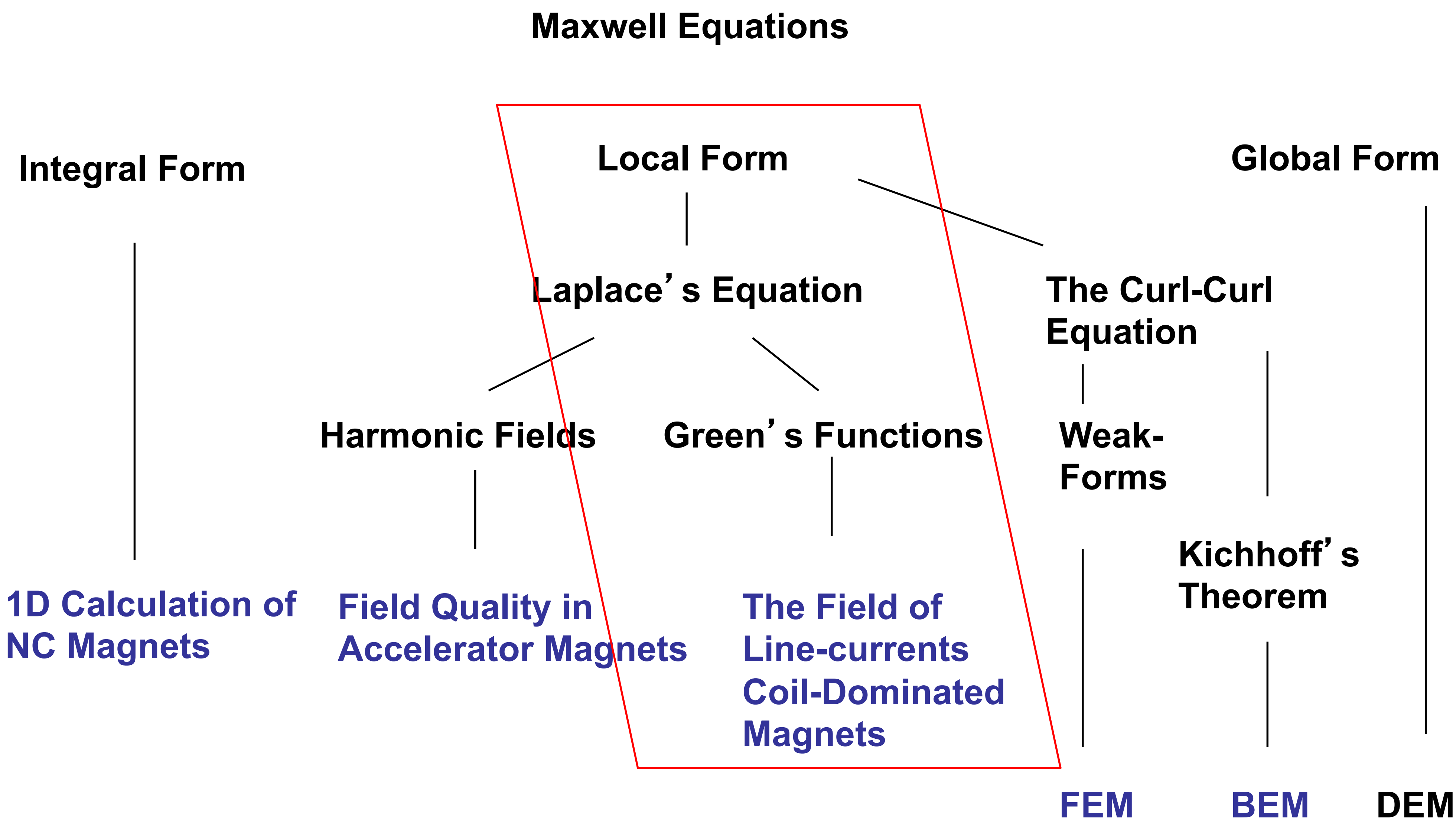
Electromagnetic Design of Accelerator Magnets and ROXIE User's Course

Theory 3

Stephan Russenschuck, CERN, 2022

Mathematical Foundations of Magnet Design



More Integral Theorems

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = \Big[g(x)f(x)\Big]_a^b - \int_a^b g(x)f'(x)\,\mathrm{d}x$$

Green's First

$$\int_{\mathcal{V}} \left(\operatorname{grad}\phi \cdot \operatorname{grad}\psi + \phi\nabla^2\psi\right)\,\mathrm{d}V = \int_{\partial\mathcal{V}} \phi\,\partial_{\mathbf{n}}\psi\,\mathrm{d}a$$

Green's Second

$$\int_{\Omega} \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\,\mathrm{d}V = \int_{\Gamma} (\phi\partial_{\mathbf{n}}\psi - \psi\partial_{\mathbf{n}}\phi)\,\mathrm{d}a$$

Vector Form of Green's Second

$$\int_{\mathcal{V}} \mathbf{a}\cdot\operatorname{curl}\mathbf{b}\,\mathrm{d}V = \int_{\mathcal{V}} \mathbf{b}\cdot\operatorname{curl}\mathbf{a}\,\mathrm{d}V - \int_{\partial\mathcal{V}} \mathbf{a}\cdot(\mathbf{b}\times\mathbf{n})\,\mathrm{d}a.$$

Generalization of the Integration by Parts Rule

$$-\int_{\mathcal{V}} \mathbf{a}\cdot\operatorname{grad}\phi\,\mathrm{d}V = \int_{\mathcal{V}} \phi\operatorname{div}\mathbf{a}\,\mathrm{d}V - \int_{\partial\mathcal{V}} \phi(\mathbf{a}\cdot\mathbf{n})\,\mathrm{d}a.$$

Stratton #1 and #2

$$\int_{\mathcal{V}} \operatorname{div}(\mathbf{a}\times\operatorname{curl}\mathbf{b})\mathrm{d}V = \int_{\partial\mathcal{V}} (\mathbf{a}\times\operatorname{curl}\mathbf{b})\cdot\mathbf{n}\,\mathrm{d}a$$

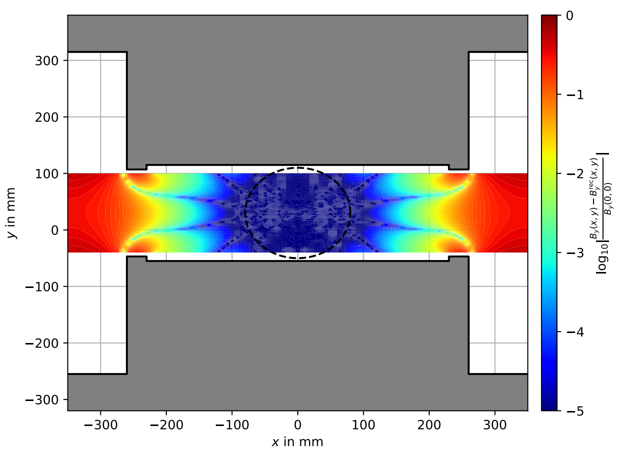
$$\int_{\mathcal{V}} (\mathbf{a}\operatorname{curl}\operatorname{curl}\mathbf{b} - \mathbf{b}\operatorname{curl}\operatorname{curl}\mathbf{a})\,\mathrm{d}V = \int_{\partial\mathcal{V}} (\mathbf{b}\times\operatorname{curl}\mathbf{a} - \mathbf{a}\times\operatorname{curl}\mathbf{b})\cdot\mathbf{n}\,\mathrm{d}a\,.$$

