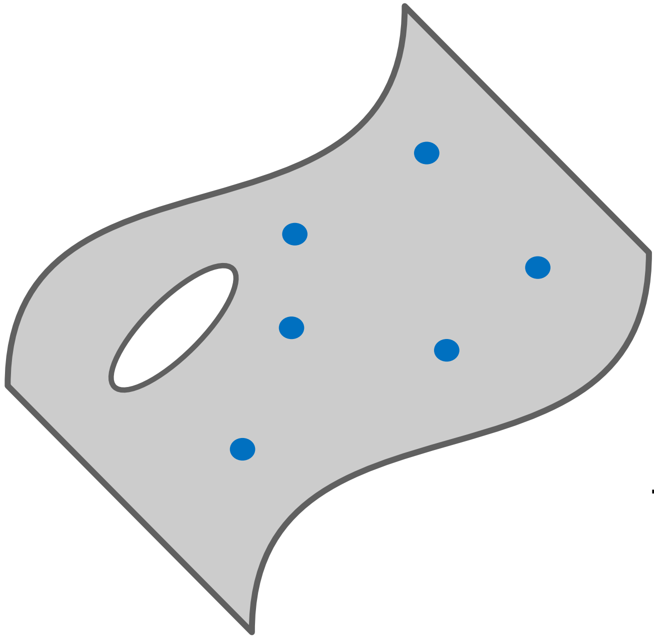
Vector space

Affine space + Origin + Basis

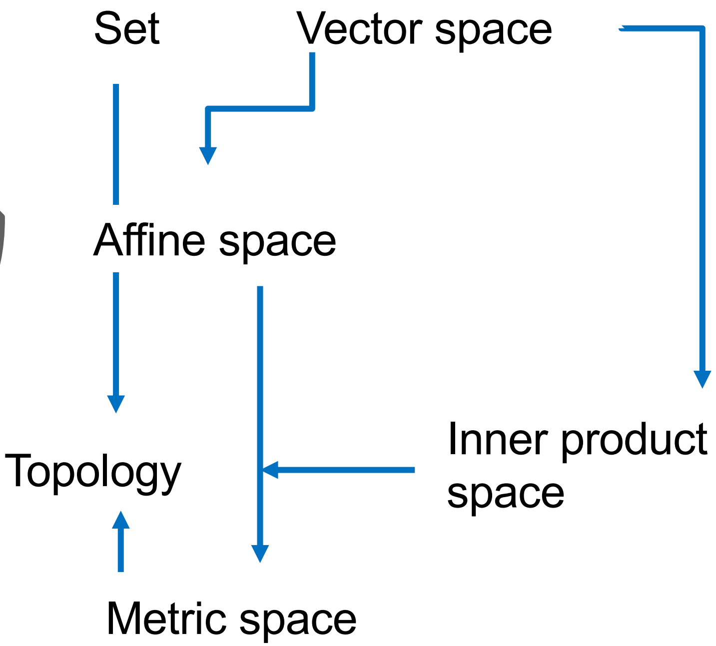
Topology

Inner product space

Metric space



Multiply connected, noncontractable



Set

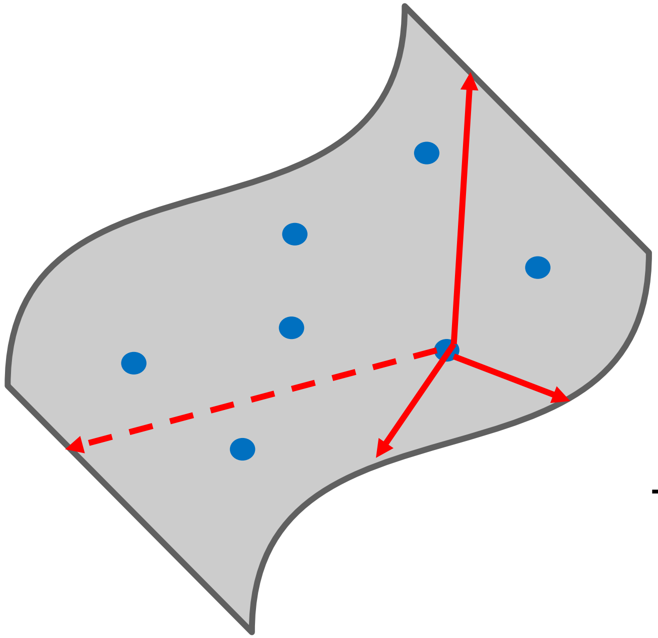
Vector space

Affine space

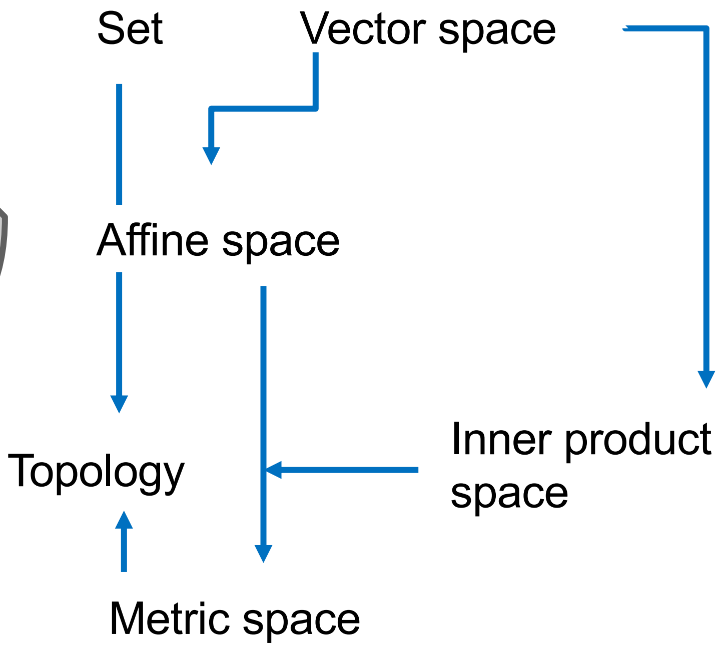
Topology

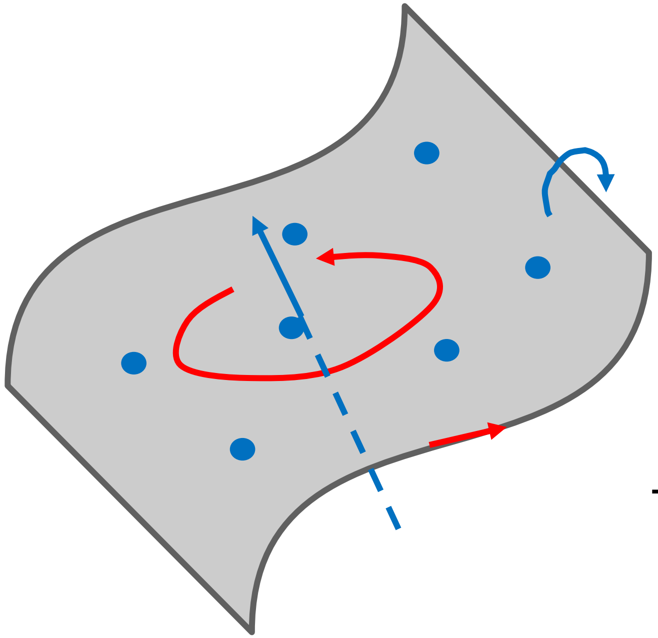
Inner product space

Metric space

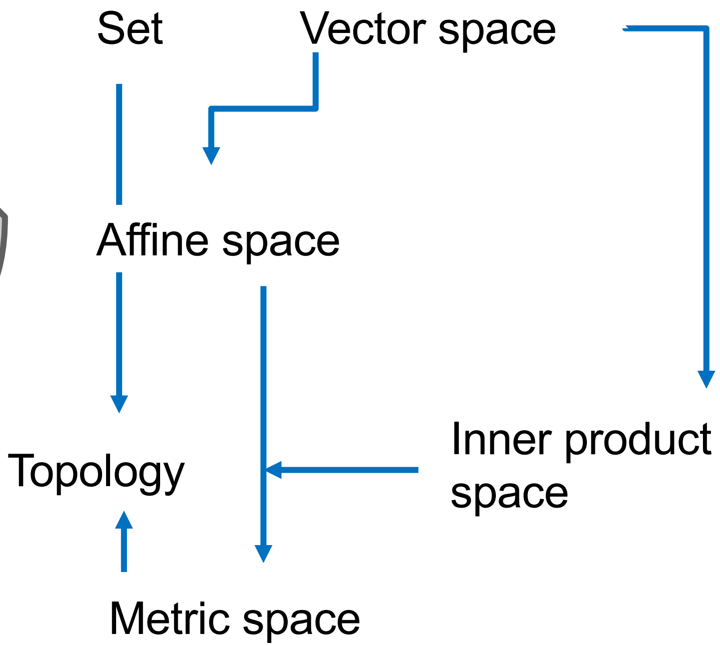


Not star shaped,
but simply connected





Inner oriented
Outer oriented



Set

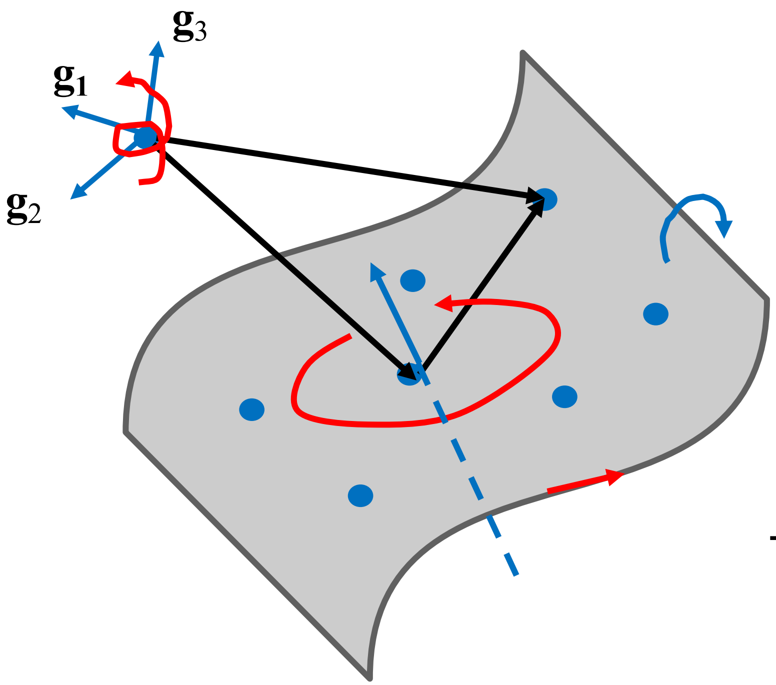
Vector space

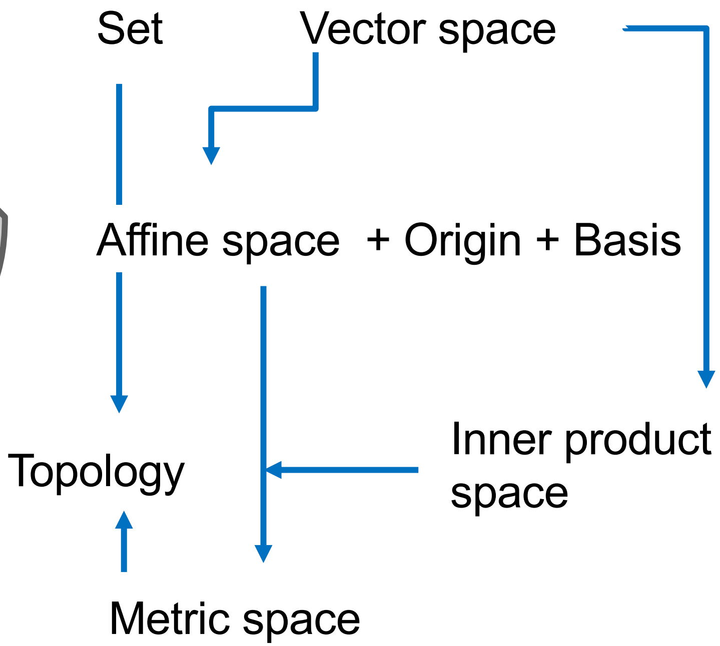
Affine space

Topology

Inner product space

Metric space





Set

Vector space

Affine space + Origin + Basis

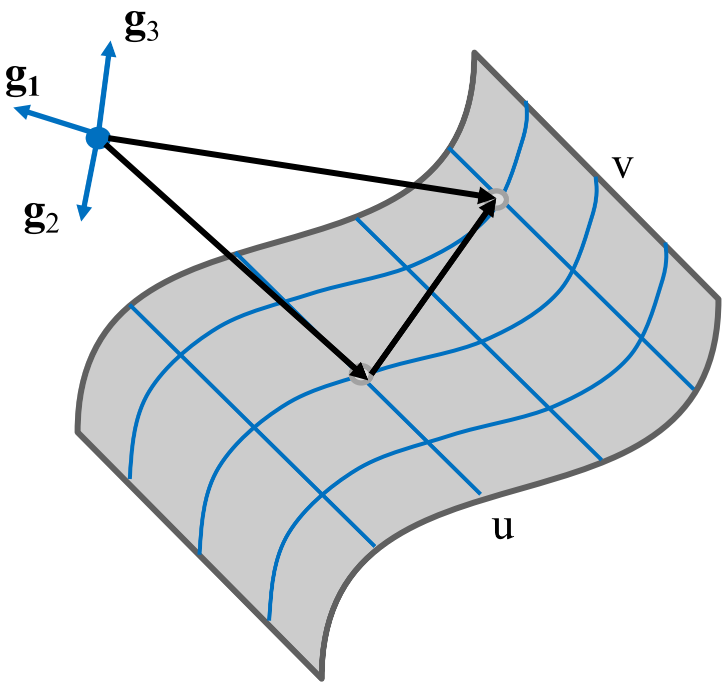
Topology

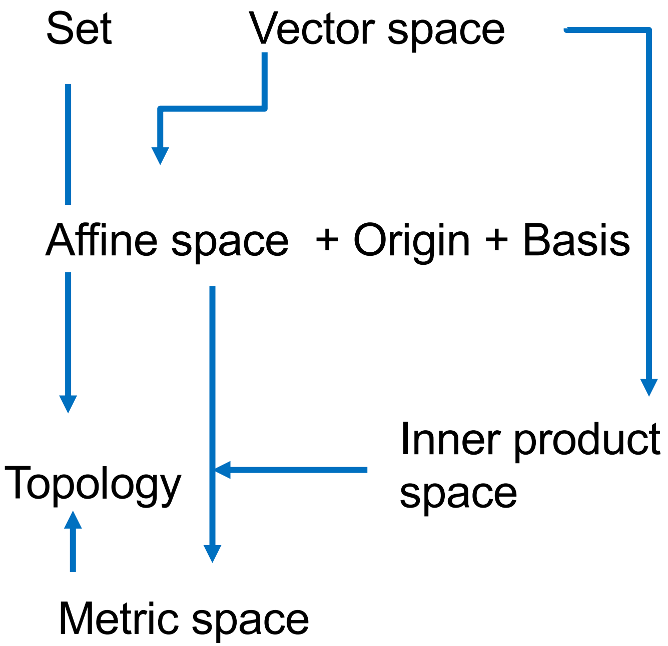
Inner product space

Metric space

Embedded, consistently inner and outer oriented

$$\| \mathbf{a} \| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \sqrt{a^i a^j g_{ij}},$$





Set

Vector space

Affine space + Origin + Basis

Topology

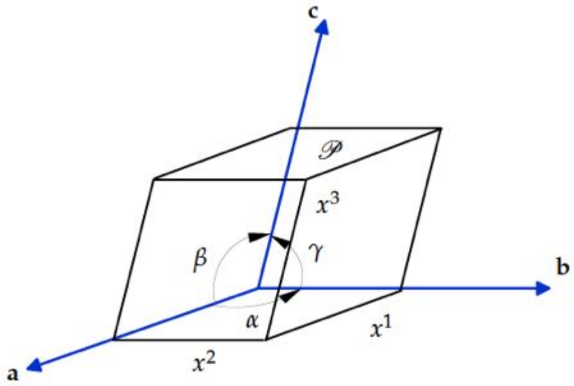
Inner product space

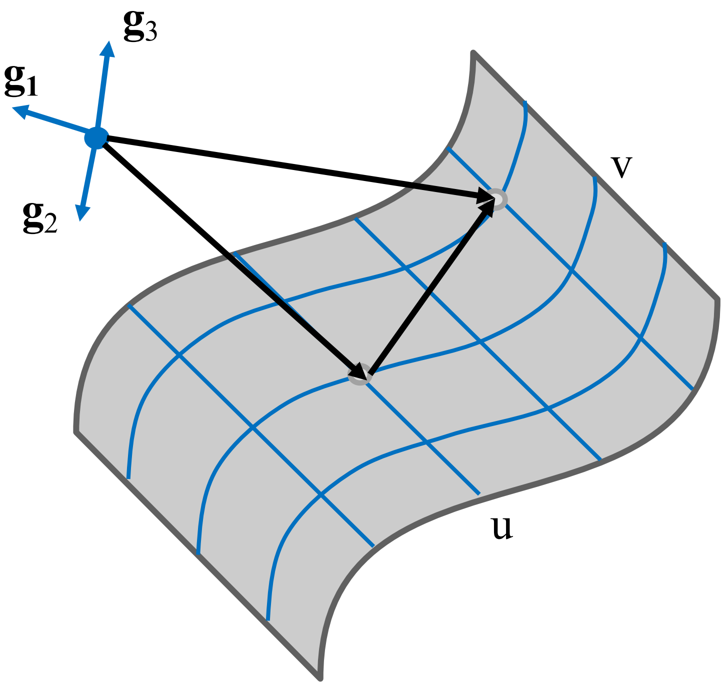
Metric space

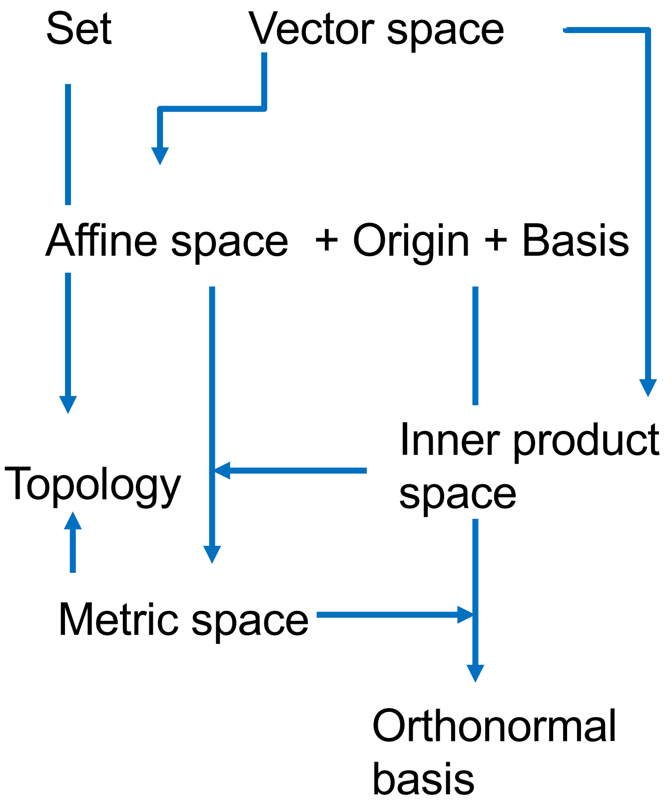
$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{3} \sum_{j=1}^{3} a^i b^j \langle \mathbf{g}_i, \mathbf{g}_j \rangle \equiv a^i b^j \langle \mathbf{g}_i, \mathbf{g}_j \rangle =: a^i b^j g_{ij},$$

$$\cos \alpha = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\| \mathbf{a} \| \, \| \mathbf{b} \|} = \frac{a^i b^j g_{ij}}{\sqrt{a^p a^q g_{pq}} \sqrt{b^r b^s g_{rs}}}.$$

$$[G] = \begin{pmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{pmatrix}$$