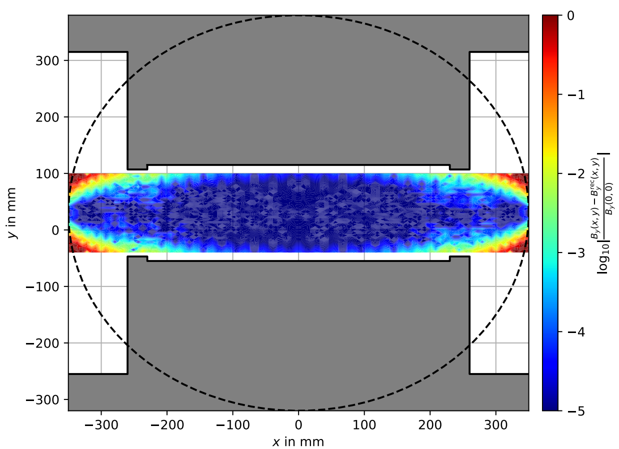
Properties of Harmonic Functions

Theorem 4.4 (Maximum principle) *If ϕ is harmonic in a closed, contractible volume $\mathscr{V} \subset \Omega$, ϕ cannot take a maximum or a minimum at any interior point of $\mathscr{V}$.*

$$\int_{\mathscr{V}_s} \partial_{\mathbf{n}} \phi \, \mathrm{d}V = \int_{\partial \mathscr{V}_s} \operatorname{grad} \phi \cdot \mathrm{d}\mathbf{a} = \int_{\mathscr{V}_s} \nabla^2 \phi \, \mathrm{d}V = 0$$

Theorem 4.5 (Liouville) *If ϕ is a harmonic scalar field in E_n with an upper (or lower) bound, ϕ is constant.*

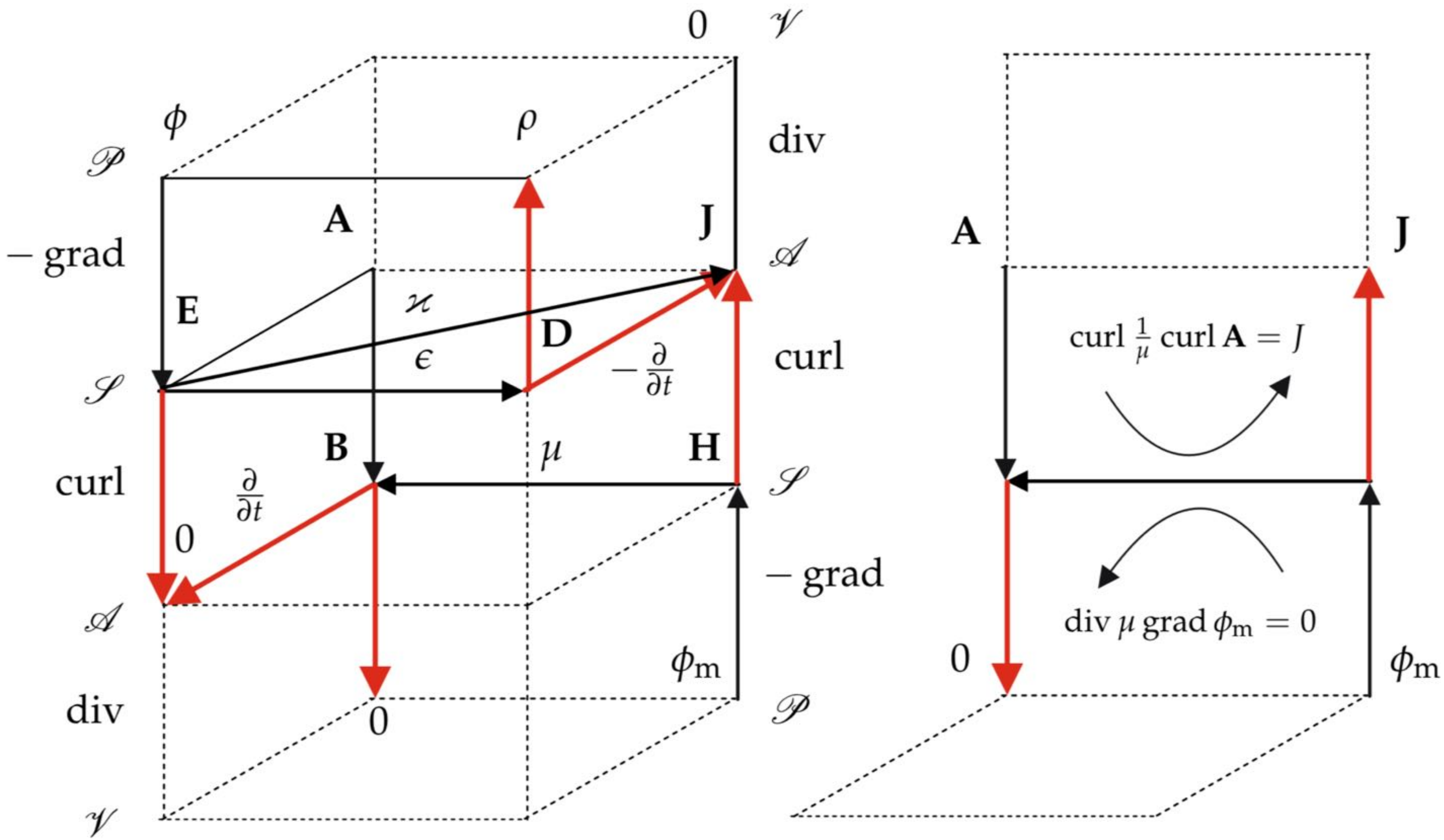




Maxwell's House

Faraday complex
Inner oriented

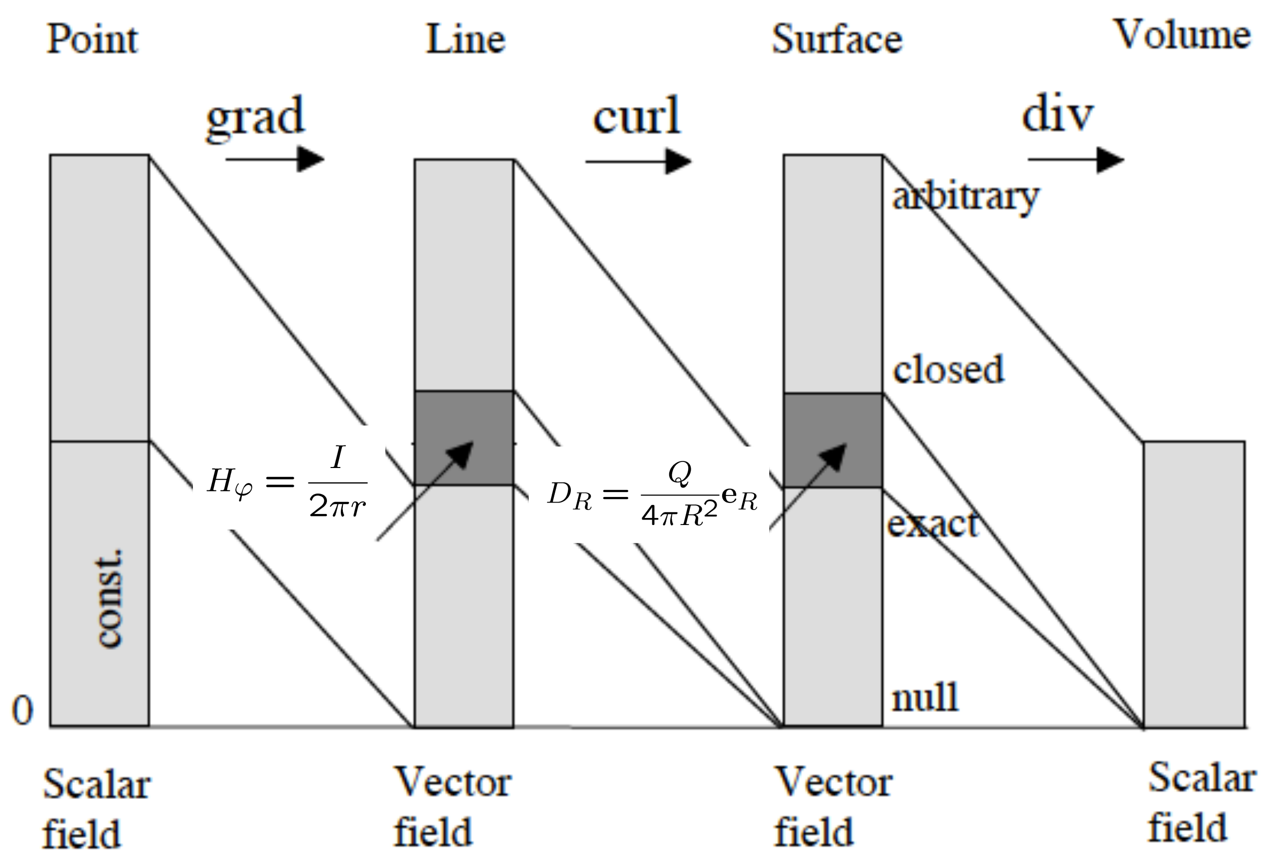
Ampere-Maxwell complex
Outer oriented



Potential Formulations

Formulation		$\Gamma_H (H_t = 0)$	$\Gamma_B (B_n = 0)$
Red. magnetic scalar potential	$\operatorname{div}(\mu \operatorname{grad} \phi_{\mathrm{m}}^{\mathrm{red}}) = \operatorname{div}(\mu \mathbf{H}_{\mathrm{s}})$	$\operatorname{grad} \phi_{\mathrm{m}}^{\mathrm{red}} \times \mathbf{n} = \mathbf{H}_{\mathrm{s}} \times \mathbf{n}$	$\mu \partial_{\mathbf{n}} \phi_{\mathrm{m}}^{\mathrm{red}} = \mu \mathbf{H}_{\mathrm{s}} \cdot \mathbf{n}$
Total magnetic scalar potential	$\operatorname{div}(\mu \operatorname{grad} \phi_{\mathrm{m}}) = 0$	$\mathbf{n} \times (\operatorname{grad} \phi_{\mathrm{m}} \times \mathbf{n}) = \mathbf{0}$	$\mu \partial_{\mathbf{n}} \phi_{\mathrm{m}} = 0$
Vector potential (curl–curl)	$\operatorname{curl}(\frac{1}{\mu} \operatorname{curl} \mathbf{A}) - \operatorname{grad}(\frac{1}{\mu} \operatorname{div} \mathbf{A}) = \mathbf{J}$	$\frac{1}{\mu} \operatorname{curl} \mathbf{A} \times \mathbf{n} = \mathbf{0}$	$\operatorname{curl} \mathbf{A} \cdot \mathbf{n} = 0$
Vector potential (Vector Poisson)	$\nabla^2 \mathbf{A} = \mu_0 (\mathbf{J} + \operatorname{curl} \mathbf{M})$	$\mathbf{n} \times (\frac{1}{\mu_0} \operatorname{curl} \mathbf{A} \times \mathbf{n}) = \mathbf{0}$	$\operatorname{curl} \mathbf{A} \cdot \mathbf{n} = 0$
Red. vector potential	$\operatorname{curl}(\frac{1}{\mu} \operatorname{curl}(\mathbf{A}_{\mathrm{r}} + \mathbf{A}_{\mathrm{s}})) - \operatorname{grad}(\frac{1}{\mu} \operatorname{div}(\mathbf{A}_{\mathrm{r}} + \mathbf{A}_{\mathrm{s}})) = \mathbf{J}$	$\frac{1}{\mu_0} \operatorname{curl} \mathbf{A}_{\mathrm{r}} \times \mathbf{n} = -\mathbf{H}_{\mathrm{s}} \times \mathbf{n}$	$\operatorname{curl} \mathbf{A}_{\mathrm{r}} \cdot \mathbf{n} = -\mu_0 \mathbf{H}_{\mathrm{s}} \cdot \mathbf{n}$

Lemmata of Poincare (Non-Contractible Domains)



Coulomb Gauge

$$\mathbf{A} \rightarrow \mathbf{A}' : \mathbf{A}' = \mathbf{A} + \operatorname{grad} \psi \, .$$

$$\operatorname{div} \mathbf{A}' = q$$

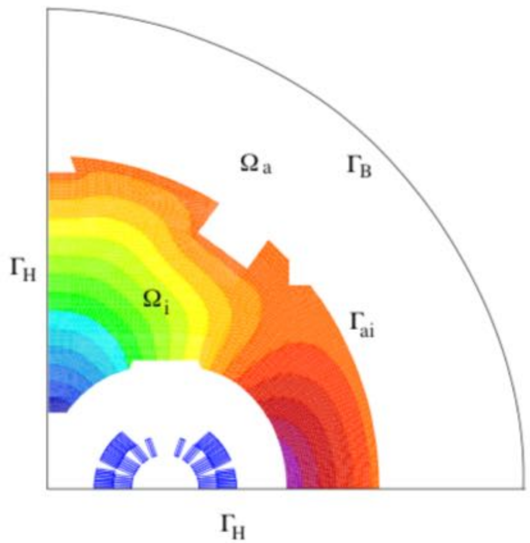
$$q = \operatorname{div} \mathbf{A} + \nabla^2 \psi \qquad q = 0 \qquad \psi = \int_\Omega \frac{\operatorname{div} \mathbf{A} - q}{4\pi |\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}V$$

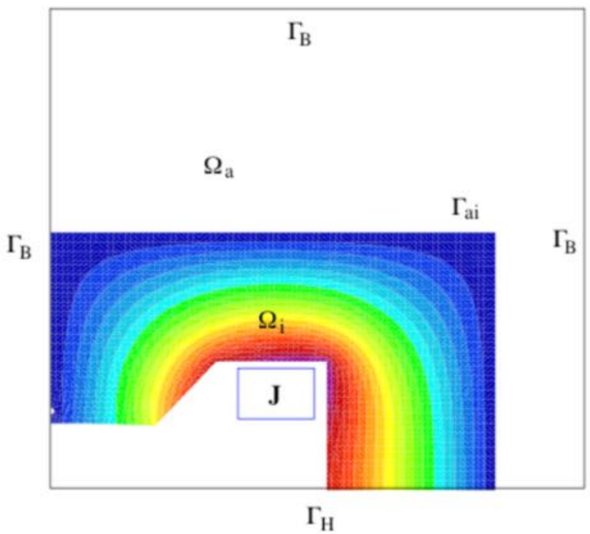
$$\frac{1}{\mu} \operatorname{div} \mathbf{A} = 0$$

$$\mathbf{A} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_H$$

$$\operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} \mathbf{A} \right) - \operatorname{grad} \left(\frac{1}{\mu} \operatorname{div} \mathbf{A} \right) = \mathbf{J} \, .$$

The Elementary Model Problem and the Curl-Curl Equation





$$\mathbf{n} \times (\mathbf{H} \times \mathbf{n}) = \mathbf{0} \qquad \text{on } \Gamma_H.$$

$$\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}\mathbf{A}\right) = \mathbf{J}$$

↑

$$\mathbf{H} \times \mathbf{n} = \boldsymbol{\alpha} \qquad \text{on } \Gamma_H,$$

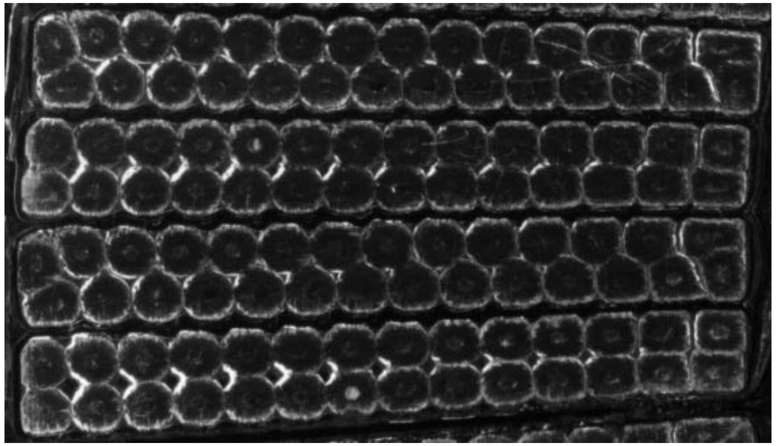
$$\mathbf{B} \cdot \mathbf{n} = \sigma_m \qquad \text{on } \Gamma_B,$$