Set

Vector space

Affine space + Origin + Basis

Topology

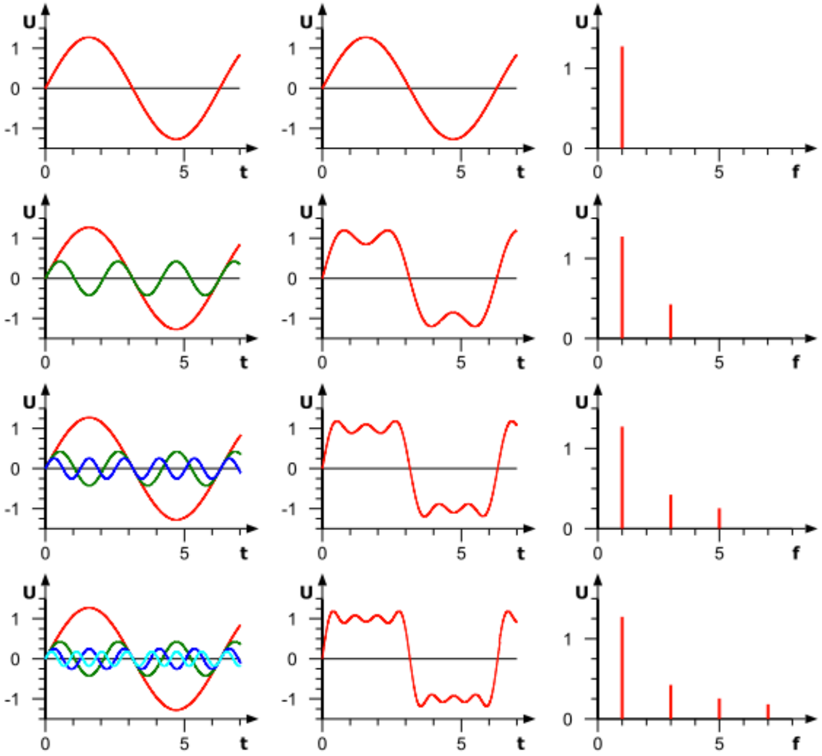
Inner product space

Metric space

Orthonormal basis

Applications: Calibration of Helmholtz coils, Calibration of 3-axis displacement stages and robots

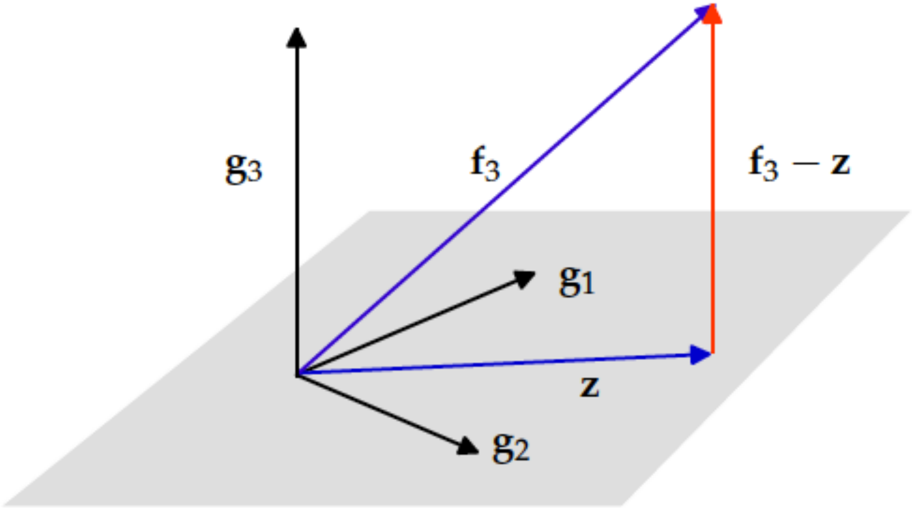
The Fourier Polynomial Pn is the best Approximation in Vn

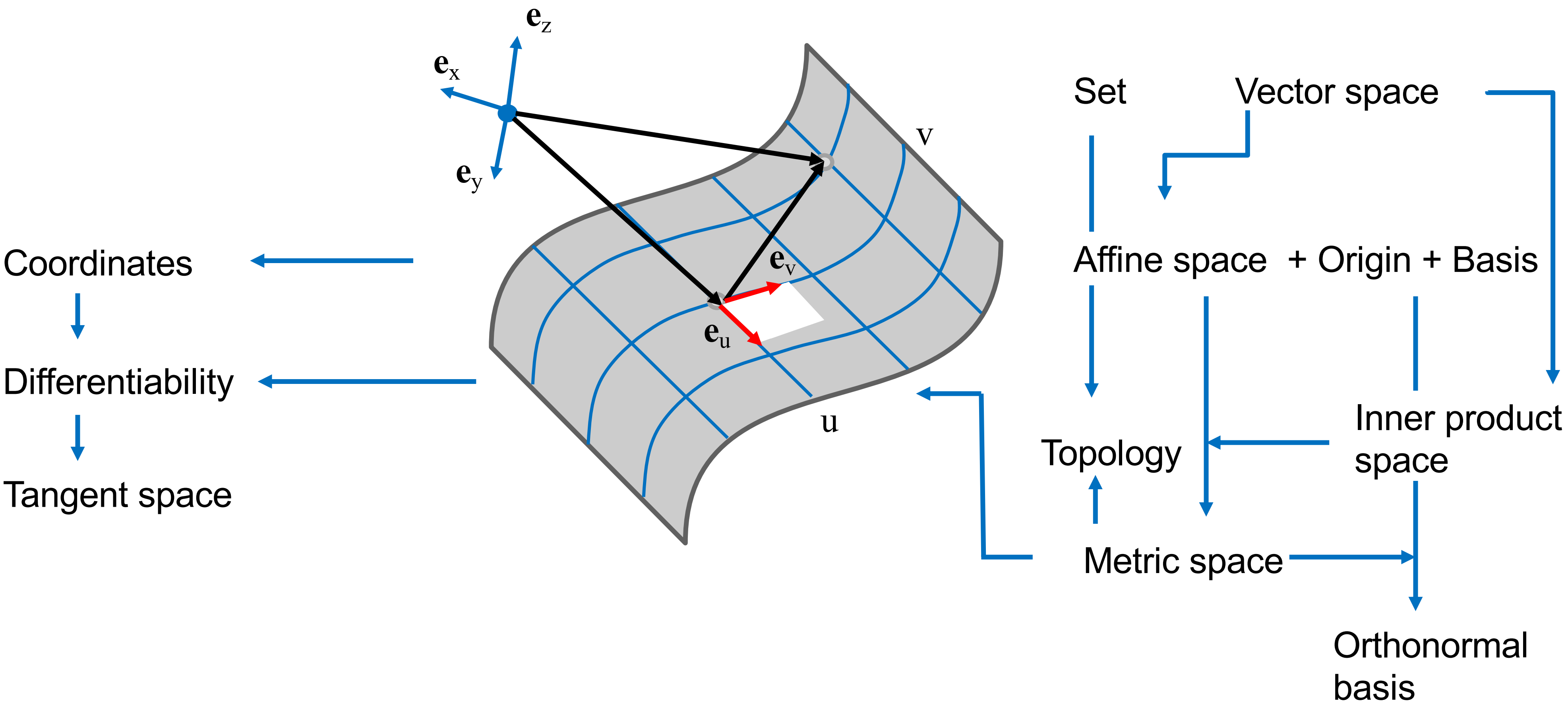


Projection of the square wave onto the “shape” of the trigonometric functions

f_3-z is the shortest distance to the projective plane

$$d(\mathscr{P}_1,\mathscr{P}_2)=\sqrt{\sum_{i=1}^{n}(x^i(\mathscr{P}_1)-x^i(\mathscr{P}_2))^2}.$$





Coordinates

Differentiability

Tangent space

Set

Vector space

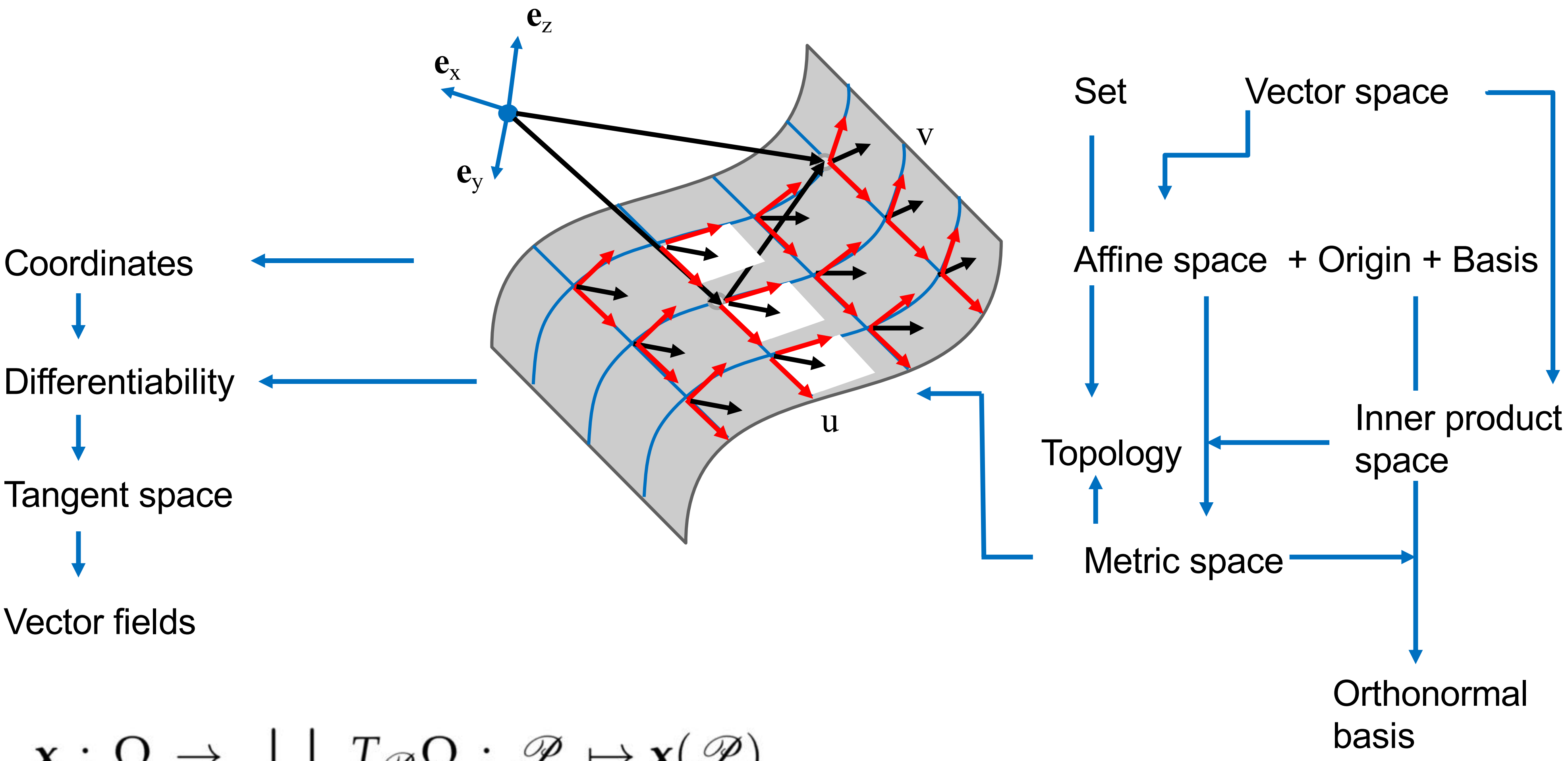
Affine space + Origin + Basis

Topology

Inner product space

Metric space

Orthonormal basis



Coordinates

Differentiability

Tangent space

Vector fields

Set

Vector space

Affine space + Origin + Basis

Topology

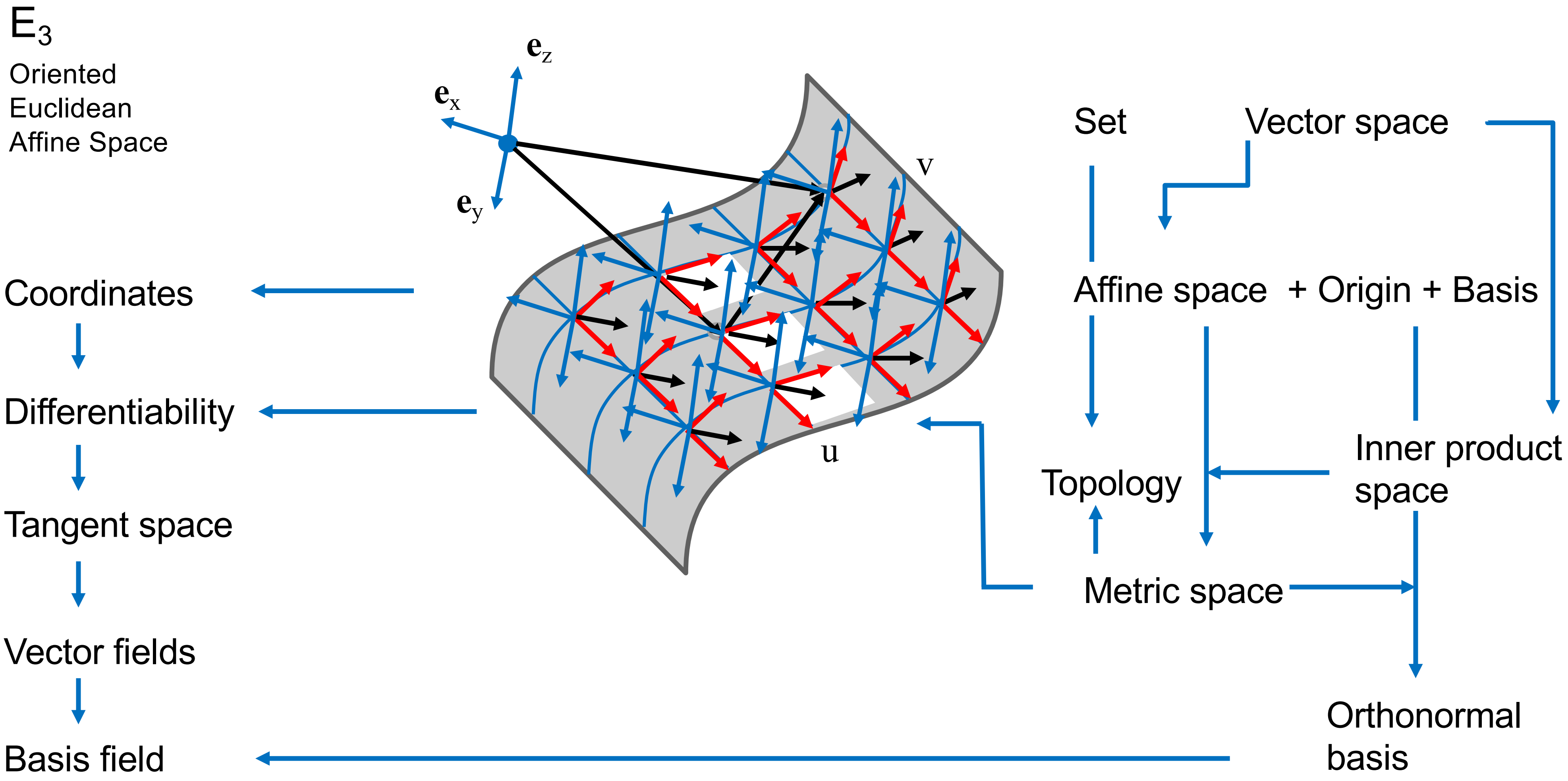
Inner product space

Metric space

Orthonormal basis

$$\mathbf{x} : \Omega \to \bigcup_{\mathscr{P}\in\Omega} T_{\mathscr{P}}\Omega : \mathscr{P} \mapsto \mathbf{x}(\mathscr{P})$$

$\mathbf{B} = \mu\,\mathbf{H}$



E_3
Oriented
Euclidean
Affine Space

Coordinates

Differentiability

Tangent space

Vector fields

Basis field

Set

Vector space

Affine space + Origin + Basis

Topology

Inner product space

Metric space

Orthonormal basis

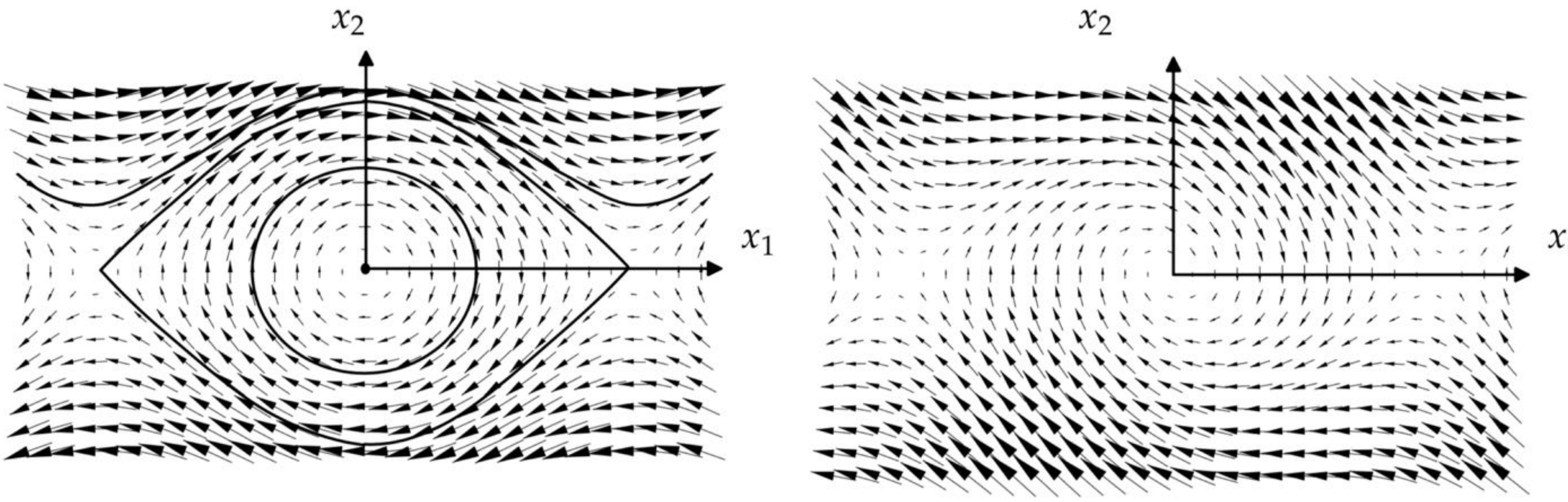
$$\mathbf{a} : \Omega \to \mathbb{R}^3 : \mathbf{r} \mapsto \mathbf{a}(\mathbf{r}) : \mathbf{a}(\mathbf{r}) = (a^1(\mathbf{r}), a^2(\mathbf{r}), a^3(\mathbf{r}))$$

2-Dimensional Trace Space

$$\mathbf{v} : P_2 \to \bigcup_{\mathbf{x}\in P_2} T_{\mathscr{P}} P_2 : \mathbf{x}(t) \mapsto \mathbf{v}(\mathbf{x}(t)) \qquad \mathbf{x}(t) := (x_1(t), x_2(t))^T = \left(\varphi(t), \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}\right)^T$$