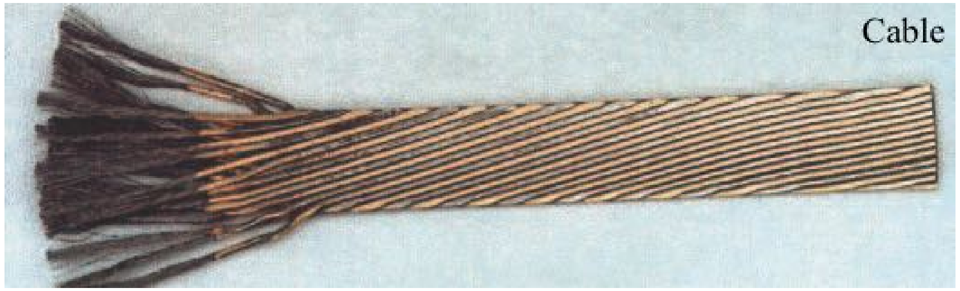
→

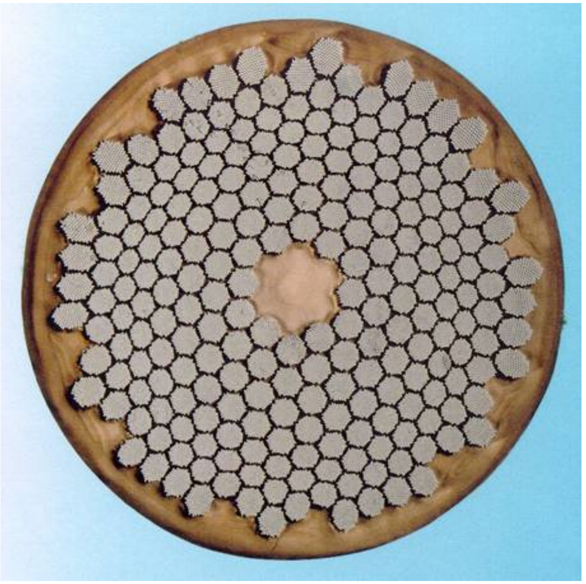
$$\frac{1}{\mu}\operatorname{curl}\mathbf{A} \times \mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma_H,$$

$$\operatorname{curl}\mathbf{A} \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma_B.$$

Rutherford (Roebel) Kabel, Strand, Nb-Ti Filament





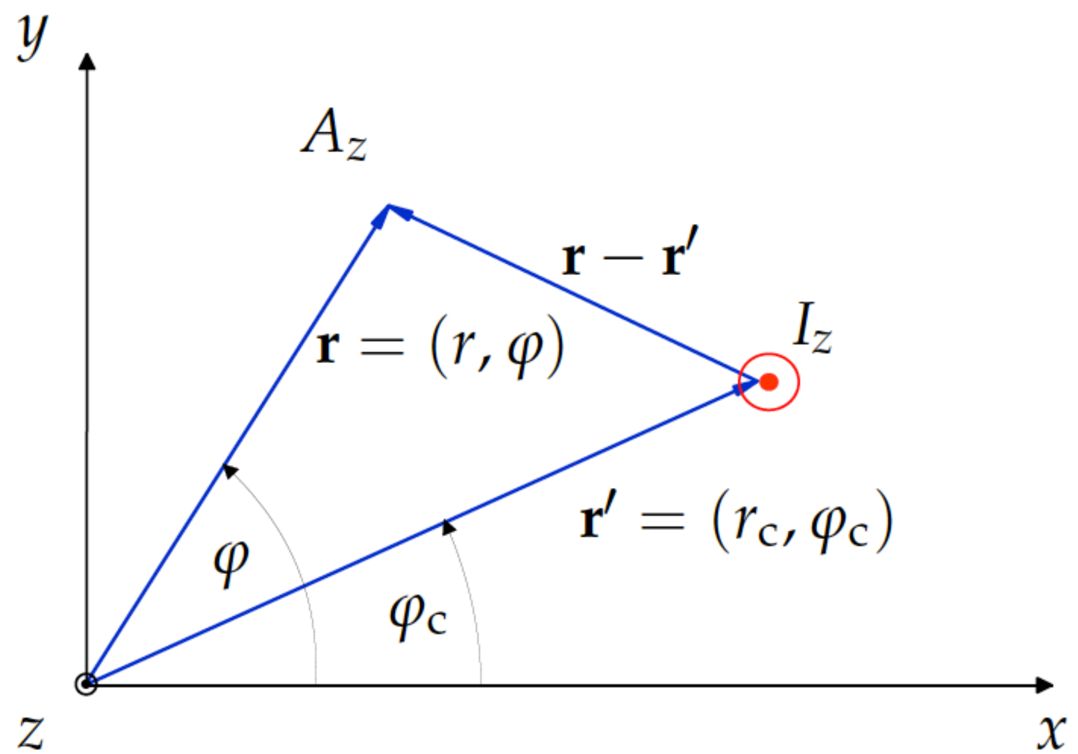


The Field of Line Currents

$$\mathbf{r} \mapsto \phi(|\mathbf{r}-\mathbf{r}'|)$$
$$\mathbf{r}' \mapsto \phi(|\mathbf{r}-\mathbf{r}'|)$$

$$\begin{aligned}
\operatorname{grad}\phi(|\mathbf{r}-\mathbf{r}'|) &= -\operatorname{grad}_{\mathbf{r}'}\phi(|\mathbf{r}-\mathbf{r}'|)\,,\\
\operatorname{div}\mathbf{a}(|\mathbf{r}-\mathbf{r}'|) &= -\operatorname{div}_{\mathbf{r}'}\mathbf{a}(|\mathbf{r}-\mathbf{r}'|)\,,\\
\operatorname{curl}\mathbf{a}(|\mathbf{r}-\mathbf{r}'|) &= -\operatorname{curl}_{\mathbf{r}'}\mathbf{a}(|\mathbf{r}-\mathbf{r}'|)\,,\\
\nabla^2\phi(|\mathbf{r}-\mathbf{r}'|) &= \nabla^2_{\mathbf{r}'}\phi(|\mathbf{r}-\mathbf{r}'|)\,.
\end{aligned}$$

Why bother?
Reciprocity; except for sign it does not matter if we exchange the source and field points



Greens Functions of Free Space

$$\mathcal{L}_{\mathbf{r}'}\phi(\mathbf{r}') = -f(\mathbf{r}')$$

$$\mathcal{L}_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'),$$

$$\int_{\mathcal{V}} \mathcal{L}_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}')\, f(\mathbf{r})\mathrm{d}V = -\int_{\mathcal{V}} \delta(\mathbf{r}-\mathbf{r}') f(\mathbf{r})\mathrm{d}V = -f(\mathbf{r}').$$

$$\mathcal{L}_{\mathbf{r}'}\phi(\mathbf{r}') = \int_{\mathcal{V}} \mathcal{L}_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}') f(\mathbf{r})\mathrm{d}V = \mathcal{L}_{\mathbf{r}'} \int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}') f(\mathbf{r})\mathrm{d}V,$$

$$\phi(\mathbf{r}') = \int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}') f(\mathbf{r})\mathrm{d}V.$$

$$G_2(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{r_{\mathrm{ref}}}\right), \qquad G_3(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Green's Functions of Free Space

$$\phi(\mathbf{r}') = \int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}') f(\mathbf{r}) \mathrm{d}V \,.$$

$$\phi(\mathbf{r}) = \int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}') f(\mathbf{r}') \mathrm{d}V' \,.$$

But what if boundaries are present?

$$\phi(\mathbf{r}) = \int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}') f(\mathbf{r}') \mathrm{d}V' + \int_{\partial\mathcal{V}} \Big(-\phi(\mathbf{r}')\, \partial_{\mathbf{n}'} G(\mathbf{r},\mathbf{r}') + G(\mathbf{r},\mathbf{r}')\, \partial_{\mathbf{n}'} \phi(\mathbf{r}') \Big) \mathrm{d}a' \,.$$

$$\phi_{\mathrm{m}}(\mathbf{r}) = \frac{1}{4\pi\mu_0} \int_{\mathcal{V}} \frac{\rho_{\mathrm{m}}}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}V' + \frac{1}{4\pi} \int_{\partial\mathcal{V}} \left(\partial_{\mathbf{n}'} \phi_{\mathrm{m}}(\mathbf{r}') \frac{1}{|\mathbf{r}-\mathbf{r}'|} + \phi_{\mathrm{m}} \frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{n}'}{|\mathbf{r}-\mathbf{r}'|^3} \right) \mathrm{d}a'$$

Biot-Savart's Law

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}\,, \qquad G_3(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \frac{J_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V'\,,$$

$$\mathbf{A}(\mathbf{r}) = A_x \mathbf{e_x} + A_y \mathbf{e_y} + A_z \mathbf{e_z} = \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V'.$$

This works only in Cartesian Coordinates

$$\mathbf{B}(\mathbf{r}) = \operatorname{curl} \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \operatorname{curl} \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) \mathrm{d}V'$$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \sum_{k=1}^{3} J_k(\mathbf{r}')(\mathbf{e}_i(\mathbf{r}) \cdot \mathbf{e}_k(\mathbf{r}')) \,\mathrm{d}V'. \quad \mathrm{d}V'$$

$$= \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \mathrm{d}V'.$$

Biot Savart's Law

But wait a minute: Are we finished? Are we sure that the divergence of the vector potential is zero as it was required for the Laplace equation?

$$
\begin{aligned}
\operatorname{div} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \operatorname{div} \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right) \mathrm{d}V' \\
&= \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \left(\mathbf{J}(\mathbf{r}') \cdot \operatorname{grad} \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) + \frac{1}{|\mathbf{r}-\mathbf{r}'|} \operatorname{div} \mathbf{J}(\mathbf{r}') \right) \mathrm{d}V' \\
&= \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \mathbf{J}(\mathbf{r}') \cdot \operatorname{grad} \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) \mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi} \int_{\mathscr{V}} \mathbf{J}(\mathbf{r}') \cdot \operatorname{grad}_{\mathbf{r}'} \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) \mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi} \int_{\mathscr{V}} \left(\operatorname{div}_{\mathbf{r}'} \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|} \operatorname{div}_{\mathbf{r}'} \mathbf{J}(\mathbf{r}') \right) \mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi} \int_{\mathscr{V}} \operatorname{div}_{\mathbf{r}'} \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right) \mathrm{d}V' = -\frac{\mu_0}{4\pi} \int_{\partial\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \cdot \mathrm{d}\mathbf{a}' \, .
\end{aligned}
$$

Current loops must always be closed and must not leave the problem domain

Biot-Savart's Law for Line Currents

$$\mathbf{A}(\mathbf{r}) = A_x \mathbf{e_x} + A_y \mathbf{e_y} + A_z \mathbf{e_z} = \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V'.$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{\mathscr{C}} \frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

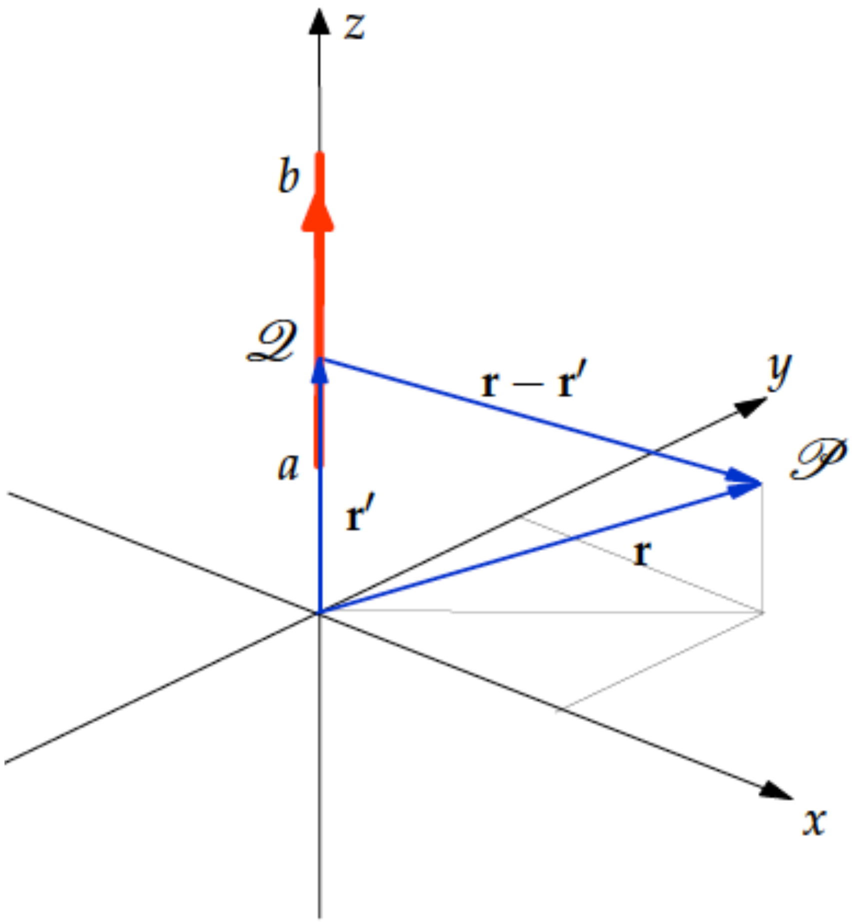
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{\mathscr{C}} \frac{\mathrm{d}\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3},$$

Vector Potential of a Line Current

$$A_z(x,y,z) = \frac{\mu_0 I}{4\pi}\int_a^b \frac{\mathrm{d}z_c}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 I}{4\pi}\int_a^b \frac{\mathrm{d}z_c}{\sqrt{x^2+y^2+(z-z_c)^2}}$$

$$\frac{-\mu_0 I}{4\pi}\ln\left((z-z_c)+\sqrt{x^2+y^2+(z-z_c)^2}\right)\Bigg|_a^b$$

$$\frac{\mu_0 I}{4\pi}\ln\frac{z-a+\sqrt{x^2+y^2+(z-a)^2}}{z-b+\sqrt{x^2+y^2+(z-b)^2}}\,.$$



Field of a Line Current (Infinitely Long)

$$\lim_{a,b\to\pm\infty} \ln \frac{z-a+\sqrt{x^2+y^2+(z-a)^2}}{z-b+\sqrt{x^2+y^2+(z-b)^2}} = \lim_{a,b\to\pm\infty} \ln \frac{-a+|a|\sqrt{1+\frac{x^2+y^2}{a^2}}}{-b+|b|\sqrt{1+\frac{x^2+y^2}{b^2}}}$$

$$= \lim_{a,b\to\pm\infty} \ln \frac{-a-a(1+\frac{x^2+y^2}{2a^2}+\cdots)}{-b+b(1+\frac{x^2+y^2}{2b^2}+\cdots)} = \lim_{a,b\to\pm\infty} \ln \frac{-2a}{-b+b+\frac{x^2+y^2}{2b}}$$

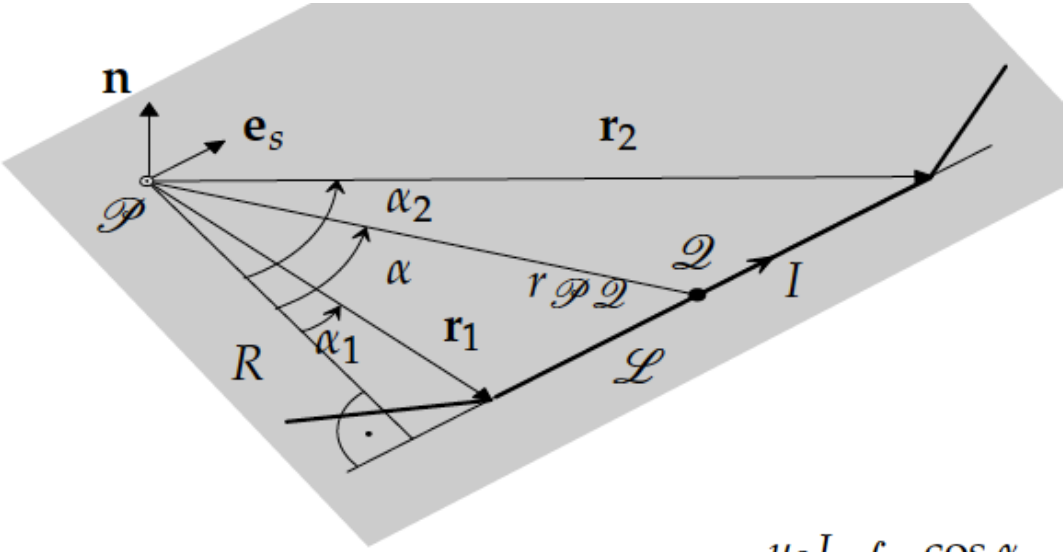
$$= \lim_{a,b\to\pm\infty} \ln \frac{-4ab}{x^2+y^2} .$$

$$A_z(x,y) = \lim_{a,b\to\pm\infty} \frac{\mu_0 I}{4\pi} \ln\left(\frac{-4ab}{x_0^2+y_0^2}\right) - \frac{\mu_0 I}{4\pi} \ln\left(\frac{x^2+y^2}{x_0^2+y_0^2}\right) .$$