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{v}(t) = \left(\frac{\mathrm{d}x_1(t)}{\mathrm{d}t}, \frac{\mathrm{d}x_2(t)}{\mathrm{d}t}\right)^T = \left(\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}, \frac{\mathrm{d}^2\varphi(t)}{\mathrm{d}t^2}\right)^T$$

$$= \left(x_2(t), -\alpha x_2(t) - \frac{g}{l}\sin x_1(t)\right)^T .$$

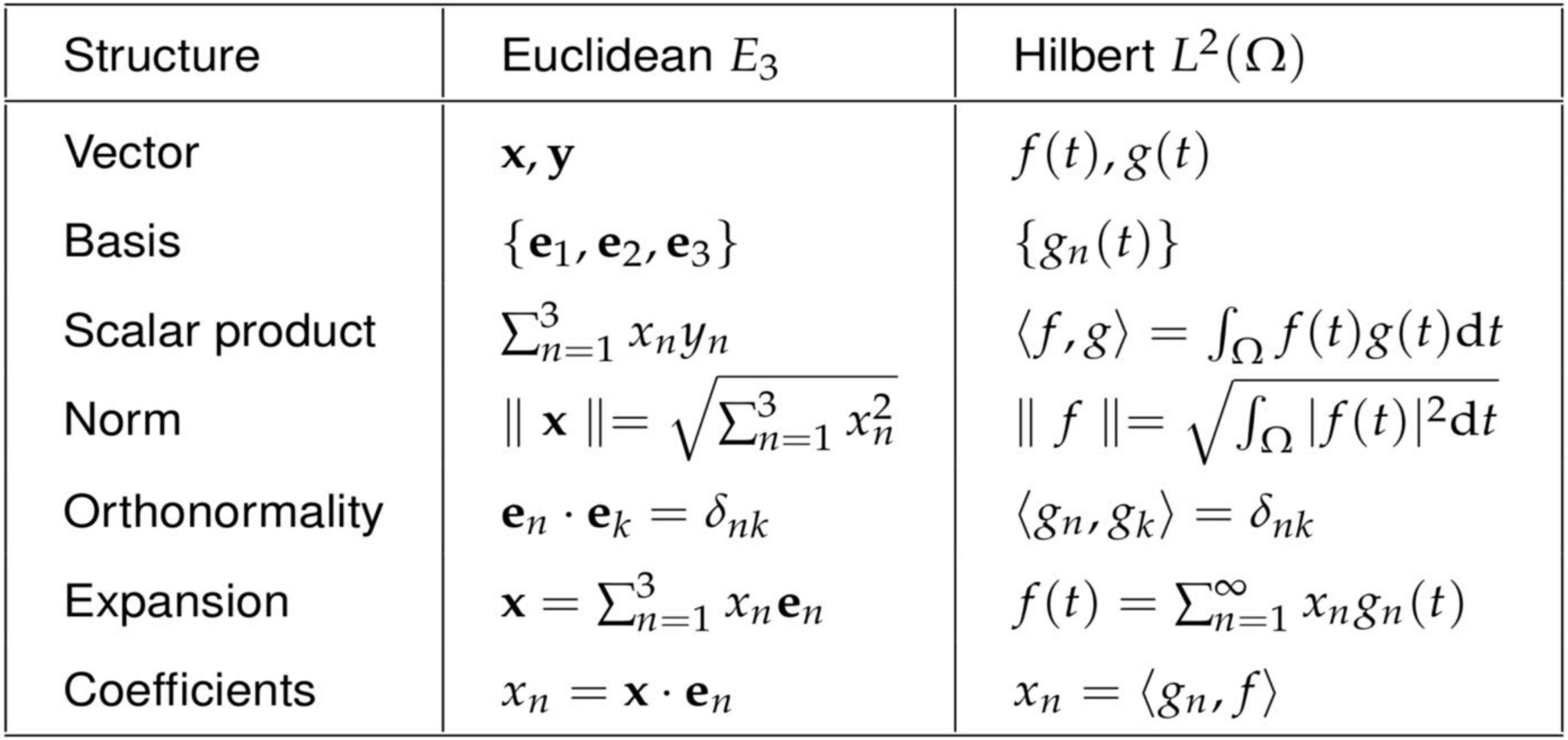


Find flow maps:

$$\phi(t,\mathbf{x}) : P_2 \to P_2 : \mathbf{x}(0) \mapsto \mathbf{x}(t)$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \frac{g}{l}\sin\varphi = 0$$

E_3 and L^2



Structure	Euclidean E_3	Hilbert $L^2(\Omega)$
Vector	$\mathbf{x}, \mathbf{y}$	$f(t), g(t)$
Basis	$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$	$\{g_n(t)\}$
Scalar product	$\sum_{n=1}^{3} x_n y_n$	$\langle f, g \rangle = \int_\Omega f(t) g(t) \mathrm{d}t$
Norm	$\parallel \mathbf{x} \parallel = \sqrt{\sum_{n=1}^{3} x_n^2}$	$\parallel f \parallel = \sqrt{\int_\Omega \lvert f(t) \rvert^2 \mathrm{d}t}$
Orthonormality	$\mathbf{e}_n \cdot \mathbf{e}_k = \delta_{nk}$	$\langle g_n, g_k \rangle = \delta_{nk}$
Expansion	$\mathbf{x} = \sum_{n=1}^{3} x_n \mathbf{e}_n$	$f(t) = \sum_{n=1}^{\infty} x_n g_n(t)$
Coefficients	$x_n = \mathbf{x} \cdot \mathbf{e}_n$	$x_n = \langle g_n, f \rangle$

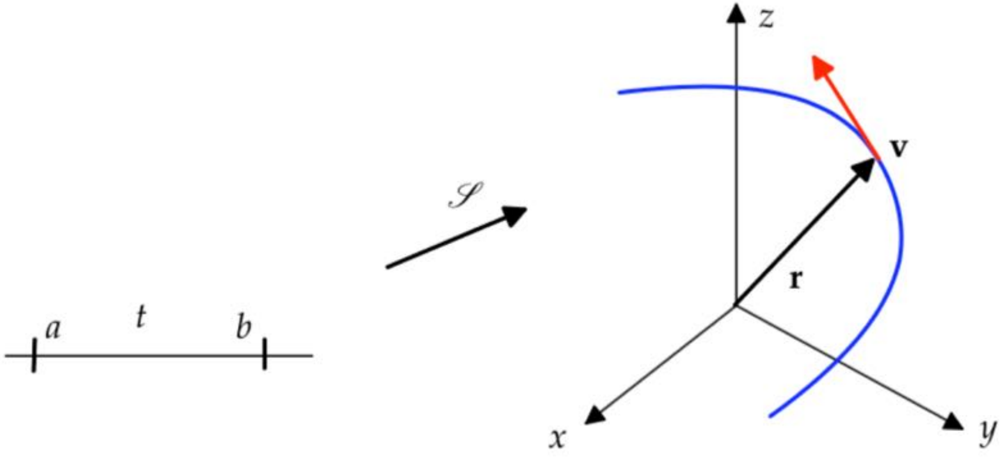
Hilbert spaces are those in which notation and concepts of ordinary Euclidean geometry hold without any restrictions on the dimension.

Space Curves (as Mappings)

$$\mathbf{r}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y + z(t)\mathbf{e}_z$$

$$\mathscr{S} : I \to E_3 : t \mapsto \mathbf{r}(t)$$

$$\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{e}_x + \frac{\mathrm{d}y}{\mathrm{d}t}\mathbf{e}_y + \frac{\mathrm{d}z}{\mathrm{d}t}\mathbf{e}_z$$



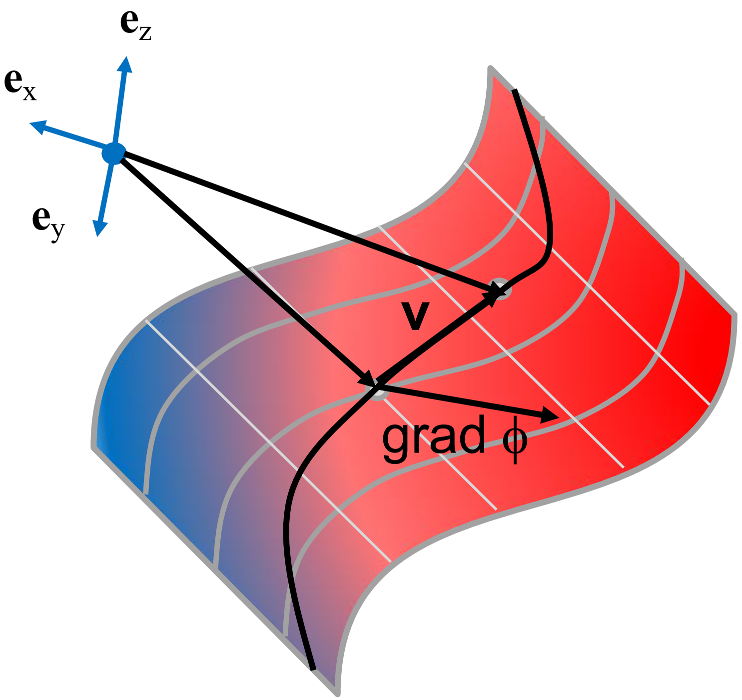
$$\frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = \mathbf{a}(t) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathbf{e}_x + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\mathbf{e}_y + \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\mathbf{e}_z$$

$$\mathbf{T}(t) := \frac{\mathbf{v}(t)}{v(t)} = T_x(t)\mathbf{e}_x + T_y(t)\mathbf{e}_y + T_z(t)\mathbf{e}_z$$

Directional Derivative and the Gradient

Space curve with $\mathbf{r}(t) = (x(t), y(t), z(t))$ parametrized such that $\mathbf{r}(0) = P$.

1-smooth scalar field $\phi : E_3 \to R : \mathbf{r} \mapsto \phi(\mathbf{r})$ expressed as $\phi(x,y,z)$, then $\phi(\mathbf{r}(t))$ at parameter (time) t.



$$\partial_{\mathbf{v}}\phi = \frac{\partial \phi}{\partial v} = \frac{\mathrm{d}}{\mathrm{d}t}[\phi(\mathbf{r} + t\mathbf{v})]_{t=0} = \lim_{t\to 0} \frac{\phi(\mathbf{r}+t\mathbf{v}) - \phi(\mathbf{r})}{t}$$

$$\partial_{\mathbf{e}_\mathrm{x}}\phi = \frac{\partial \phi(x,y,z)}{\partial x} = \lim_{\Delta x \to 0} \frac{\phi(x + \Delta x, y, z) - \phi(x,y,z)}{\Delta x}$$

$$\partial_{\mathbf{v}}\phi = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\mathbf{r}(t)) = \frac{\partial \phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{grad}\,\phi \cdot \mathbf{v}$$

$$\mathrm{grad}\phi = \frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

$$\xcancel{\frac{\partial \phi}{\partial \mathbf{n}}}$$

Best linear approximation of ϕ over displacement distance d**r**

Grad, Curl and Div in Cartesian Coordinates

$$\operatorname{grad}\phi := \frac{\partial\phi}{\partial x}\mathbf{e}_x + \frac{\partial\phi}{\partial y}\mathbf{e}_y + \frac{\partial\phi}{\partial z}\mathbf{e}_z$$

$$\operatorname{curl}\mathbf{g} = \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right)\mathbf{e}_z .$$

$$\operatorname{div}\mathbf{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} .$$

The First Lemma of Poincare

$$\begin{aligned}\operatorname{curl}\operatorname{grad}\phi = {} & \operatorname{curl}\left[\frac{1}{h_1}\frac{\partial\phi}{\partial u^1}\mathbf{e}_{u^1} + \frac{1}{h_2}\frac{\partial\phi}{\partial u^2}\mathbf{e}_{u^2} + \frac{1}{h_3}\frac{\partial\phi}{\partial u^3}\mathbf{e}_{u^3}\right] \\ = {} & \frac{1}{h_2h_3}\left(\frac{\partial^2\phi}{\partial u^2\partial u^3} - \frac{\partial^2\phi}{\partial u^3\partial u^2}\right)\mathbf{e}_{u^1} \\ & + \frac{1}{h_3h_1}\left(\frac{\partial^2\phi}{\partial u^3\partial u^1} - \frac{\partial^2\phi}{\partial u^1\partial u^3}\right)\mathbf{e}_{u^2} \\ & + \frac{1}{h_1h_2}\left(\frac{\partial^2\phi}{\partial u^1\partial u^2} - \frac{\partial^2\phi}{\partial u^2\partial u^1}\right)\mathbf{e}_{u^3} = 0\,,\end{aligned}$$

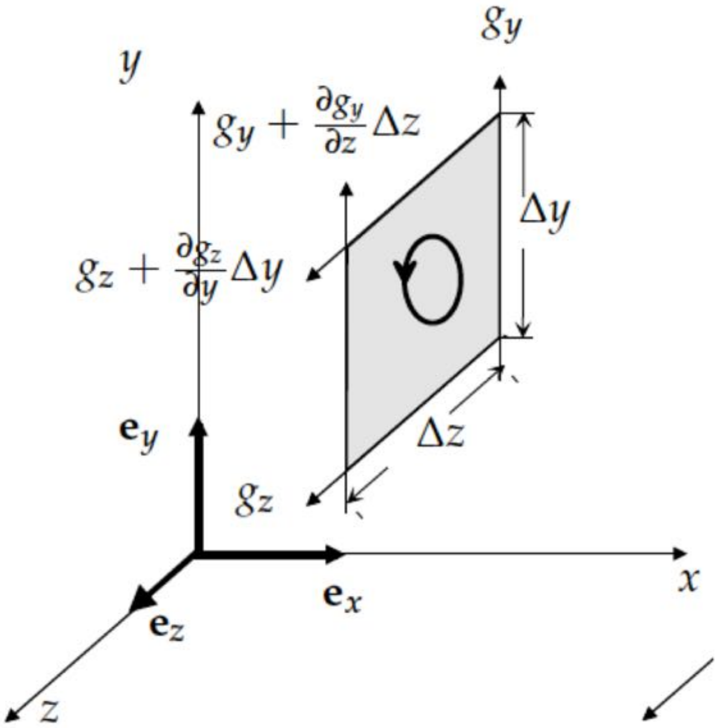
Ugly and not even a universal proof (orthogonality assumed)

Coordinate Free Definition of Grad, Curl, and Div

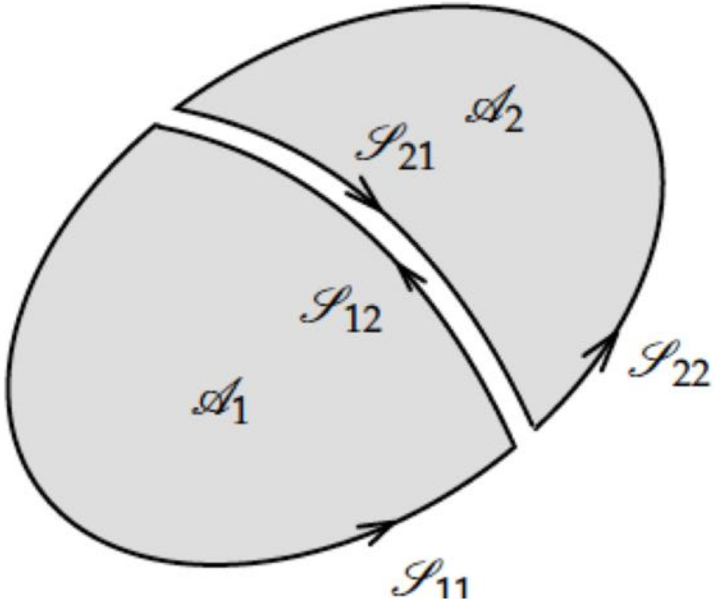
$$\int_{\mathscr{P}_1}^{\mathscr{P}_2} \mathbf{a}\cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{P}_1}^{\mathscr{P}_2} \operatorname{grad}\phi\cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{P}_1}^{\mathscr{P}_2} \mathrm{d}\phi = \phi(\mathscr{P}_2)-\phi(\mathscr{P}_1),$$

$$\mathbf{n}\cdot \operatorname{curl}\mathbf{g} = \lim_{a\to 0}\frac{\int_{\partial\mathscr{A}}\mathbf{g}\cdot\mathrm{d}\mathbf{r}}{a},$$

$$\operatorname{div}\,\mathbf{g} = \lim_{V\to 0}\frac{\int_{\partial\mathscr{V}}\mathbf{g}\cdot\mathrm{d}\mathbf{a}}{V},$$

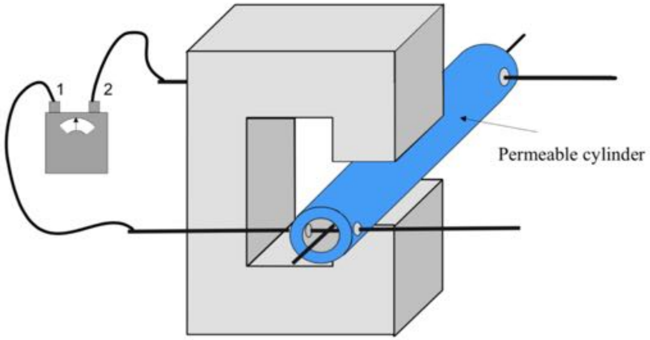


Kelvin-Stokes Theorem



Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

No jump discontinuities (for example, co-moving shielding devices)

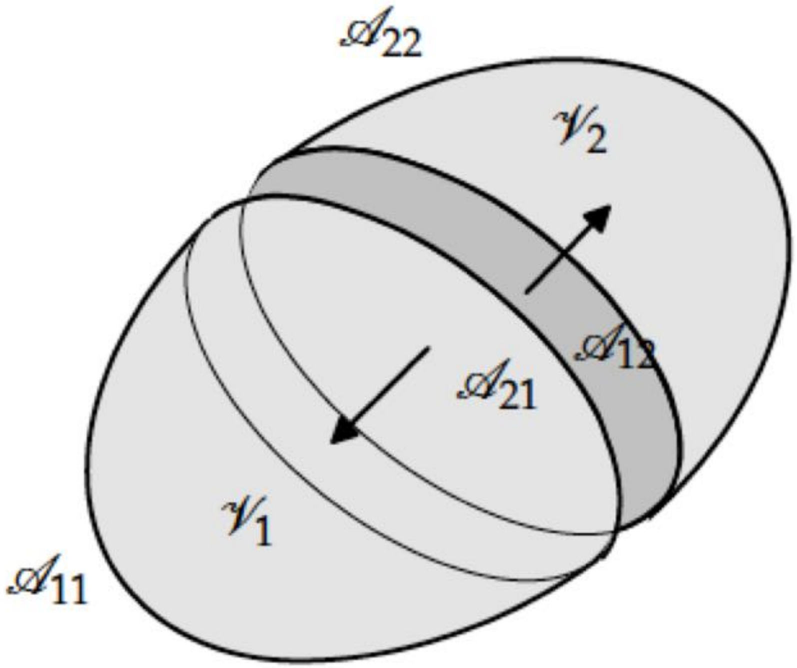


$$\int_{\partial\mathscr{A}} \mathbf{g}\cdot\mathrm{d}\mathbf{r} = \int_{\mathscr{S}_1} \mathbf{g}\cdot\mathrm{d}\mathbf{r} + \int_{\mathscr{S}_2} \mathbf{g}\cdot\mathrm{d}\mathbf{r} = \int_{\mathscr{S}_{11}} \mathbf{g}\cdot\mathrm{d}\mathbf{r} + \int_{\mathscr{S}_{22}} \mathbf{g}\cdot\mathrm{d}\mathbf{r},$$

$$\int_{\partial\mathscr{A}} \mathbf{g}\cdot\mathrm{d}\mathbf{r} = \lim_{I\to\infty}\sum_{i=1}^{I}\int_{\partial\mathscr{A}_i} \mathbf{g}\cdot\mathrm{d}\mathbf{r} = \lim_{I\to\infty}\sum_{i=1}^{I}\Delta a_i \frac{1}{\Delta a_i}\int_{\partial\mathscr{A}_i} \mathbf{g}\cdot\mathrm{d}\mathbf{r}$$

$$= \lim_{I\to\infty}\sum_{i=1}^{I}(\operatorname{curl}\mathbf{g})_i\cdot\mathbf{n}\,\Delta a_i = \int_{\mathscr{A}} \operatorname{curl}\mathbf{g}\cdot\mathrm{d}\mathbf{a}\,.$$

Gauss' Theorem



Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

$$\int_{\partial\mathscr{V}} \mathbf{g}\cdot\mathbf{da} = \lim_{I\to\infty}\sum_{i=1}^{I}\int_{\partial\mathscr{V}_i}\mathbf{g}\cdot\mathbf{da} = \lim_{I\to\infty}\sum_{i=1}^{I}\Delta V_i\frac{1}{\Delta V_i}\int_{\partial\mathscr{V}_i}\mathbf{g}\cdot\mathbf{da}$$

$$= \lim_{I\to\infty}\sum_{i=1}^{I}(\operatorname{div}\mathbf{g})_i\,\Delta V_i = \int_{\mathscr{V}}\operatorname{div}\mathbf{g}\,\mathrm{d}V\,.$$