Arbitrarily large but constant

$$\mathbf{A}(x,y) = -\frac{\mu_0 I}{4\pi} \ln\left(\frac{x^2+y^2}{x_0^2+y_0^2}\right) \mathbf{e}_z = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{r}{r_{\text{ref}}}\right) \mathbf{e}_z ,$$

Field of a Line Current Segment



$$\mathbf{B}(\mathscr{P}) = \frac{\mu_0 I}{4\pi}\int_{\mathscr{L}} \frac{\cos\alpha}{r^2_{\mathscr{P}\mathscr{Q}}}\mathrm{d}r' = \frac{\mu_0 I}{4\pi R}\mathbf{n}\int_{\alpha_1}^{\alpha_2}\cos\alpha\,\mathrm{d}\alpha = \frac{\mu_0 I}{4\pi R}\left(\sin\alpha_2 - \sin\alpha_1\right)\mathbf{n}$$

$$= \frac{\mu_0 I}{4\pi}\,\frac{\cos\alpha_2 + \cos\alpha_1}{R}\,\frac{\sin\alpha_2 - \sin\alpha_1}{\cos\alpha_2 + \cos\alpha_1}\,\mathbf{n}$$

$$= \frac{\mu_0 I}{4\pi}\left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|}\right)\frac{\sin(\alpha_2 - \alpha_1)}{1 + \cos(\alpha_2 - \alpha_1)}\,\mathbf{n}$$

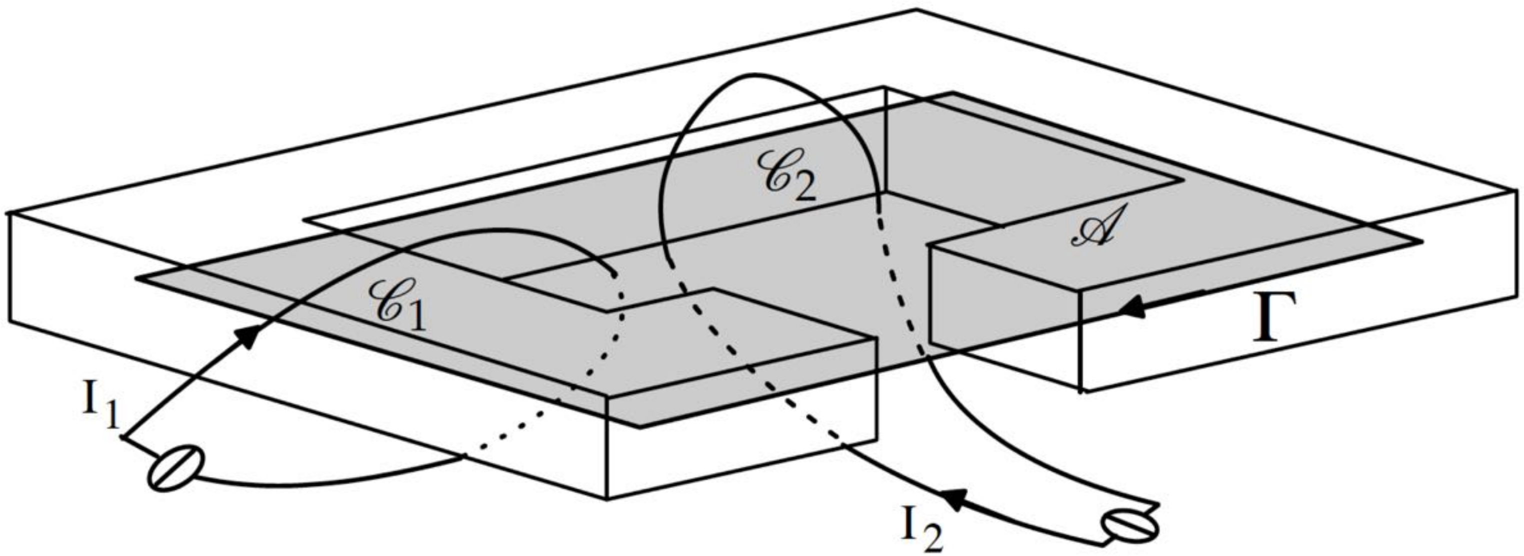
$$= \frac{\mu_0 I}{4\pi}\left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|}\right)\frac{\sin(\alpha_2 - \alpha_1)}{1 + \frac{\mathbf{r}_1\cdot\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|}}\,\frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|\sin(\alpha_2 - \alpha_1)}$$

$$= \frac{\mu_0 I}{4\pi}\,\frac{|\mathbf{r}_1| + |\mathbf{r}_2|}{|\mathbf{r}_1|\,|\mathbf{r}_2| + \mathbf{r}_1\cdot\mathbf{r}_2}\,\frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|},$$

The Linking Number

$$\int_{\partial\mathscr{A}} \mathbf{H}\cdot d\mathbf{r} = \int_{\mathscr{A}} \breve{\mathbf{J}}\cdot d\mathbf{a} = N\,I$$

$$N\,I = I\sum_{k=1}^{K} \operatorname{link}(\partial\mathscr{A},\mathscr{C}_k) = I\sum_{k=1}^{K} \operatorname{int}(\mathscr{A},\mathscr{C}_k)$$



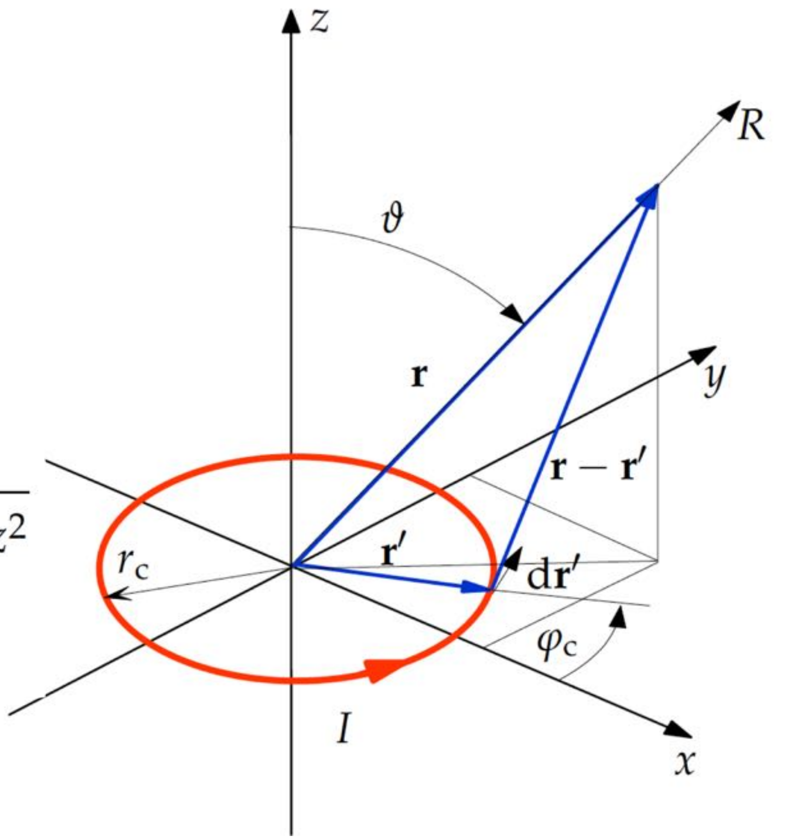
$$\operatorname{link}(\partial\mathscr{A},\mathscr{C}_k) = \frac{1}{4\pi}\int_{\partial\mathscr{A}}\int_{\mathscr{C}_k} \frac{d\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\cdot d\mathbf{r},$$

Field of a Ring Current

$$\mathbf{r}' = \cos\varphi_c r_c\,\mathbf{e}_x + \sin\varphi_c r_c\,\mathbf{e}_y$$

$$\mathrm{d}\mathbf{r}' = -\sin\varphi_c r_c\,\mathrm{d}\varphi_c\mathbf{e}_x + \cos\varphi_c r_c\,\mathrm{d}\varphi_c\mathbf{e}_y$$

$$\begin{aligned}|\mathbf{r}-\mathbf{r}'| &= \sqrt{(x-x_c)^2+(y-y_c)^2+z^2}\\ &= \sqrt{(r\cos\varphi - r_c\cos\varphi_c)^2 + (r\sin\varphi - r_c\sin\varphi_c)^2 + z^2}\\ &= \sqrt{r^2+r_c^2+z^2-2rr_c\cos\varphi_c}\,,\end{aligned}$$



$$A_y(r,z) = \frac{\mu_0 I r_c}{2\pi}\int_0^{\pi}\frac{\cos\varphi_c\mathrm{d}\varphi_c}{\sqrt{r^2+r_c^2+z^2-2rr_c\cos\varphi_c}}$$

Field of a Ring Current

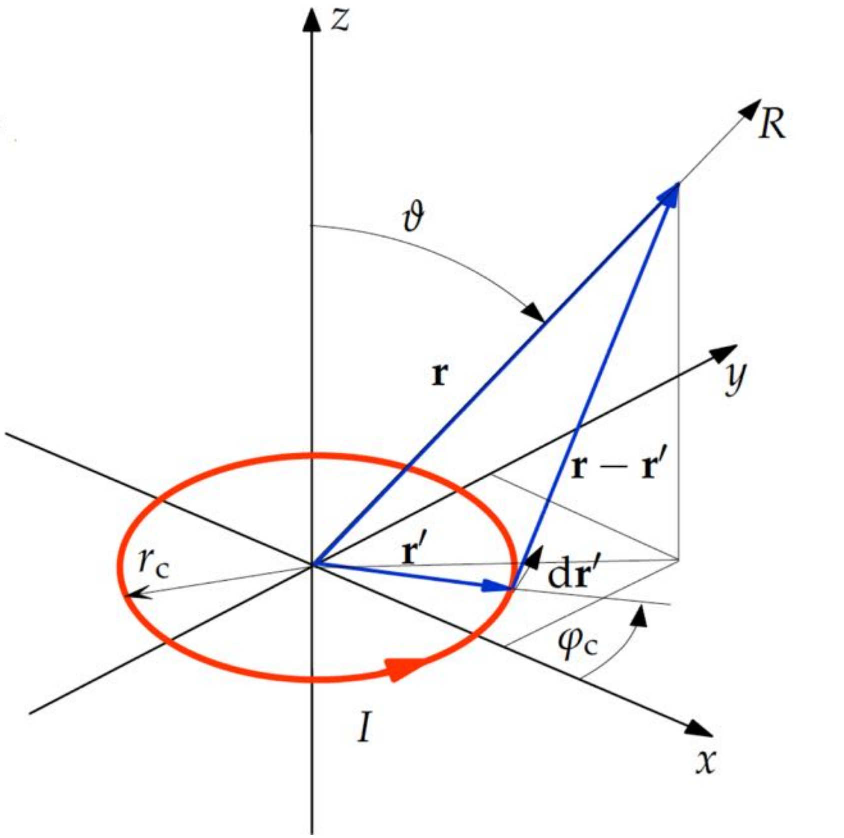
$$A_y(r,z) = \frac{\mu_0 I r_c}{2\pi}\int_0^{\pi} \frac{\cos\varphi_c \, d\varphi_c}{\sqrt{r^2 + r_c^2 + z^2 - 2 r r_c \cos\varphi_c}}$$

$$\psi := (\pi + \varphi_c)/2, \qquad k^2 := \frac{4 r r_c}{(r + r_c)^2 + z^2}$$

$$A_\varphi(r,z) = \frac{\mu_0 I r_c}{\pi\sqrt{(r+r_c)^2 + z^2}} \int_0^{\pi/2} \frac{2\sin^2\psi - 1}{\sqrt{1 - k^2\sin^2\psi}} \, d\psi$$

$$K\left(\frac{\pi}{2}, k\right) := \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2\psi}},$$

$$E\left(\frac{\pi}{2}, k\right) := \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\psi}\, d\psi,$$



$$A_\varphi(r,z) = \frac{\mu_0 I}{2\pi r}\sqrt{(r+r_c)^2 + z^2}\left[\left(1 - \frac{k^2}{2}\right) K\left(\frac{\pi}{2}, k\right) - E\left(\frac{\pi}{2}, k\right)\right]$$

Why Elliptic Integrals

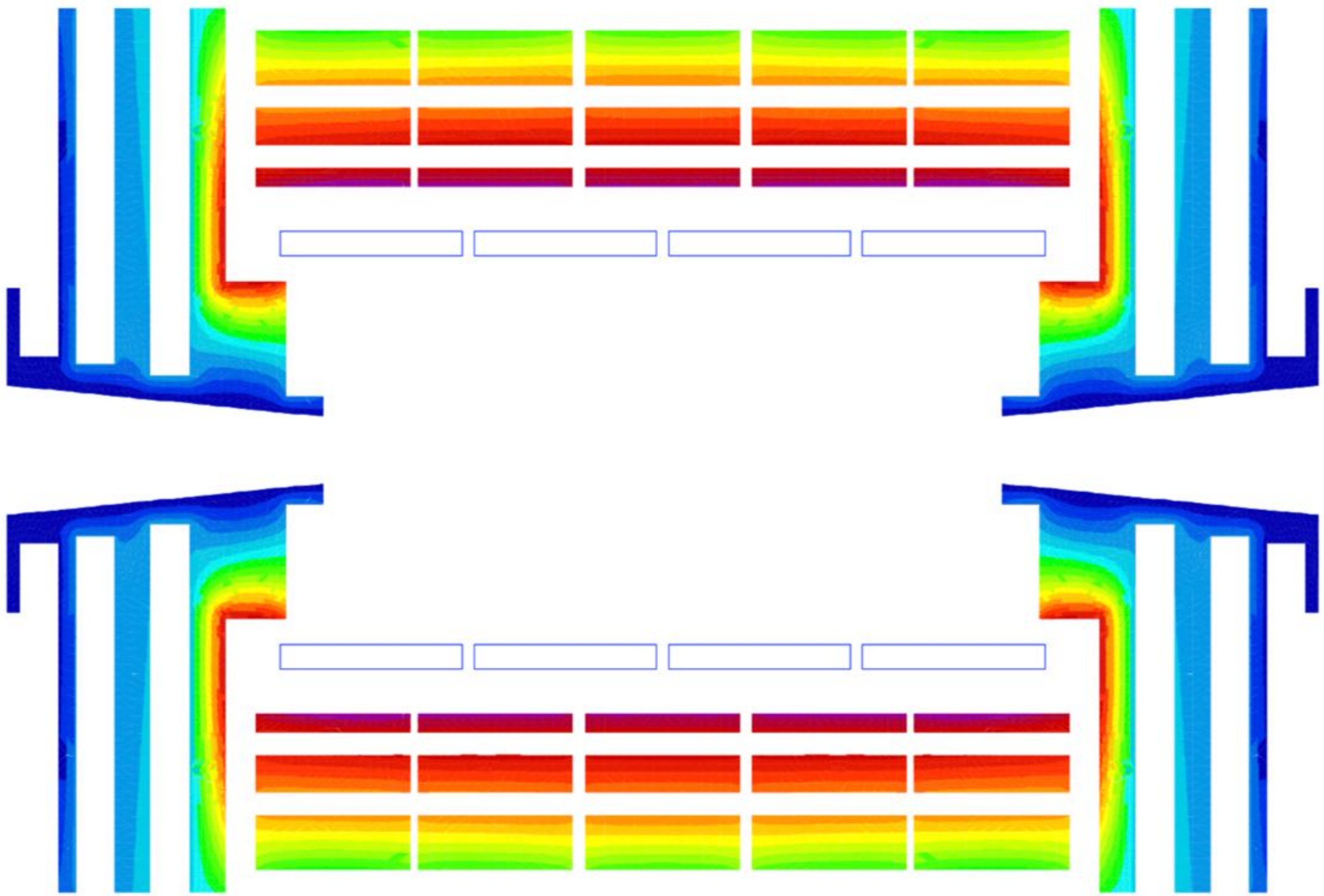
$$\mathbf{r}(t) = a\cos t\,\mathbf{e}_x + b\sin t\,\mathbf{e}_y \qquad |\mathbf{v}| = \sqrt{a^2\sin^2 t + b^2\cos^2 t},$$

$$s(t) = \int_0^t |\mathbf{v}(\tau)|\mathrm{d}\tau = \int_0^t b\sqrt{1 + \frac{a^2 - b^2}{b^2}\sin^2\tau}\,\mathrm{d}\tau$$
$$= \int_0^t b\sqrt{1 - e^2\sin^2\tau}\,\mathrm{d}\tau\,,$$

$$K\left(\frac{\pi}{2},k\right) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots + \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 k^{2n} + \cdots\right]$$

$$E\left(\frac{\pi}{2},k\right) = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \cdots - \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 \frac{k^{2n}}{2n-1} - \cdots\right]$$