The Boundary Operator

$$\partial(\partial\mathscr{V}) = \emptyset\,, \qquad \partial(\partial\mathscr{A}) = \emptyset\,,$$

$$\int_{\mathscr{V}} \operatorname{div}\operatorname{curl}\mathbf{g}\,\mathrm{d}V = \int_{\partial\mathscr{V}} \operatorname{curl}\mathbf{g}\cdot\mathrm{d}\mathbf{a} = \int_{\partial(\partial\mathscr{V})} \mathbf{g}\cdot\mathrm{d}\mathbf{r} = 0\,,$$

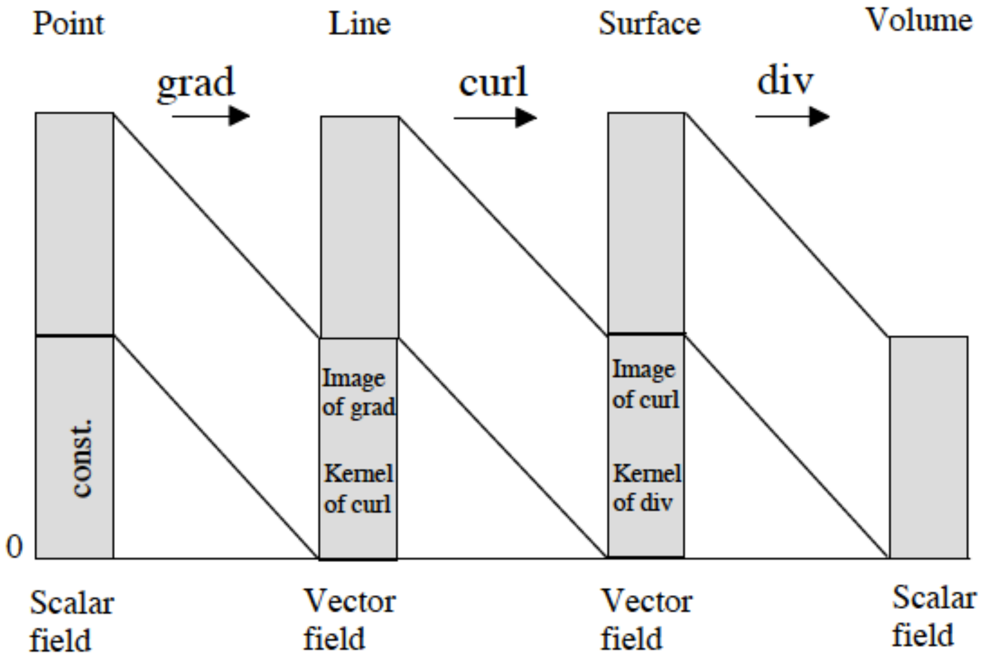
$$\int_{\mathscr{A}} \operatorname{curl}\operatorname{grad}\phi\cdot\mathrm{d}\mathbf{a} = \int_{\partial\mathscr{A}} \operatorname{grad}\phi\cdot\mathrm{d}\mathbf{r} = \phi|_{\partial(\partial\mathscr{A})} = 0\,,$$

Reversal of arguments yields two important statements (next slides):
Much nicer than writing it in coordinates

The second Lemma of Poincare (Contractible Domains)

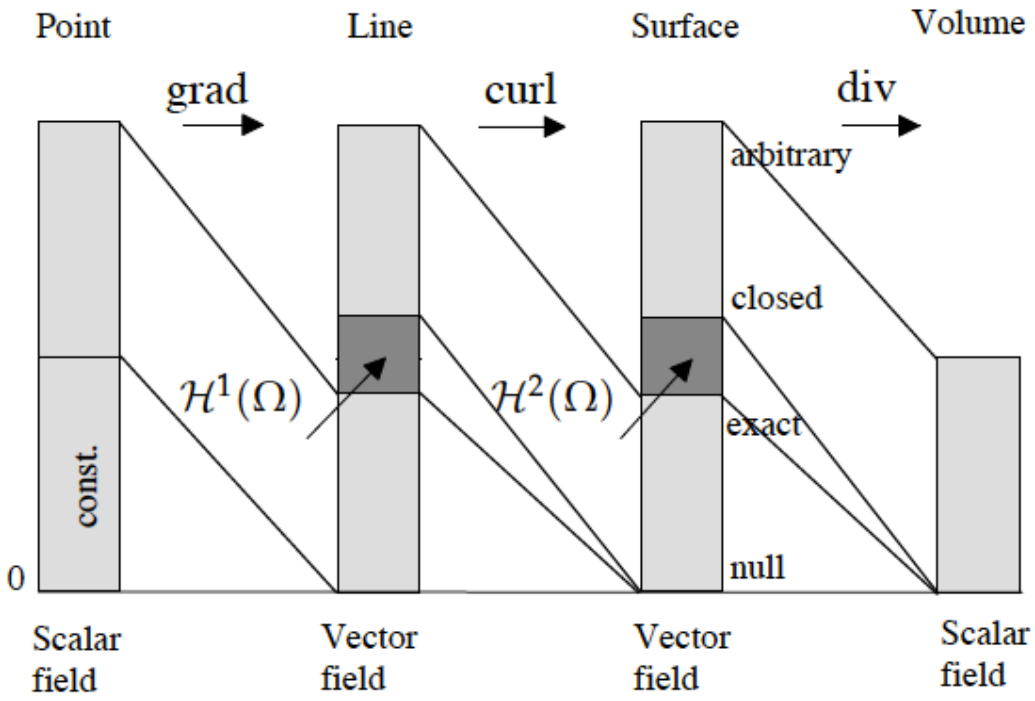
$$\operatorname{div} \mathbf{b} = 0 \quad \rightarrow \quad \mathbf{b} = \operatorname{curl} \mathbf{a}.$$

$$\operatorname{curl} \mathbf{h} = 0 \quad \rightarrow \quad \mathbf{h} = \operatorname{grad} \phi.$$



Lemmata of Poincare (Non-Contractible Domains)

$$\mathcal{H}^1(\Omega) := \frac{\ker(\,\mathrm{curl}\,)}{\mathrm{im}(\,\mathrm{grad}\,)}$$



$$\mathcal{H}^2(\Omega) := \frac{\ker(\mathrm{div})}{\mathrm{im}(\,\mathrm{curl}\,)}$$

Toroidal domain Ω in a cylindrical coordinate system (r, φ, z):

$$H_\varphi = \frac{I}{2\pi r}$$

$$\mathrm{curl}\,\mathbf{H} = \frac{1}{r}\frac{\partial}{\partial r}(rH_\varphi) = 0$$

But $\oint_C \mathbf{H} \cdot \mathrm{ds} = I$ and Ω, with $\oint_C \mathrm{grad}\,\phi \cdot \mathrm{ds} = 0$

Domain Ω between two nested spheres centered at the origin.

$$D_R = \frac{Q}{4\pi R^2}\mathbf{e}_R$$

$$\mathrm{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial R}(R^2 D_R) = 0$$

But $\oint_a \mathbf{D} \cdot \mathrm{da} = Q$ and $\oint_a \mathrm{curl}\,\mathbf{A} \cdot \mathrm{da} = 0$

Maxwell's Equations in Differential Form

$$\int_{\mathscr{A}} \operatorname{curl} \mathbf{g} \cdot \mathrm{d}\mathbf{a} = \int_{\partial\mathscr{A}} \mathbf{g} \cdot \mathrm{d}\mathbf{r},$$

$$\int_{\mathscr{V}} \operatorname{div} \mathbf{g} \, \mathrm{d}V = \int_{\partial\mathscr{V}} \mathbf{g} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{A}} \mathbf{H} \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{A}} \mathbf{J} \cdot \mathrm{d}\mathbf{a} + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{A}} \mathbf{D} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{A}} \mathbf{E} \cdot \mathrm{d}\mathbf{r} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{A}} \mathbf{B} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{V}} \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0,$$

$$\int_{\partial\mathscr{V}} \mathbf{D} \cdot \mathrm{d}\mathbf{a} = \int_{\mathscr{V}} \rho \, \mathrm{d}V.$$

$$\int_{\mathscr{A}} \operatorname{curl} \mathbf{H} \cdot \mathrm{d}\mathbf{a} = \int_{\mathscr{A}} \left(\mathbf{J} + \frac{\partial}{\partial t} \mathbf{D} \right) \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\mathscr{A}} \operatorname{curl} \mathbf{E} \cdot \mathrm{d}\mathbf{a} = -\int_{\mathscr{A}} \frac{\partial}{\partial t} \mathbf{B} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\mathscr{V}} \operatorname{div} \mathbf{B} \, \mathrm{d}V = 0,$$

$$\int_{\mathscr{V}} \operatorname{div} \mathbf{D} \, \mathrm{d}V = \int_{\mathscr{V}} \rho \, \mathrm{d}V.$$

$$\operatorname{curl} \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t} \mathbf{D},$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B},$$

$$\operatorname{div} \mathbf{B} = 0,$$

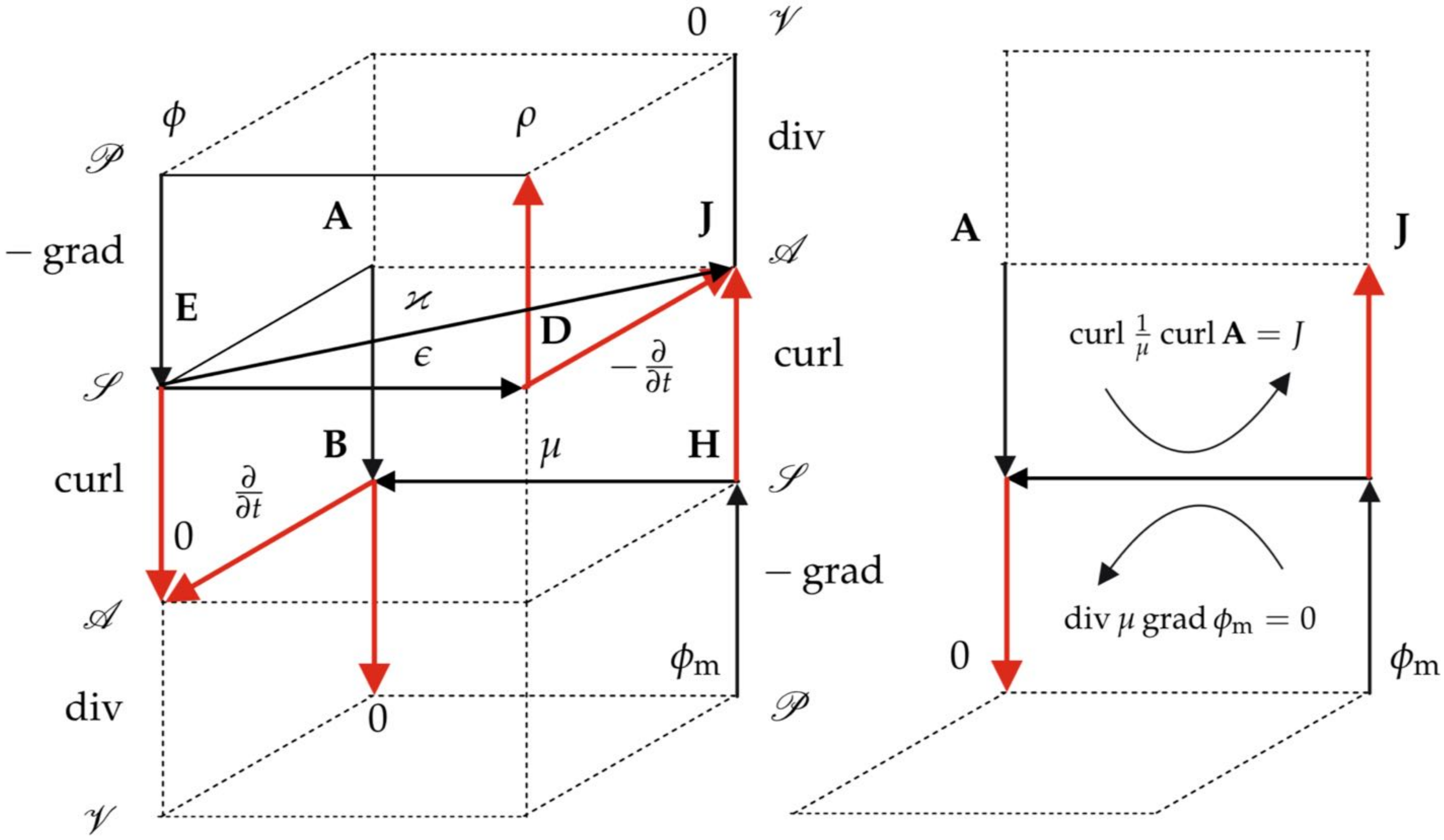
$$\operatorname{div} \mathbf{D} = \rho.$$

Maxwell’s House

Faraday complex
Inner oriented

Ampere-Maxwell complex
Outer oriented

curl **B** is ugly !



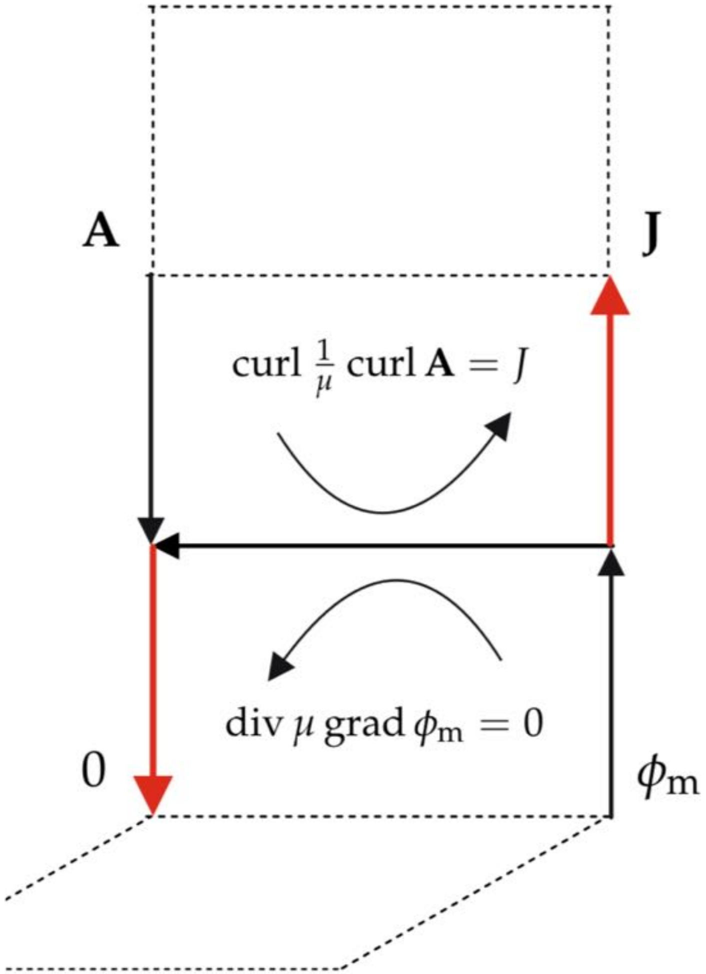
Maxwell’s Facade

$$\operatorname{curl}\frac{1}{\mu}\operatorname{curl}\mathbf{A}=\mathbf{J}$$

$$\frac{1}{\mu_0}\operatorname{curl}\operatorname{curl}\mathbf{A}=\mathbf{J}$$

$$\nabla^2\mathbf{A}-\operatorname{grad}\operatorname{div}\mathbf{A}=0$$

$$\nabla^2 A_z=0$$

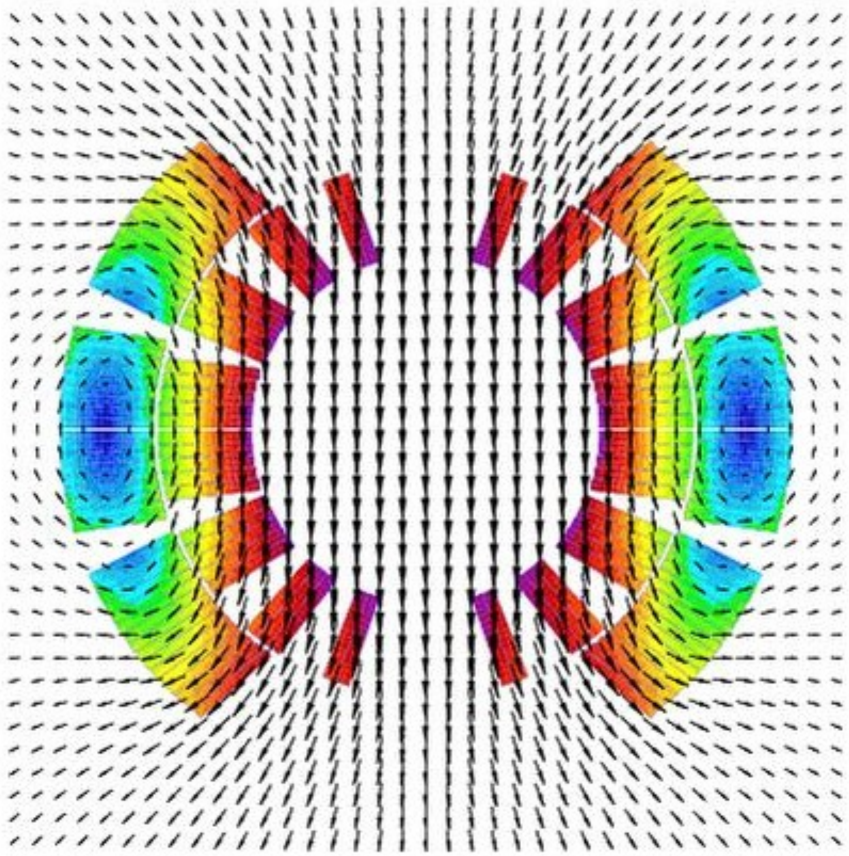


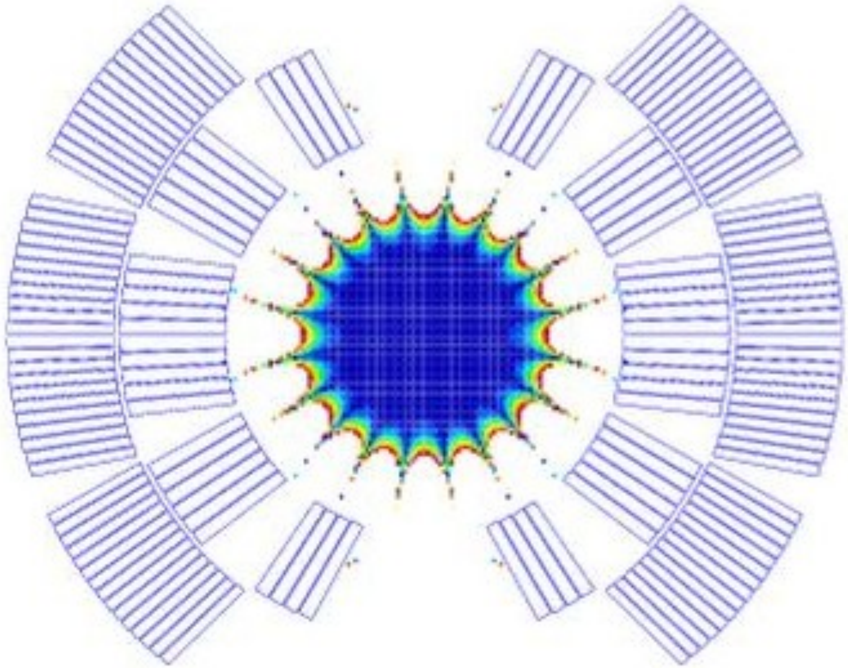
$$\operatorname{div}\mu\operatorname{grad}\phi_{\mathrm{m}}=0$$

$$\mu_0\operatorname{div}\operatorname{grad}\phi_{\mathrm{m}}=0$$

$$\nabla^2\phi_{\mathrm{m}}=0$$

Field Quality





Field map

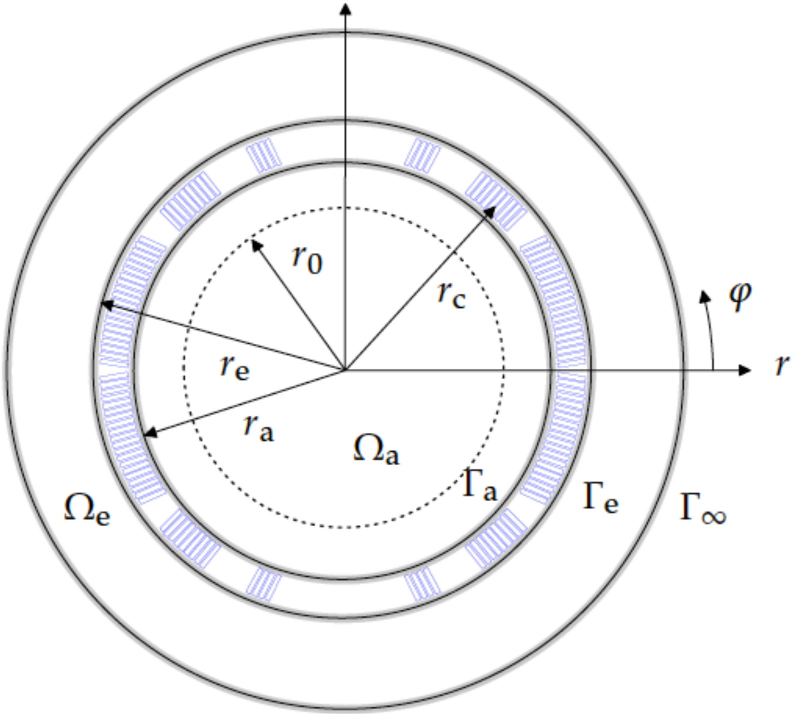
Good field region

Solving of Boundary Value Problems

1. Governing equation in the air domain

$$\nabla^2 A_z = 0,$$

2. Chose a suitable coordinate system



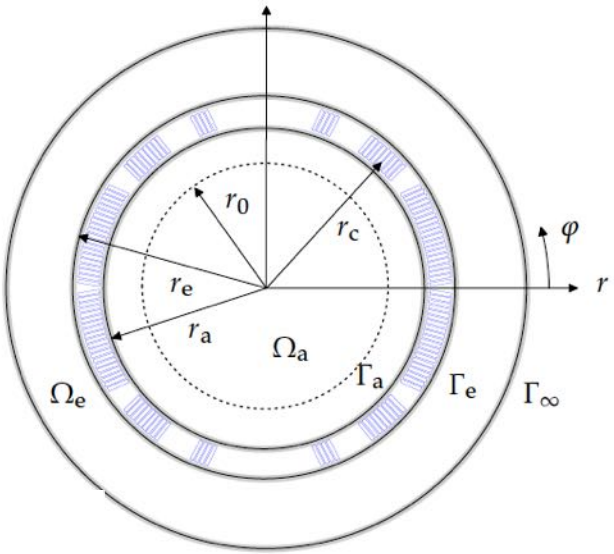
$$r^2\frac{\partial^2 A_z}{\partial r^2} + r\frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0,$$

3. Make a guess, look it up in a book, use the method of separation:
   That is: find eigenfunctions. Coefficients are not know yet

$$A_z(r,\varphi) = \sum_{n=1}^{\infty}(\mathcal{E}_n r^n + \mathcal{F}_n r^{-n})(\mathcal{G}_n \sin n\varphi + \mathcal{H}_n \cos n\varphi).$$

Solving of Boundary Value Problems

4. Incorporate a bit of knowledge and rename

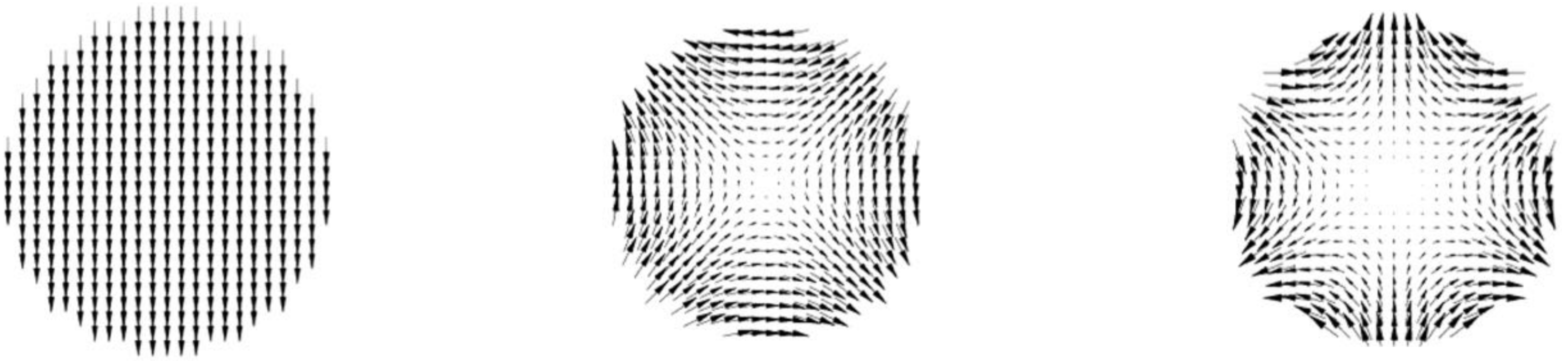


$$A_z(r,\varphi) = \sum_{n=1}^{\infty} r^n (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi).$$

5. Calculate a field component

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

$$B_\varphi(r,\varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi),$$

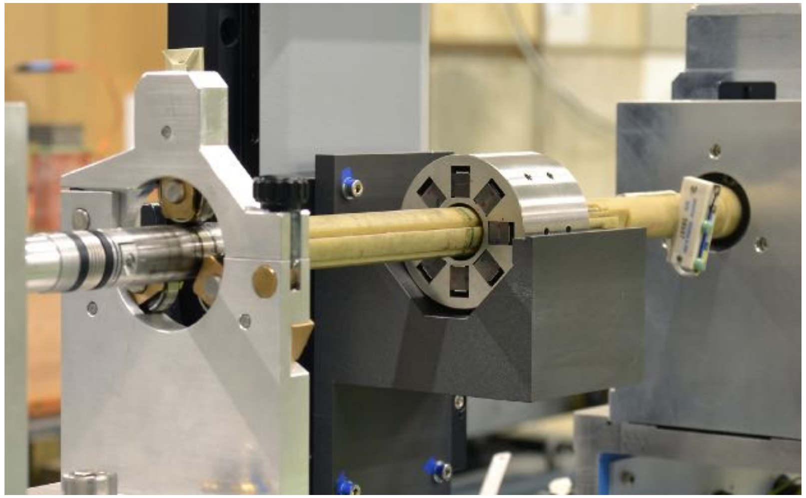


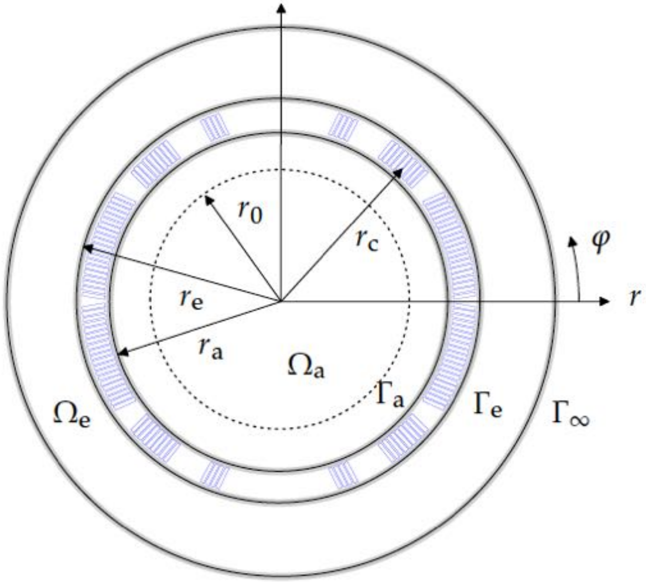
Solving of Boundary Value Problems

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1}(\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

6. Measure or calculate the field on a reference radius and perform Fourier analysis (develop into the eigenfunctions). Coefficients known here.

$$B_r(r_0,\varphi) = \sum_{n=1}^{\infty} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi),$$





Solving the Boundary Value Problem

7: Compare the known and unknown coefficients

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1}(\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

$$B_r(r_0,\varphi) = \sum_{n=1}^{\infty} (B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi),$$

$$\mathcal{A}_n = \frac{1}{n\, r_0^{n-1}} A_n(r_0)\,, \qquad \mathcal{B}_n = \frac{-1}{n\, r_0^{n-1}} B_n(r_0)\,.$$

8. Put this into the original solution for the entire air domain

$$A_z(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r_0}{n}\left(\frac{r}{r_0}\right)^n (B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi).$$

Solving the Boundary Value Problem

9: Calculate fields and potential in the entire air domain

$$A_z(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r_0}{n}\left(\frac{r}{r_0}\right)^n \left(B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi\right).$$

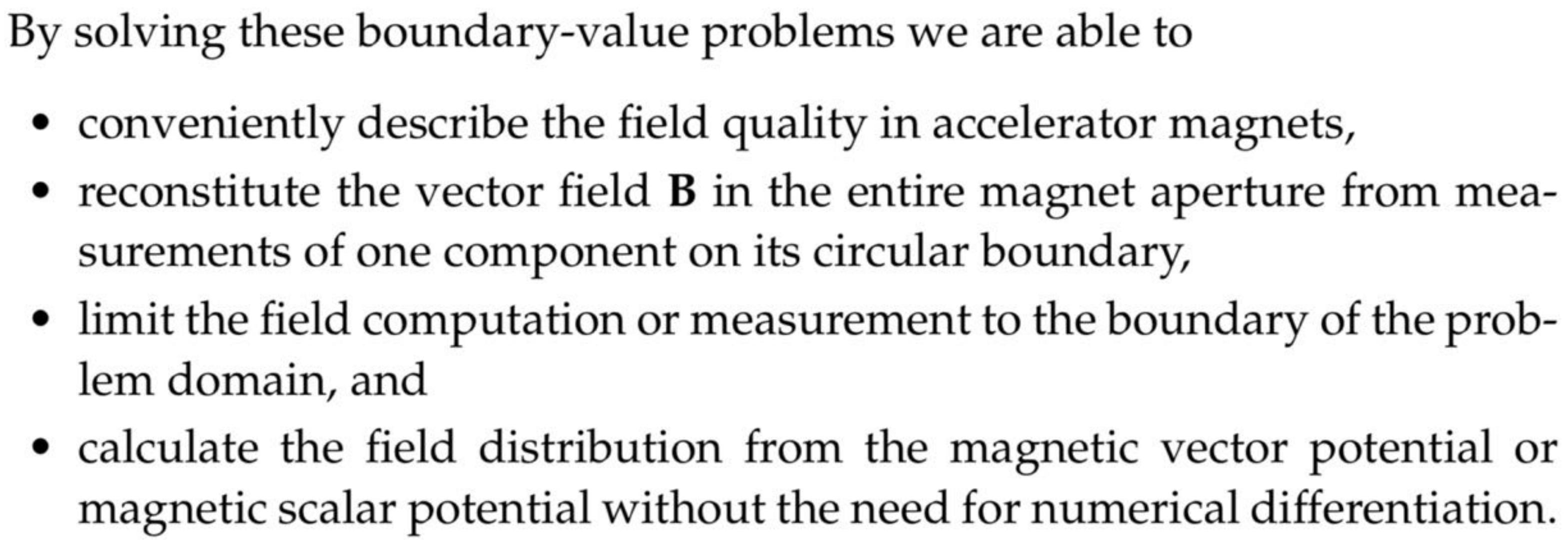
$$B_r(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \left(B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi\right)$$

$$B_\varphi(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \left(B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi\right)$$

$$B_x(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \left(B_n(r_0)\sin(n-1)\varphi + A_n(r_0)\cos(n-1)\varphi\right)$$

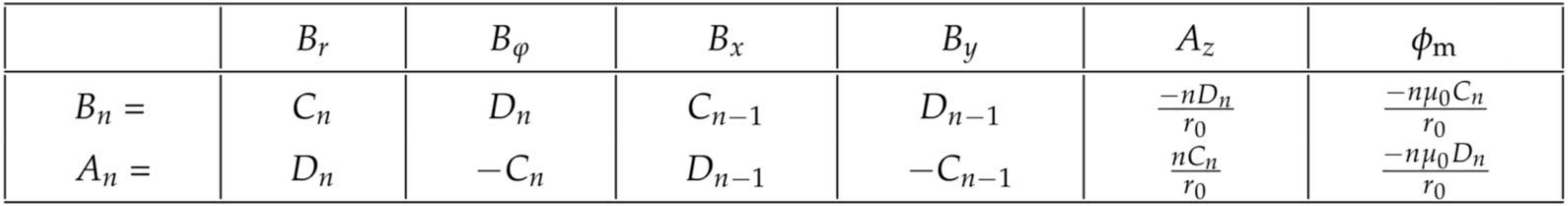
$$B_y(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \left(B_n(r_0)\cos(n-1)\varphi - A_n(r_0)\sin(n-1)\varphi\right)$$

Conclusion



By solving these boundary-value problems we are able to

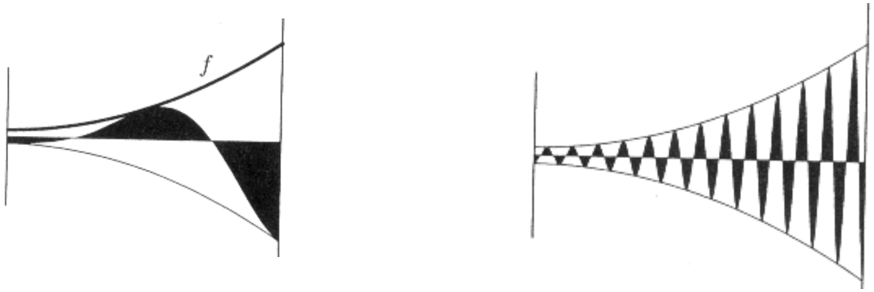
- conveniently describe the field quality in accelerator magnets,
- reconstitute the vector field **B** in the entire magnet aperture from measurements of one component on its circular boundary,
- limit the field computation or measurement to the boundary of the problem domain, and
- calculate the field distribution from the magnetic vector potential or magnetic scalar potential without the need for numerical differentiation.



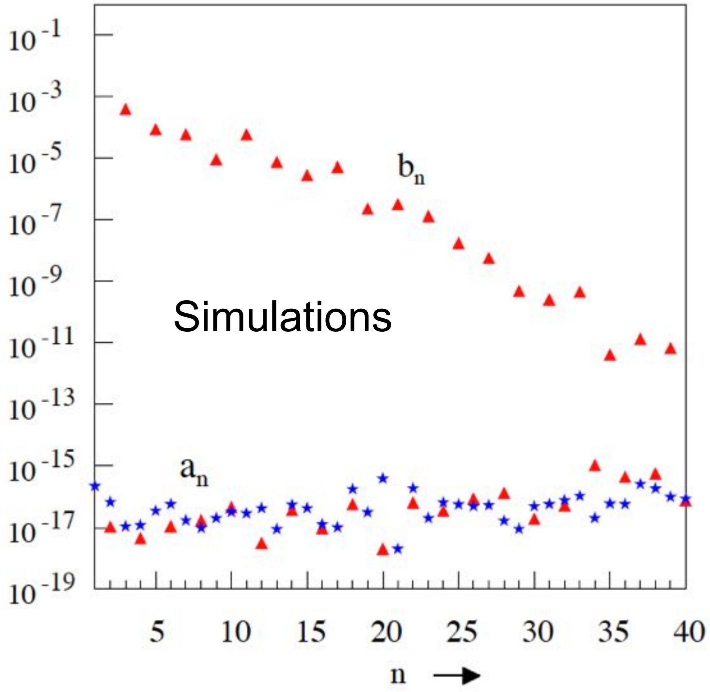
	B_r	B_φ	B_x	B_y	A_z	ϕ_{m}
$B_n =$	C_n	D_n	C_{n-1}	D_{n-1}	$\frac{-nD_n}{r_0}$	$\frac{-n\mu_0 C_n}{r_0}$
$A_n =$	D_n	$-C_n$	D_{n-1}	$-C_{n-1}$	$\frac{nC_n}{r_0}$	$\frac{-n\mu_0 D_n}{r_0}$

The Riemann Lebesque Lemma

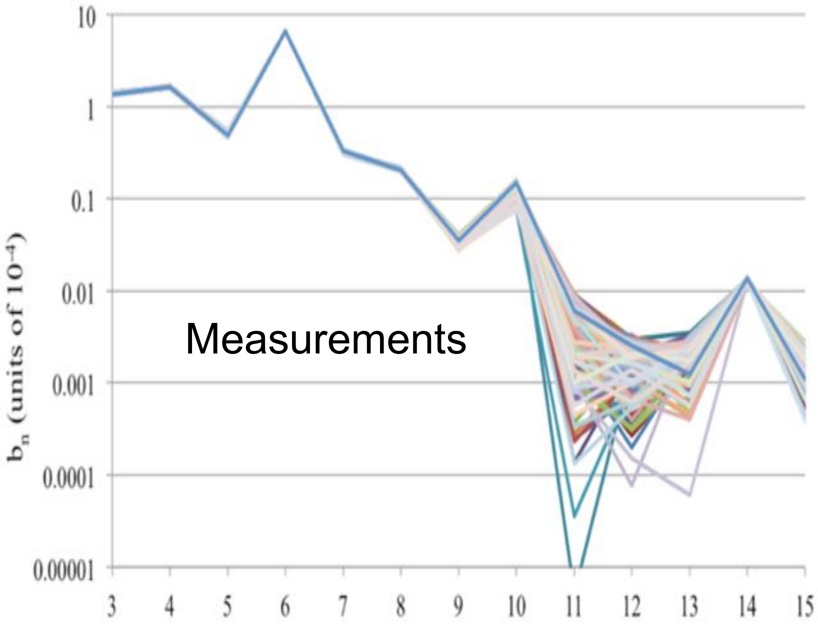
The Fourier coefficients tend to zero as n goes to infinity



Limits: 10^{-6} T, 10^{-8} Vs



Simulations



Measurements

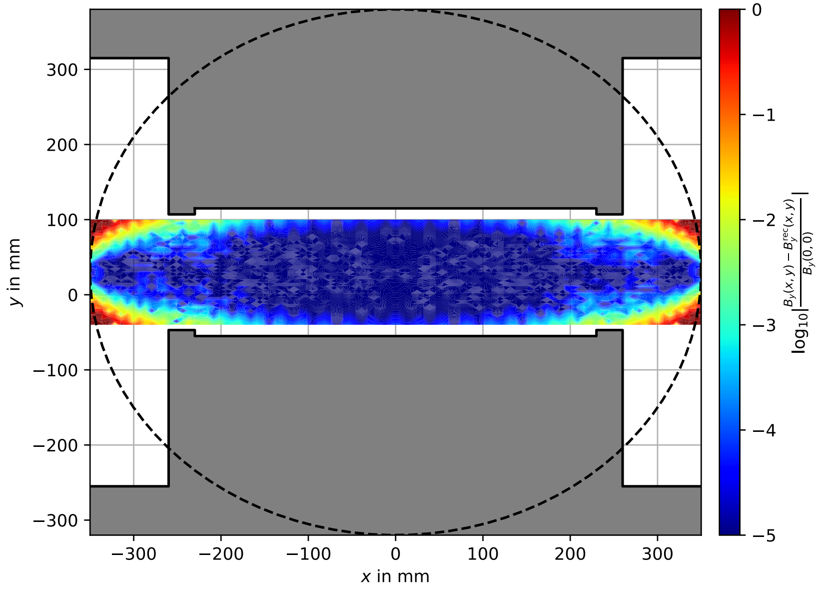
Always plot your results in logarithmic scale

Maclaurin Series

$$B_y(x) = B_0 + \left.\frac{\mathrm{d}B_y}{\mathrm{d}x}\right|_{x=y=0} x + \cdots + \frac{1}{n!}\left.\frac{\mathrm{d}^n B_y}{\mathrm{d}x^n}\right|_{x=y=0} x^n + \cdots$$

$$B_y(r,\varphi) = \sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^{n-1}\left(B_n(r_0)\cos(n-1)\varphi - A_n(r_0)\sin(n-1)\varphi\right)$$

$$b_n(r_0) = \frac{{r_0}^{n-1}}{B_N}\frac{1}{(n-1)!}\left.\frac{\mathrm{d}^{n-1}B_y}{\mathrm{d}x^{n-1}}\right|_{x=y=0}$$



Ideal Pole Shape of Conventional Magnets

Remember the Cauchy Schwarz inequality $| \langle \mathbf{a}, \mathbf{b} \rangle | \leq \| \mathbf{a} \| \| \mathbf{b} \|,$

Thus for the directional derivative $|\partial_{\mathbf{v}} \phi| \leq |\operatorname{grad} \phi| |\mathbf{v}|$

The directional derivative takes its maximum when **v** points in the direction of the gradient. Therefore the gradient points in the direction of the steepest ascent of *Φ* and is thus normal to the surface of equipotential.

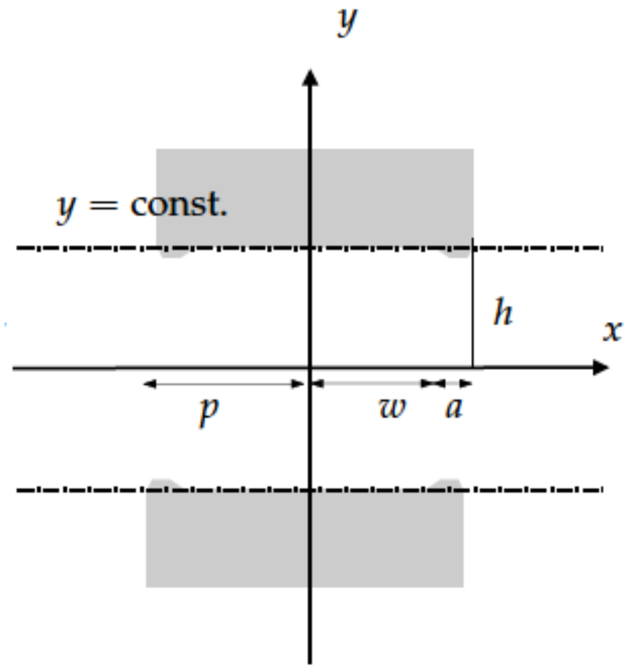
The flux density **B** exits a highly permeable surface in normal direction. Therefore the pole shape of normal conducting magnets can be seen as an equipotential of the magnetic scalar potential.

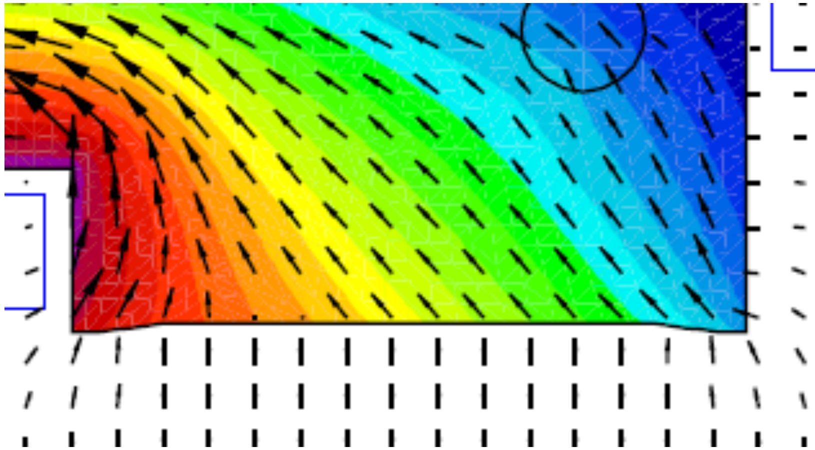
Ideal Pole Shape of Conventional Magnets

$$\phi_{\mathrm{m}}(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r^n}{\mu_0}(\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi).$$

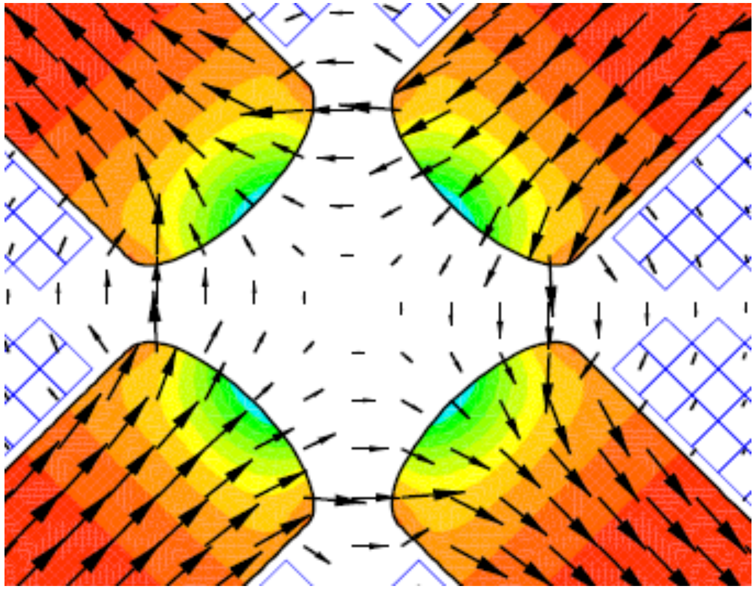
$$\phi_{\mathrm{m}}(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r_0}{n\mu_0}\left(\frac{r}{r_0}\right)^n (A_n(r_0)\cos n\varphi + B_n(r_0)\sin n\varphi).$$

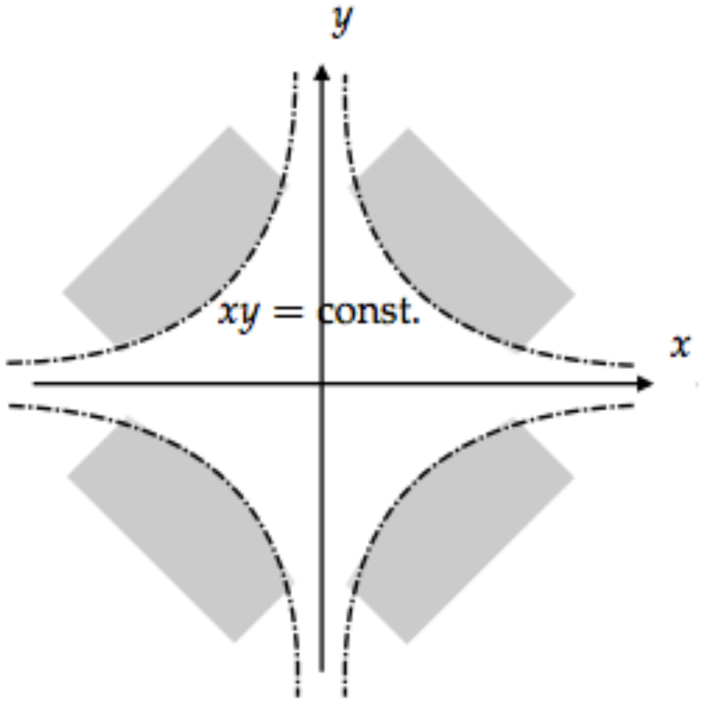
$$\phi_{\mathrm{m}}(x,y) = -\frac{1}{\mu_0}(B_1 y + A_1 x)\,.$$





Ideal Pole Shape of Conventional Magnets





$$\phi_{\mathrm{m}}(r,\varphi) = \frac{1}{2\mu_0 r_0}\left(B_2(r_0)\,2xy + A_2(r_0)\,(x^2 - y^2)\right).$$

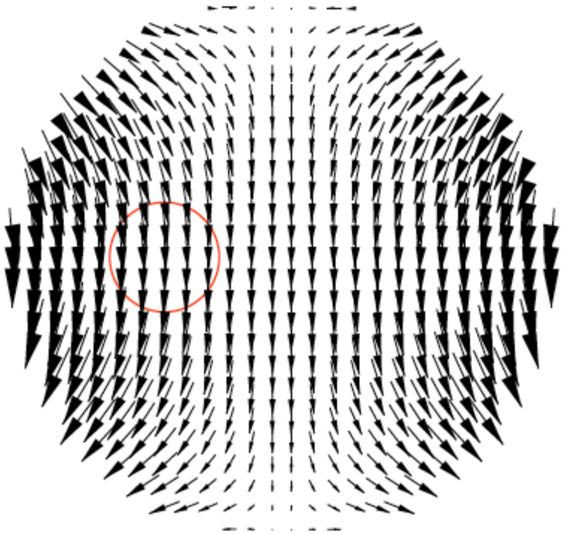
Complex Representation of the Field in Accelerator Magnets

$$B_x = B_r \cos\varphi - B_\varphi \sin\varphi, \qquad B_y = B_r \sin\varphi + B_\varphi \cos\varphi,$$

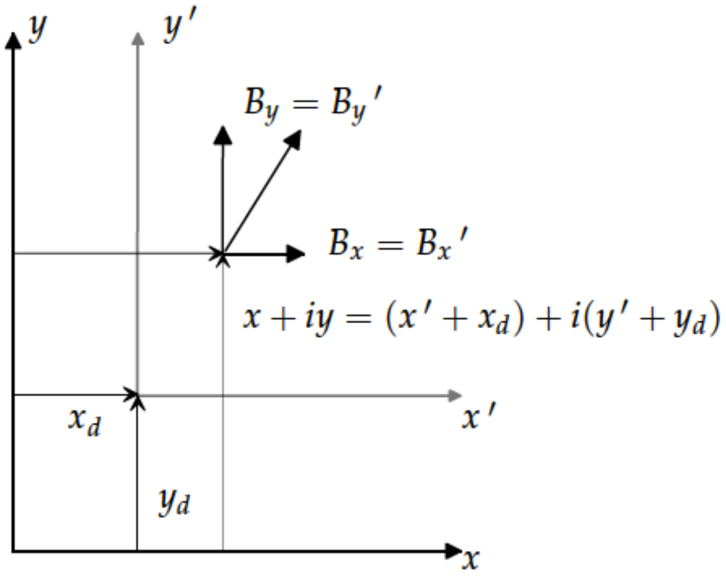
$$B_y + iB_x = (B_\varphi + iB_r)e^{-i\varphi}.$$

$$\begin{aligned} B_y + iB_x &= \sum_{n=1}^{\infty} (B_n(r_0) + i\,A_n(r_0)) \left(\frac{r}{r_0}\right)^{n-1} e^{i(n-1)\varphi} \\ &= \sum_{n=1}^{\infty} (B_n(r_0) + i\,A_n(r_0)) \left(\frac{z}{r_0}\right)^{n-1} \\ &= B_N \sum_{n=1}^{\infty} (b_n(r_0) + i\,a_n(r_0)) \left(\frac{z}{r_0}\right)^{n-1}, \end{aligned}$$

Feed-down: Proof

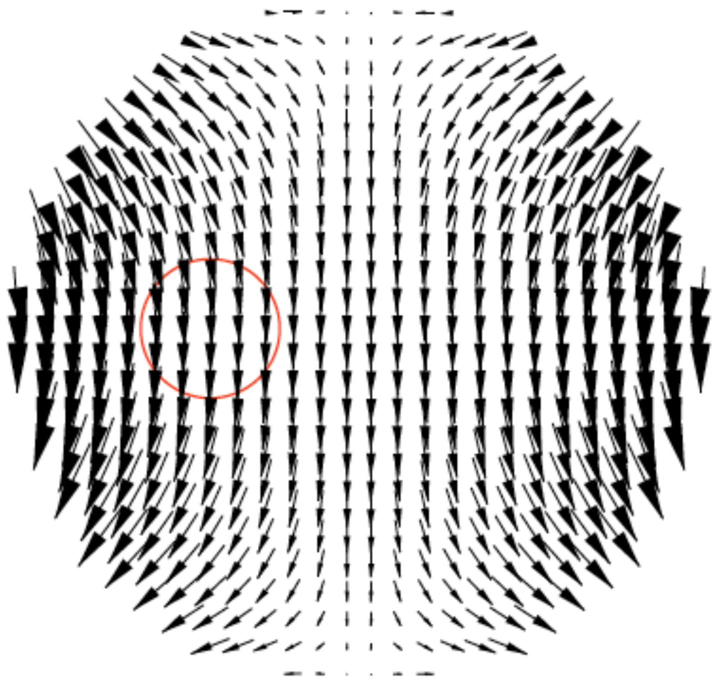


$$\begin{aligned}\sum_{n=1}^{\infty} C_n \left(\frac{z}{r_0}\right)^{n-1} &= \sum_{n=1}^{\infty} \frac{C_n}{{r_0}^{n-1}} \left(z' + z_\mathrm{d}\right)^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{C_n}{{r_0}^{n-1}} \sum_{k=1}^{n} \binom{n-1}{k-1} (z')^{k-1} {z_\mathrm{d}}^{n-k} \\ &= \sum_{k=1}^{\infty} \left[\sum_{n=1}^{n} \frac{C_n}{{r_0}^{n-1}} \binom{n-1}{k-1} {z_\mathrm{d}}^{n-k} \right] (z')^{k-1} \\ &= \sum_{k=1}^{\infty} \left[\sum_{n=k}^{\infty} C_n \binom{n-1}{k-1} \left(\frac{z_\mathrm{d}}{r_0}\right)^{n-k} \right] \left(\frac{z'}{r_0}\right)^{k-1} \\ &= \sum_{n=1}^{\infty} \left[\sum_{k=n}^{\infty} C_k \binom{k-1}{n-1} \left(\frac{z_\mathrm{d}}{r_0}\right)^{k-n} \right] \left(\frac{z'}{r_0}\right)^{n-1} .\end{aligned}$$

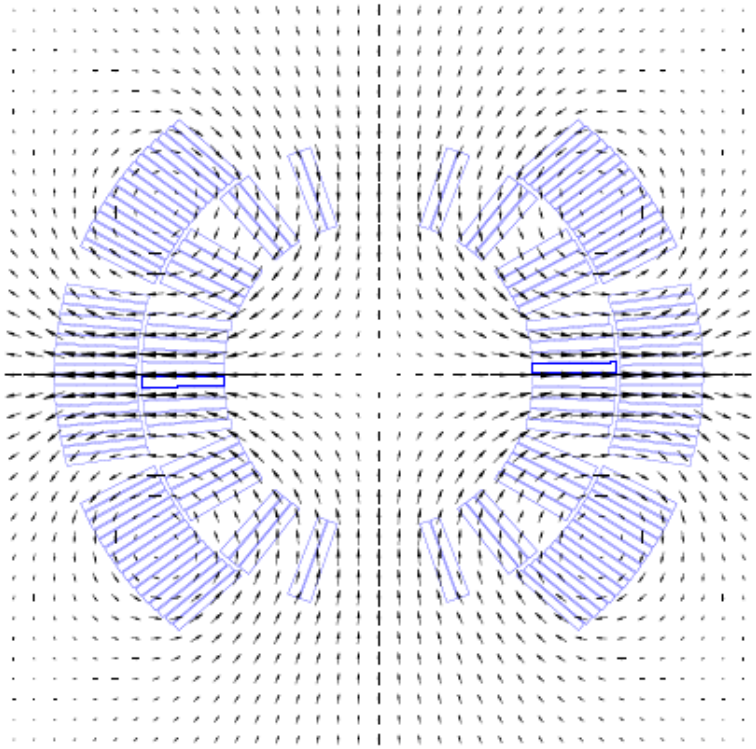


$$C_2' = C_2 + 2\,C_3 \left(\frac{z_\mathrm{d}}{r_0}\right) + 3\,C_4 \left(\frac{z_\mathrm{d}}{r_0}\right)^2 + \cdots ,$$

Feed-down: Enemy and Friend

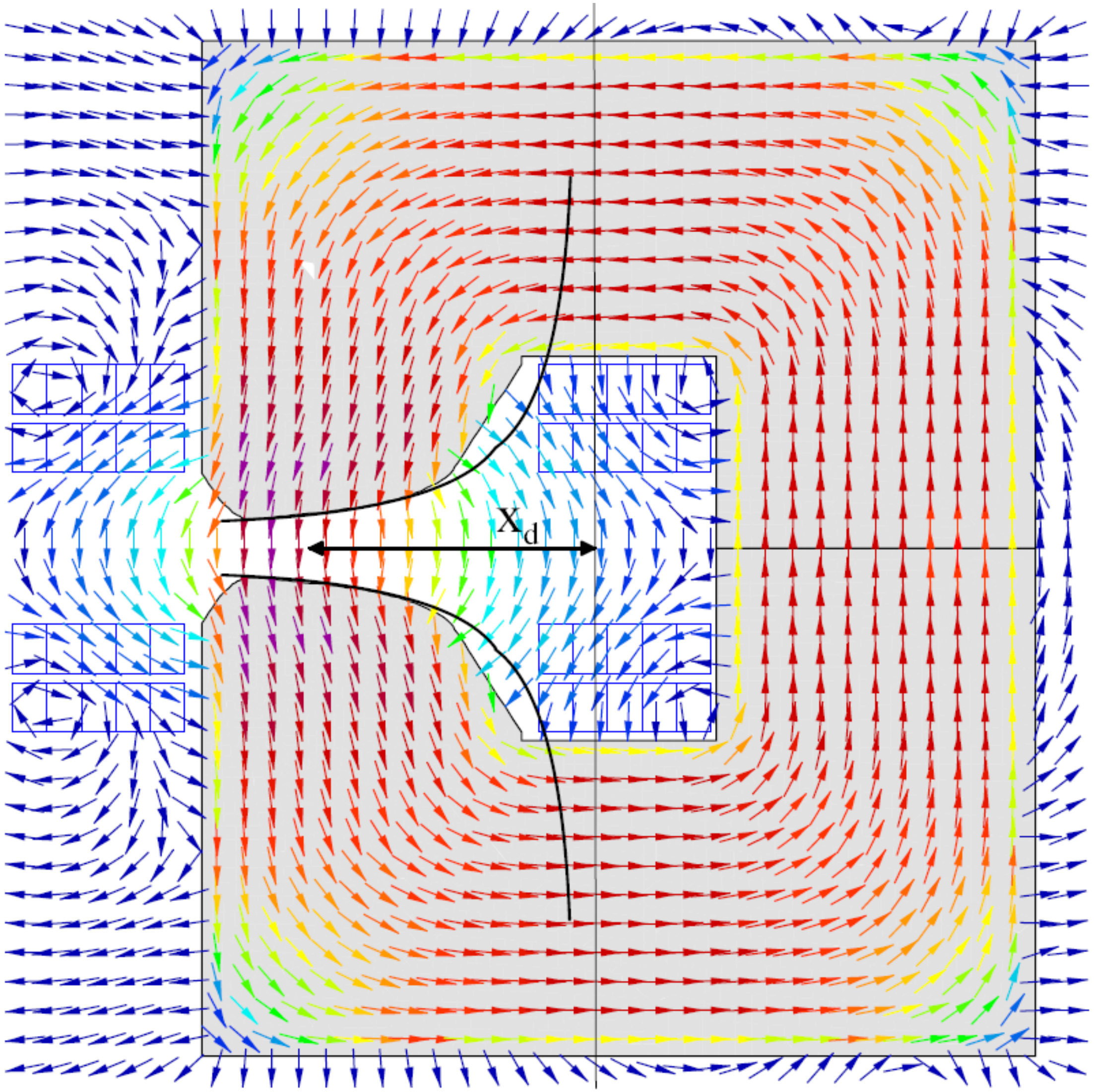


- ➔ Measurement of magnetic axis in dipole by powering the coil as a quadrupole
- ➔ Feed-down can be used to center the measurement coil
  - – Minimizing B_{10} which can only occur as feed-down from B_{11}



- ➔ Alignment tolerances of MCS and MCDO correctors w.r.t. MB
  - – 0.3 mm radially
- ➔ Dipole magnetic axis has to be well aligned with respect to the closed orbit
  - – ± 0.1 mm systematic, ± 0.5 mm random (r.m.s)

Combined Function Magnets



Gradient: 5 T/m
Dipole field: 1.5 T
Displacement: 0.33 m

$$x' = x - x_{\mathrm{d}}$$

$$B_y(x') = 2\mathcal{D}_2 x' + 2\mathcal{D}_2 x_{\mathrm{d}}$$

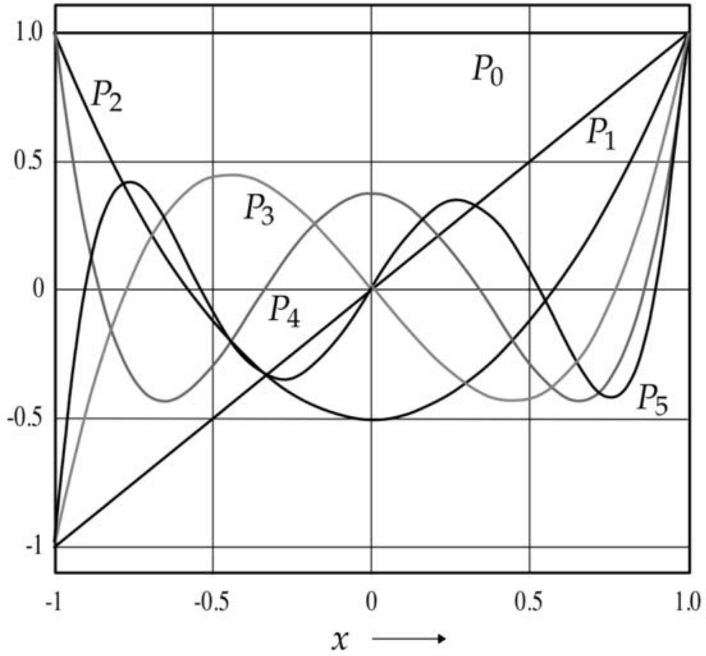
Zonal Harmonics (Solenoids)

$$\nabla^2\phi_\mathrm{m} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi_\mathrm{m}}{\partial R}\right) + \frac{1}{R^2\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\left(\sin(\vartheta)\frac{\partial\phi_\mathrm{m}}{\partial\vartheta}\right) = 0.$$

$$x = \cos(\vartheta)$$

$$\mathrm{d}x = -\sin(\vartheta)\,\mathrm{d}\vartheta.$$

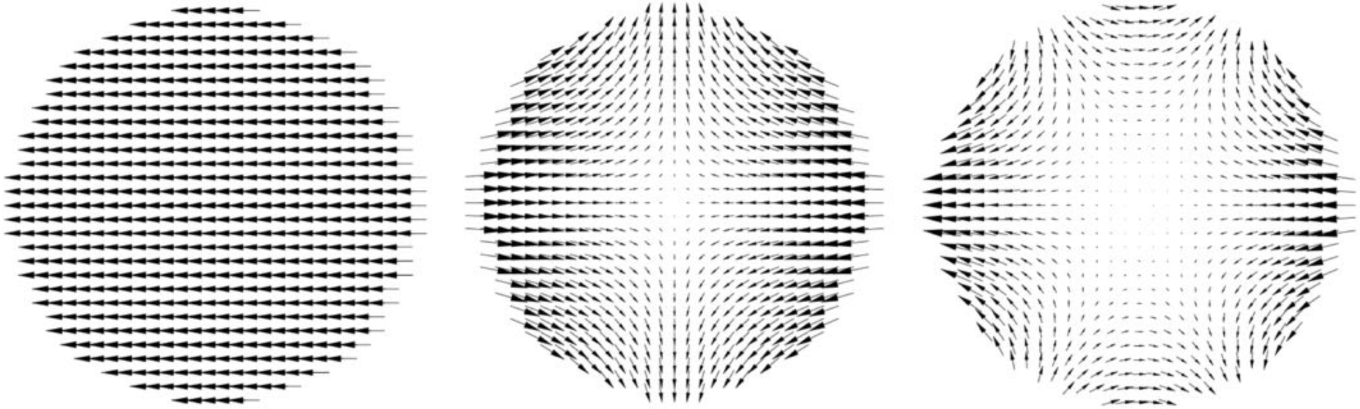
$$\phi_\mathrm{m}(R,x) = \sum_{n=0}^{\infty} \mathcal{A}_n R^n P_n(x)$$



$$B_R = -\mu_0 \sum_{n=0}^{\infty} \frac{\partial}{\partial R}\left(\mathcal{A}_n R^n P_n(\cos(\vartheta))\right) = -\mu_0 \sum_{n=1}^{\infty} \mathcal{A}_n n R^{n-1} P_n(\cos(\vartheta)).$$

$$B_\vartheta = -\mu_0 \sum_{n=0}^{\infty} \frac{1}{R}\frac{\partial}{\partial\vartheta}\left(\mathcal{A}_n R^n P_n(\cos(\vartheta))\right) = \mu_0 \sum_{n=1}^{\infty} \mathcal{A}_n R^{n-1}\, P_n^1(\cos(\vartheta)),$$

Zonal Harmonics



$$\mathcal{A}_n(R_0) = \frac{2n+1}{2}\int_0^{\pi} B_R(R_0,\vartheta)P_n(\cos(\vartheta))\sin(\vartheta)\,\mathrm{d}\vartheta$$

$$\mathcal{A}_n = -\frac{\mathcal{A}_n(R_0)}{\mu_0 n R_0^{n-1}}$$

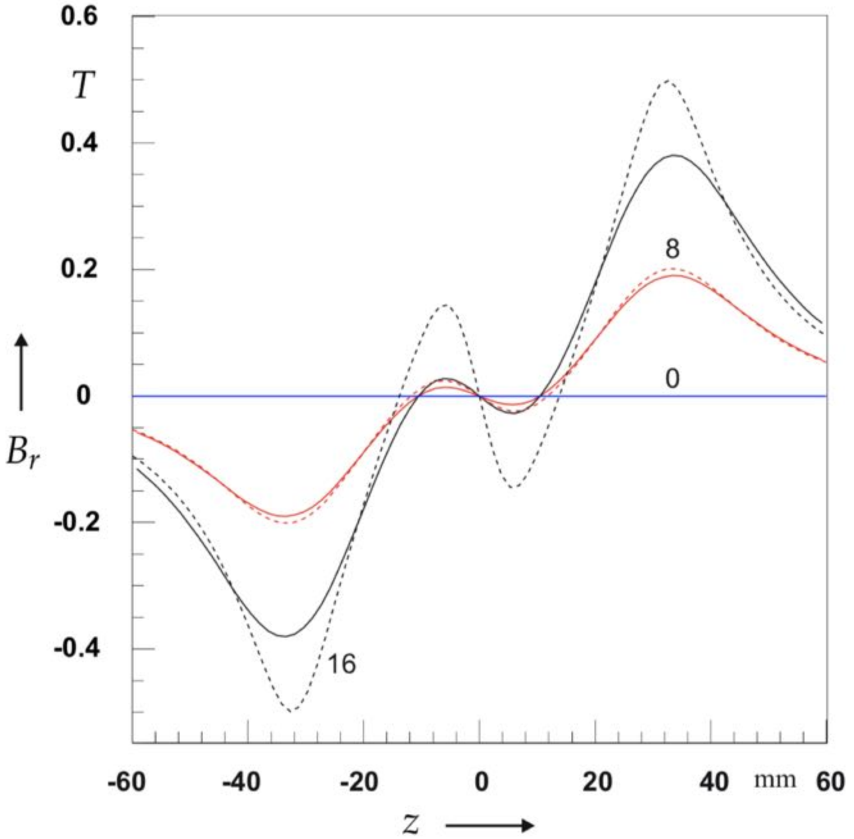
$$B_z(z) = \sum_{n=0}^{\infty}\frac{1}{n!}B_z^{(n)}(z_0)(z-z_0)^n.$$

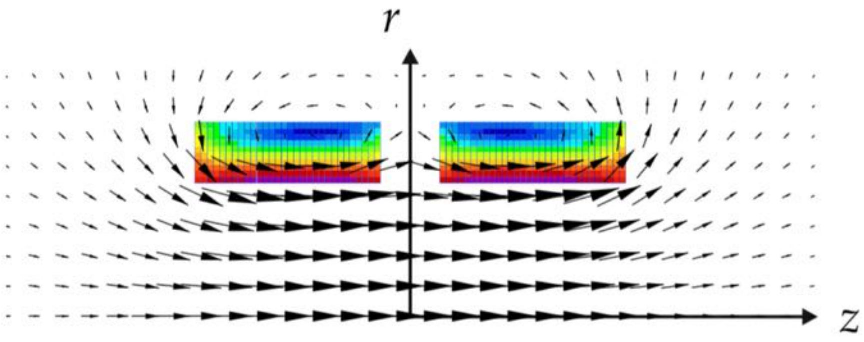
$$\mathcal{A}_n = \frac{-B_z^{(n-1)}(z_0)}{\mu_0\, n\,(n-1)!}.$$

Zonal Harmonics

$$B_r(r, z_0) = \mu_0 \left(r\mathcal{A}_2 - \frac{3}{2} r^3 \mathcal{A}_4 - \cdots \right)$$

$$= -\frac{r}{2} \frac{\mathrm{d}B_z(r=0)}{\mathrm{d}z}\bigg|_{z=z_0} + \frac{r^3}{16} \frac{\mathrm{d}^3 B_z(r=0)}{\mathrm{d}z^3}\bigg|_{z=z_0} - \cdots$$





Cartesian Coordinates (Eigensolutions for the Ideal Dipole)

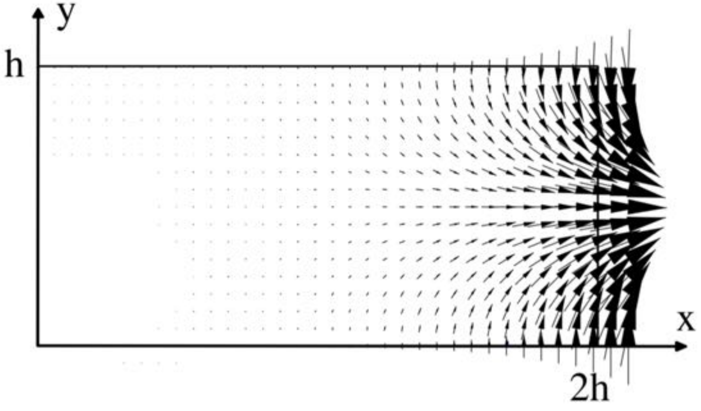
$$\phi_{\mathrm{m}} = X(x)Y(y)$$

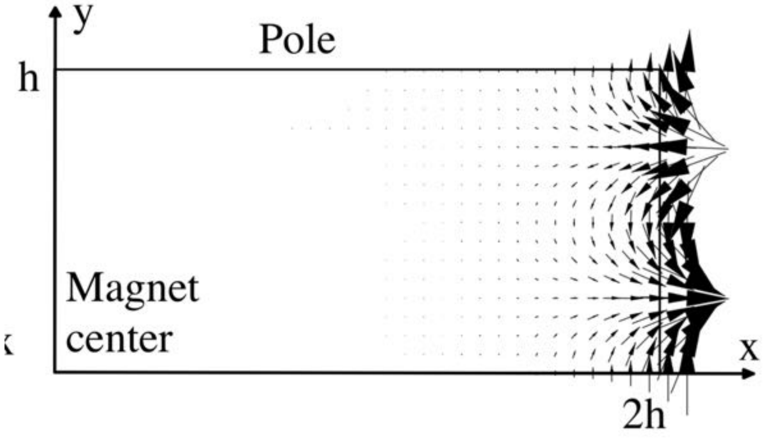
$$\underbrace{\frac{1}{X(x)}\frac{\mathrm{d}^2X(x)}{\mathrm{d}x^2}}_{p^2} + \underbrace{\frac{1}{Y(y)}\frac{\mathrm{d}^2Y(y)}{\mathrm{d}y^2}}_{-p^2} = 0$$

$$X_p(x) = \mathcal{C}_p \cos px + \mathcal{D}_p \sin px\,,$$
$$Y_p(y) = \mathcal{E}_p \cosh py + \mathcal{F}_p \sinh py\,,$$

$$p = n\,\frac{2\pi}{\lambda} =: nk_0\,.$$

$$B_x(x,y) = \mu_0 \sum_{n=1}^{\infty} \mathcal{A}_n \sinh\left(\frac{n\pi}{h}x\right)\,\sin\left(\frac{n\pi}{h}y\right)\,,$$
$$B_y(x,y) = B_0 + \mu_0 \sum_{n=1}^{\infty} \mathcal{A}_n \cosh\left(\frac{n\pi}{h}x\right)\,\cos\left(\frac{n\pi}{h}y\right)\,.$$

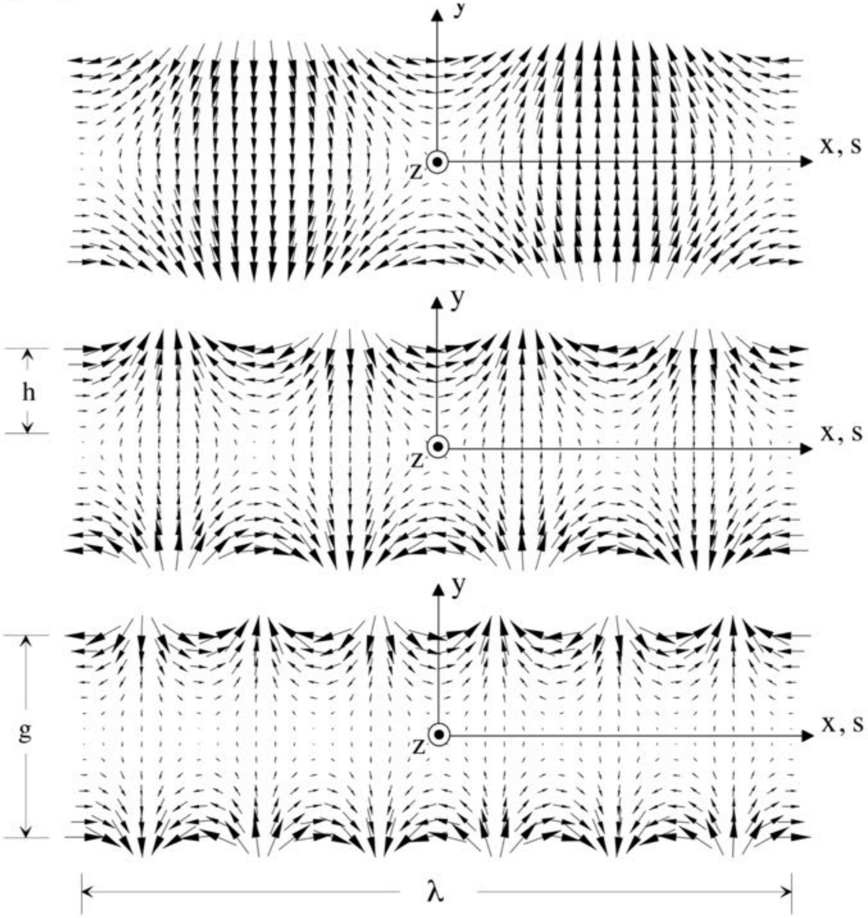




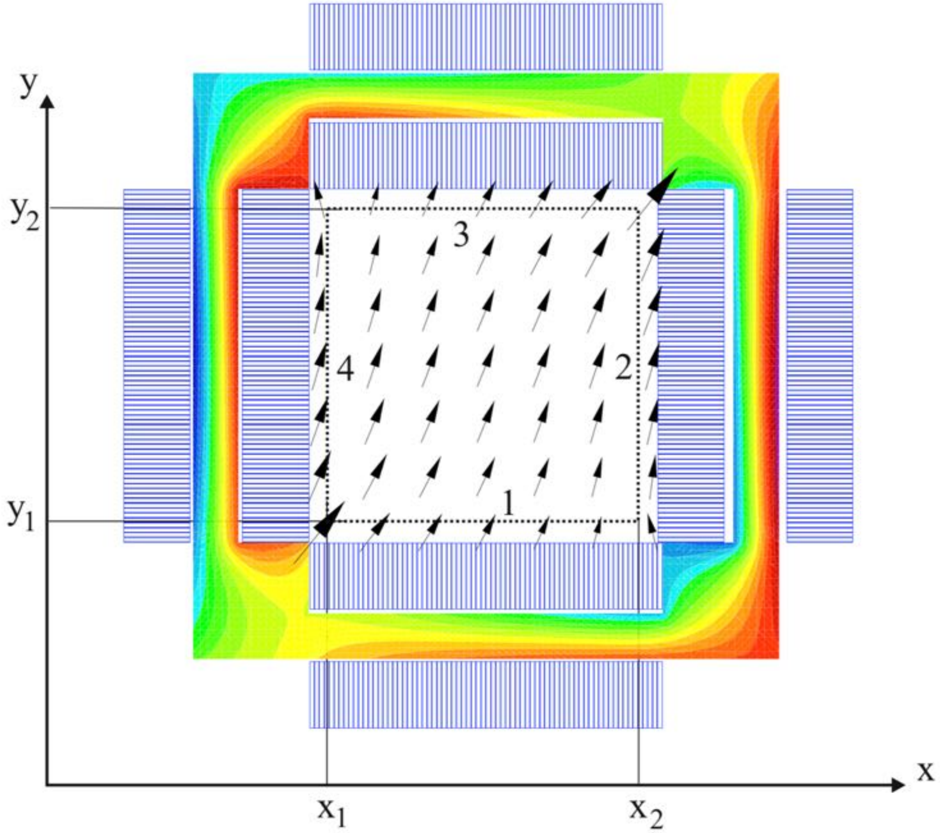
Cartesian Coordinates (Eigensolutions for the Wiggler)

$$B_x(x,y) = \mu_0 \sum_{n=1}^{\infty} \left(-\mathcal{A}_n \sin(nk_0 x) + \mathcal{B}_n \cos(nk_0 x)\right) \sinh(nk_0 y) \ ,$$

$$B_y(x,y) = \mu_0 \sum_{n=1}^{\infty} \left(\mathcal{A}_n \cos(nk_0 x) + \mathcal{B}_n \sin(nk_0 x)\right) \cosh(nk_0 y) \ .$$

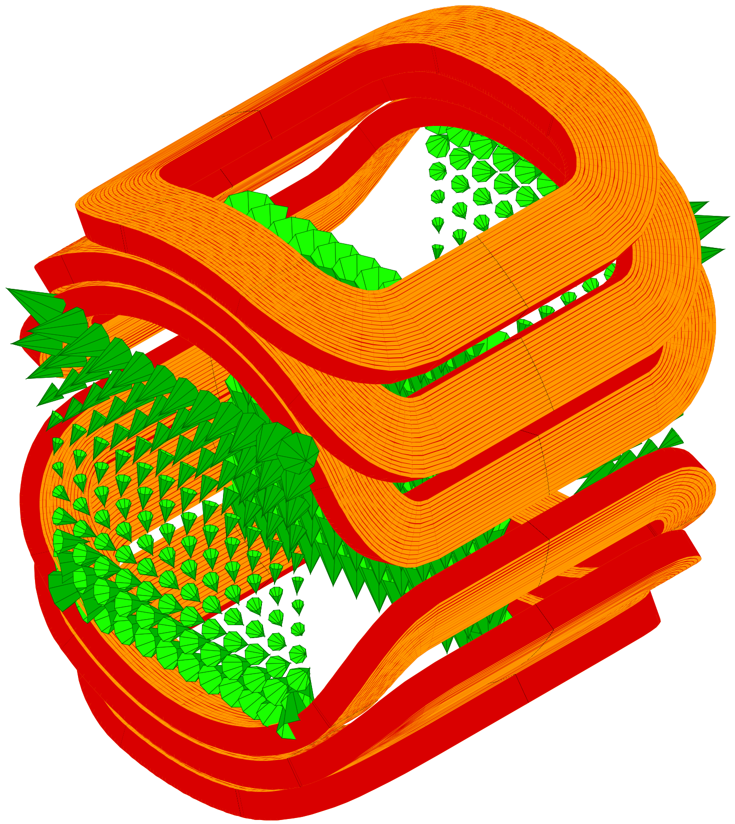


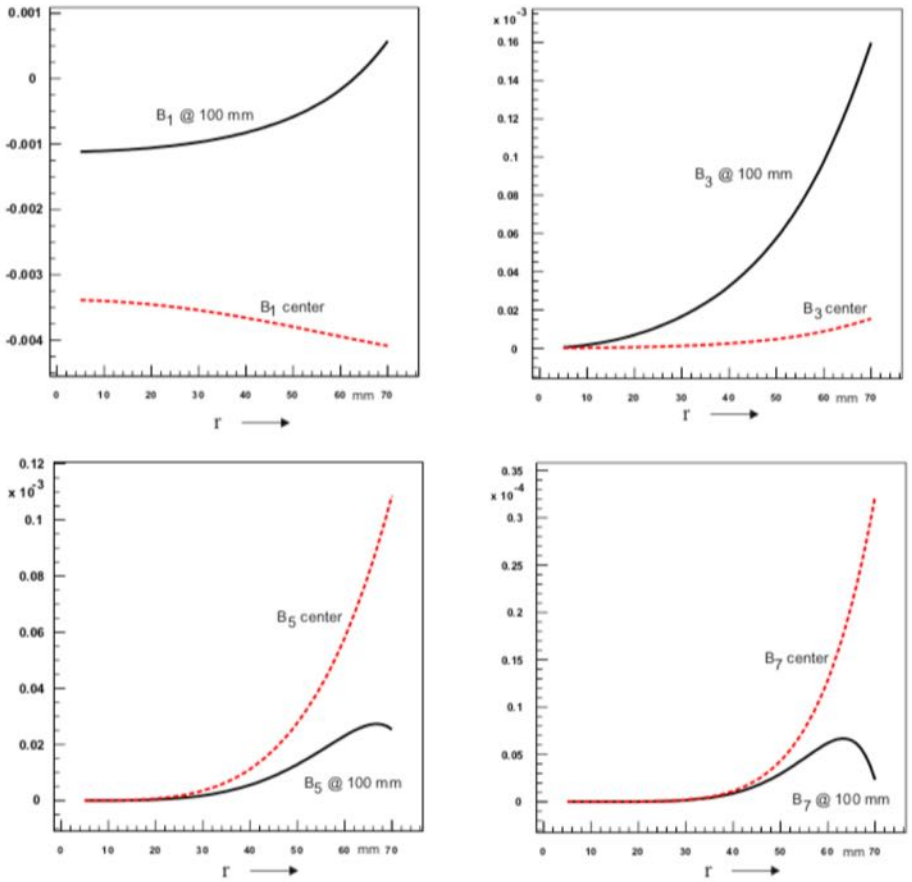
Determining the Coefficients



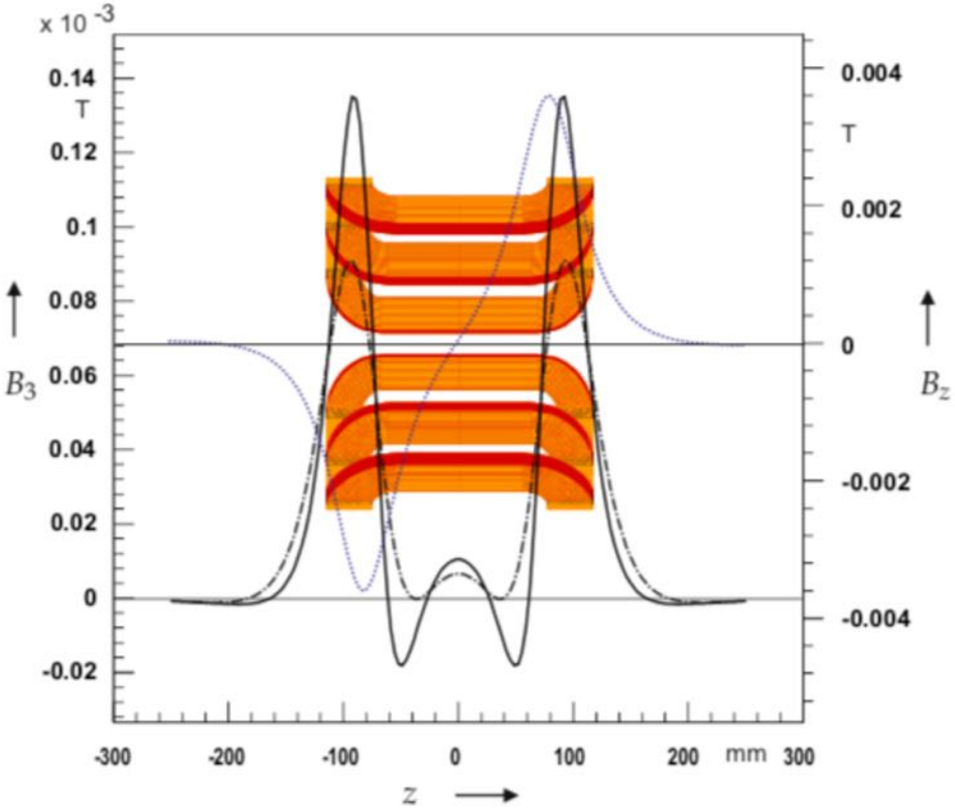
$$A_z^{(1)}(x,y) = \sum_n A_n^{(1)} \frac{\sinh\left(n\pi \frac{y_2-y}{x_2-x_1}\right)}{\sinh\left(n\pi \frac{y_2-y_1}{x_2-x_1}\right)} \sin\left(n\pi \frac{x_2-x}{x_2-x_1}\right)$$

3D Field Harmonics





Integrated Harmonics



Local transverse harmonics calculated at different reference radii and scaled with the 2D laws

$$b_n(r_1) \cdot = \left(\frac{r_1}{r_0}\right)^{n-N} b_n(r_0),$$

wrong

Integrated Harmonics

$$\nabla^2 \phi_{\mathrm{m}}(x,y,z) = \frac{\partial^2 \phi_{\mathrm{m}}(x,y,z)}{\partial x^2} + \frac{\partial^2 \phi_{\mathrm{m}}(x,y,z)}{\partial y^2} + \frac{\partial^2 \phi_{\mathrm{m}}(x,y,z)}{\partial z^2} = 0\,.$$

$$\overline{\phi}_{\mathrm{m}}(x,y) := \int_{-z_0}^{z_0} \phi_{\mathrm{m}}(x,y,z)\mathrm{dz}.$$

$$\begin{aligned}\frac{\partial^2 \overline{\phi}_{\mathrm{m}}(x,y)}{\partial x^2} + \frac{\partial^2 \overline{\phi}_{\mathrm{m}}(x,y)}{\partial y^2} &= \int_{-z_0}^{z_0} \left(\frac{\partial^2 \phi_{\mathrm{m}}}{\partial x^2} + \frac{\partial^2 \phi_{\mathrm{m}}}{\partial y^2} \right) \mathrm{dz} \\ &= \int_{-z_0}^{z_0} \left(-\frac{\partial^2 \phi_{\mathrm{m}}}{\partial z^2} \right) \mathrm{dz} = - \left. \frac{\partial \phi_{\mathrm{m}}}{\partial z} \right|_{-z_0}^{z_0} \\ &= H_z(-z_0) - H_z(z_0) \overset{!}{=} 0\,.\end{aligned}$$

The 2D scaling laws hold for the integrated harmonics

Pseudo-Multipoles (Fourier Bessel Series)

$$\phi_{\mathrm{m}}(r,\varphi,z) = \left\{ \begin{array}{c} \cos n\varphi \\ \sin n\varphi \end{array} \right\} I_n(pr) \left\{ \begin{array}{c} \cos pz \\ \sin pz \end{array} \right\}$$

$$I_n(pr) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+n+1)} \left(\frac{pr}{2}\right)^{n+2k}$$

$$\phi_{\mathrm{m}} = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} r^{n+2k} (\mathcal{C}_{n+2k,n}(z) \sin n\varphi + \mathcal{D}_{n+2k,n}(z) \cos n\varphi)$$

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\Big\{\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}(n+2k)r^{n+2k}\left(\mathcal{C}_{n+2k,n}(z)\sin n\varphi+\mathcal{D}_{n+2k,n}(z)\cos n\varphi\right)\Big\}\\
&\qquad-\frac{1}{r^2}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}n^2\,r^{n+2k}\left(\mathcal{C}_{n+2k,n}(z)\sin n\varphi+\mathcal{D}_{n+2k,n}(z)\cos n\varphi\right)\\
&\qquad+\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}r^{n+2k}\left(\mathcal{C}^{(2)}_{n+2k,n}(z)\sin n\varphi+\mathcal{D}^{(2)}_{n+2k,n}(z)\cos n\varphi\right)\\
=&\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}(n+2k)^2\,r^{n+2k-2}\left(\mathcal{C}_{n+2k,n}(z)\sin n\varphi+\mathcal{D}_{n+2k,n}(z)\cos n\varphi\right)\\
&\qquad-\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}n^2\,r^{n+2k-2}\left(\mathcal{C}_{n+2k,n}(z)\sin n\varphi+\mathcal{D}_{n+2k,n}(z)\cos n\varphi\right)\\
&\qquad+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}r^{n+2k-2}\left(\mathcal{C}^{(2)}_{n+2k-2,n}(z)\sin n\varphi+\mathcal{D}^{(2)}_{n+2k-2,n}(z)\cos n\varphi\right)\\
&=0\,, \qquad\qquad (
\end{aligned}
$$

Recursion for Cn,n

$$\mathcal{C}_{n+2k,n}(z)\left((n+2k)^2-n^2\right)+\mathcal{C}^{(2)}_{n+2k-2,n}(z)=0\,,$$

$$\mathcal{D}_{n+2k,n}(z)\left((n+2k)^2-n^2\right)+\mathcal{D}^{(2)}_{n+2k-2,n}(z)=0\,.$$

$$\mathcal{C}_{n+2k,n}(z)=\frac{1}{\prod_{m=1}^{k}(n^2-(n+2m)^2)}\,\mathcal{C}^{(2k)}_{n,n}(z)\,,$$

$$\phi_{\mathrm{m}} = \sum_{n=1}^{\infty} \Big\{ \sum_{k=0}^{\infty} \frac{1}{\prod_{m=1}^{k} (n^2 - (n+2m)^2)} \, \mathcal{C}_{n,n}^{(2k)}(z) \Big\} \, r^n \sin n\varphi$$
$$+ \sum_{n=1}^{\infty} \Big\{ \sum_{k=0}^{\infty} \frac{1}{\prod_{m=1}^{k} (n^2 - (n+2m)^2)} \, \mathcal{D}_{n,n}^{(2k)}(z) \Big\} \, r^n \cos n\varphi \, ,$$

$$\phi_{\mathrm{m}} = \sum_{n=1}^{\infty} \Big\{ \mathcal{C}_{n,n}(z) - \frac{\mathcal{C}_{n,n}^{(2)}(z)}{4(n+1)} r^2$$
$$+ \frac{\mathcal{C}_{n,n}^{(4)}(z)}{32(n+1)(n+2)} r^4 - \frac{\mathcal{C}_{n,n}^{(6)}(z)}{384(n+1)(n+2)(n+3)} r^6 + \ldots \Big\} \, r^n \sin n\varphi$$
$$+ \sum_{n=1}^{\infty} \Big\{ \mathcal{D}_{n,n}(z) - \frac{\mathcal{D}_{n,n}^{(2)}(z)}{4(n+1)} r^2$$
$$+ \frac{\mathcal{D}_{n,n}^{(4)}(z)}{32(n+1)(n+2)} r^4 - \frac{\mathcal{D}_{n,n}^{(6)}(z)}{384(n+1)(n+2)(n+3)} r^6 + \ldots \Big\} \, r^n \cos n\varphi \, ,$$

$$
\begin{aligned}
\frac{-1}{\mu_0} B_y(x,0,z) \approx \quad & \mathcal{C}_{1,1}(z) \\
& + \; x \; \left(2\,\mathcal{C}_{2,2}(z)\right) \\
& + x^2 \left(3\,\mathcal{C}_{3,3,}(z) - \frac{1}{8}\mathcal{C}^{(2)}_{1,1}(z)\right) \\
& + x^3 \left(4\,\mathcal{C}_{4,4}(z) - \frac{2}{12}\,\mathcal{C}^{(2)}_{2,2}(z)\right) \\
& + x^4 \left(5\,\mathcal{C}_{5,5}(z) - \frac{3}{16}\,\mathcal{C}^{(2)}_{3,3}(z) + \frac{1}{192}\mathcal{C}^{(4)}_{1,1}(z)\right) \\
& + x^5 \left(6\,\mathcal{C}_{6,6}(z) - \frac{4}{20}\,\mathcal{C}^{(2)}_{4,4}(z) + \frac{2}{384}\,\mathcal{C}^{(4)}_{2,2}(z)\right) \\
& + x^6 \left(7\,\mathcal{C}_{7,7}(z) - \frac{5}{24}\,\mathcal{C}^{(2)}_{5,5}(z) + \frac{3}{640}\,\mathcal{C}^{(4)}_{3,3}(z) - \frac{1}{9216}\,\mathcal{C}^{(6)}_{1,1}(z)\right) .
\end{aligned}
$$

Field Components from Pseudo-Multipoles

$$\phi_{\mathrm{m}}(r,\varphi) = \sum_{n=1}^{\infty} r^n (\widetilde{\mathcal{C}}_n(r,z)\sin n\varphi + \widetilde{\mathcal{D}}_n(z)\cos n\varphi)\,.$$

$$B_r(r,\varphi,z) = -\mu_0 \sum_{n=1}^{\infty} r^{n-1} (\overline{\mathcal{C}}_n(r,z)\sin n\varphi + \overline{\mathcal{D}}_n(r,z)\cos n\varphi)\,,$$

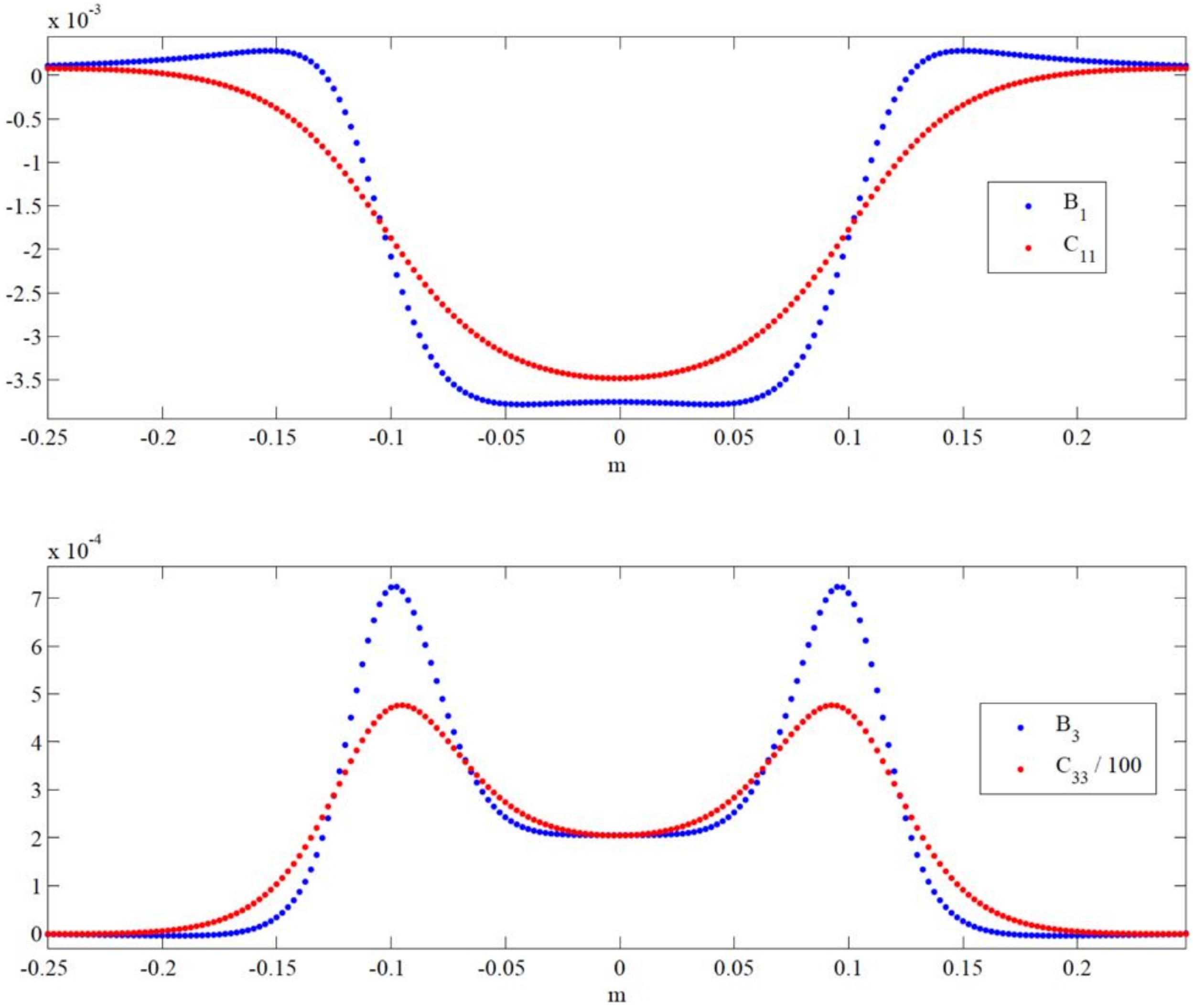
$$B_\varphi(r,\varphi,z) = -\mu_0 \sum_{n=1}^{\infty} n\, r^{n-1} (\widetilde{\mathcal{C}}_n(r,z)\cos n\varphi - \widetilde{\mathcal{D}}_n(r,z)\sin n\varphi)\,,$$

$$B_z(r,\varphi,z) = -\mu_0 \sum_{n=1}^{\infty} r^n \left(\frac{\partial \widetilde{\mathcal{C}}_n(r,z)}{\partial z} \sin n\varphi + \frac{\partial \widetilde{\mathcal{D}}_n(r,z)}{\partial z} \cos n\varphi \right)\,,$$

$$\overline{\mathcal{C}}_n(r,z) = n\,\mathcal{C}_{n,n}(z) - \frac{(n+2)\mathcal{C}_{n,n}^{(2)}(z)}{4(n+1)} r^2 + \frac{(n+4)\mathcal{C}_{n,n}^{(4)}(z)}{32(n+1)(n+2)} r^4 - \dots$$

$$\widetilde{\mathcal{C}}_n(r,z) := \mathcal{C}_{n,n}(z) - \frac{\mathcal{C}_{n,n}^{(2)}(z)}{4(n+1)} r^2 + \frac{\mathcal{C}_{n,n}^{(4)}(z)}{32(n+1)(n+2)} r^4 - \dots\,,$$

The Leading Term is NOT the Measured One



Fourier Transform for the Extractions of Cn,n

$$B_n(r_0,z) = -\mu_0\, r_0^{n-1}\overline{\mathcal{C}}_n(r_0,z) =$$

$$-\,\mu_0\, r_0^{n-1}\left(n\,\mathcal{C}_{n,n}(z) - \frac{(n+2)\mathcal{C}^{(2)}_{n,n}(z)}{4(n+1)}r_0^2 + \frac{(n+4)\mathcal{C}^{(4)}_{n,n}(z)}{32(n+1)(n+2)}r_0^4 - \dots\right).$$

$$\mathcal{F}\{\mathcal{C}_{n,n}(z)\} = \frac{-\mathcal{F}\{B_n(r_0,z)\}}{\mu_0\, r_0^{n-1}\left(n - \frac{(n+2)(i\omega)^2}{4(n+1)}r_0^2 + \frac{(n+4)(i\omega)^4}{32(n+1)(n+2)}r_0^4 - \dots\right)}$$