Field in the Return Yoke of CMS



Field of a Ring Current

On axis:

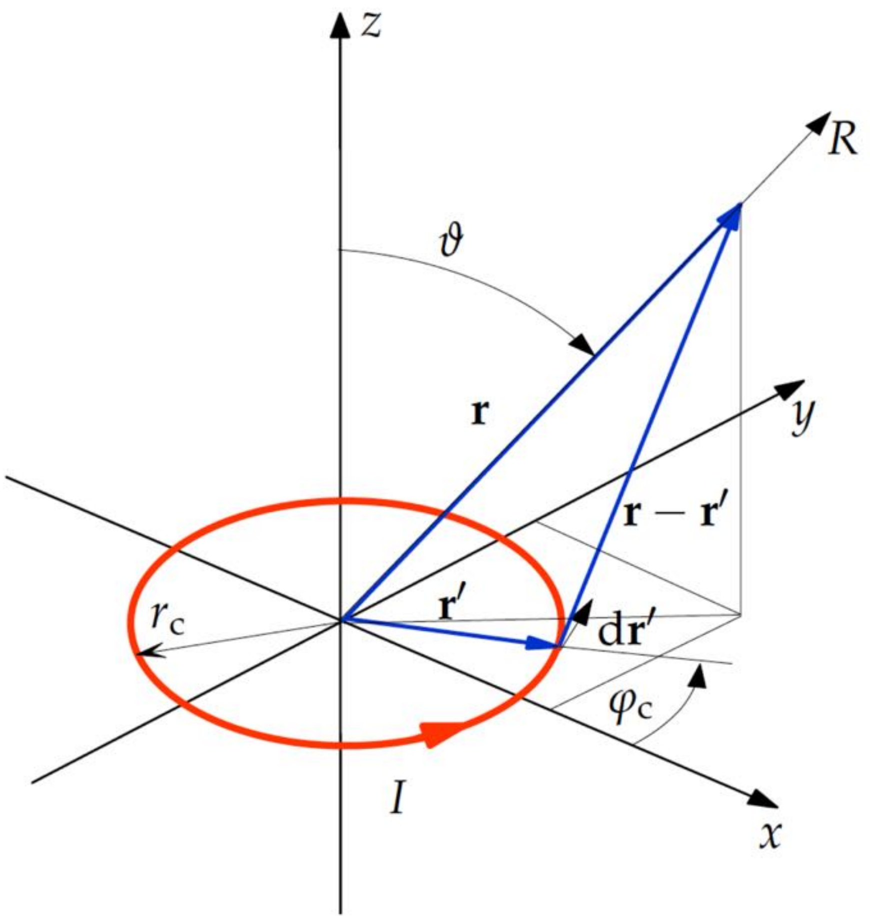
$: k \ll 1$

$$A_\varphi(r,z) = \frac{\mu_0 I r_c^2}{4} \frac{r}{(r_c^2 + z^2)^{\frac{3}{2}}},$$

$$B_z(z) = \frac{\mu_0 I}{2} \frac{r_c^2}{(r_c^2 + z^2)^{\frac{3}{2}}}.$$

In the center:

$$B_z(z=0) = \frac{\mu_0 I}{2 r_c}.$$



Far field approximation

$$A_\varphi(R,\vartheta) \approx \frac{\mu_0 I r_c^2 \pi}{4\pi} \frac{\sin\vartheta}{R^2} = \frac{\mu_0 m}{4\pi} \frac{\sin\vartheta}{R^2},$$

$$R = \sqrt{r^2 + z^2} \text{ and } \sin\vartheta = r/R,$$

$$[m] = 1\,\mathrm{A\,m}^2.$$

Definition

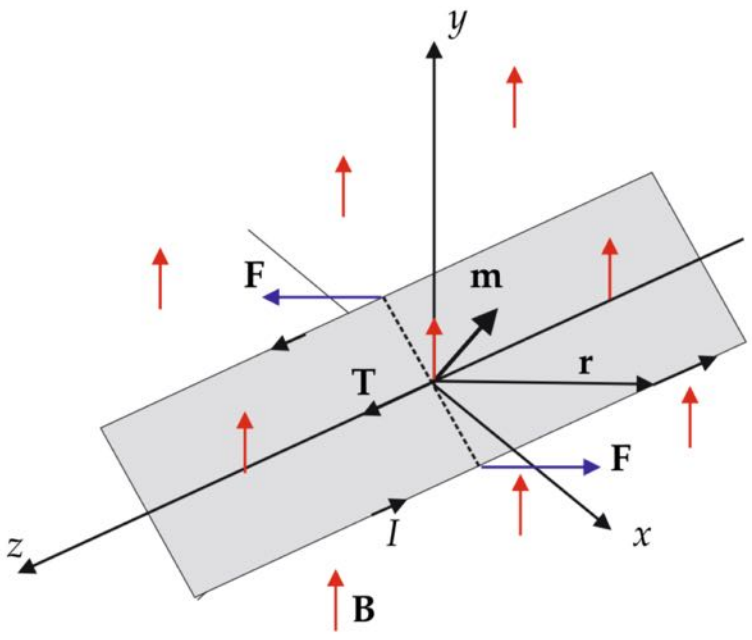
$$m := I r_c^2 \pi$$

$$\mathbf{m} = I\mathbf{a},$$

$$\mathbf{m} = \frac{I}{2}\int_{\mathscr{C}} \mathbf{r} \times \mathrm{d}\mathbf{r},$$

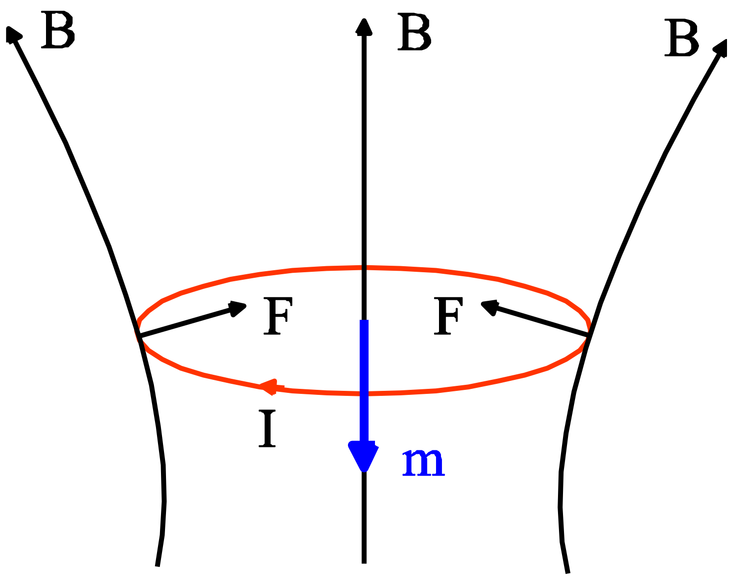
$$\mathbf{M}(\mathbf{r}) := \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}V} = \frac{1}{2}\mathbf{r} \times \mathbf{J}(\mathbf{r}),$$

Torque on a Magnetic Moment



$$\mathbf{T} = \mathbf{m} \times \mathbf{B}$$

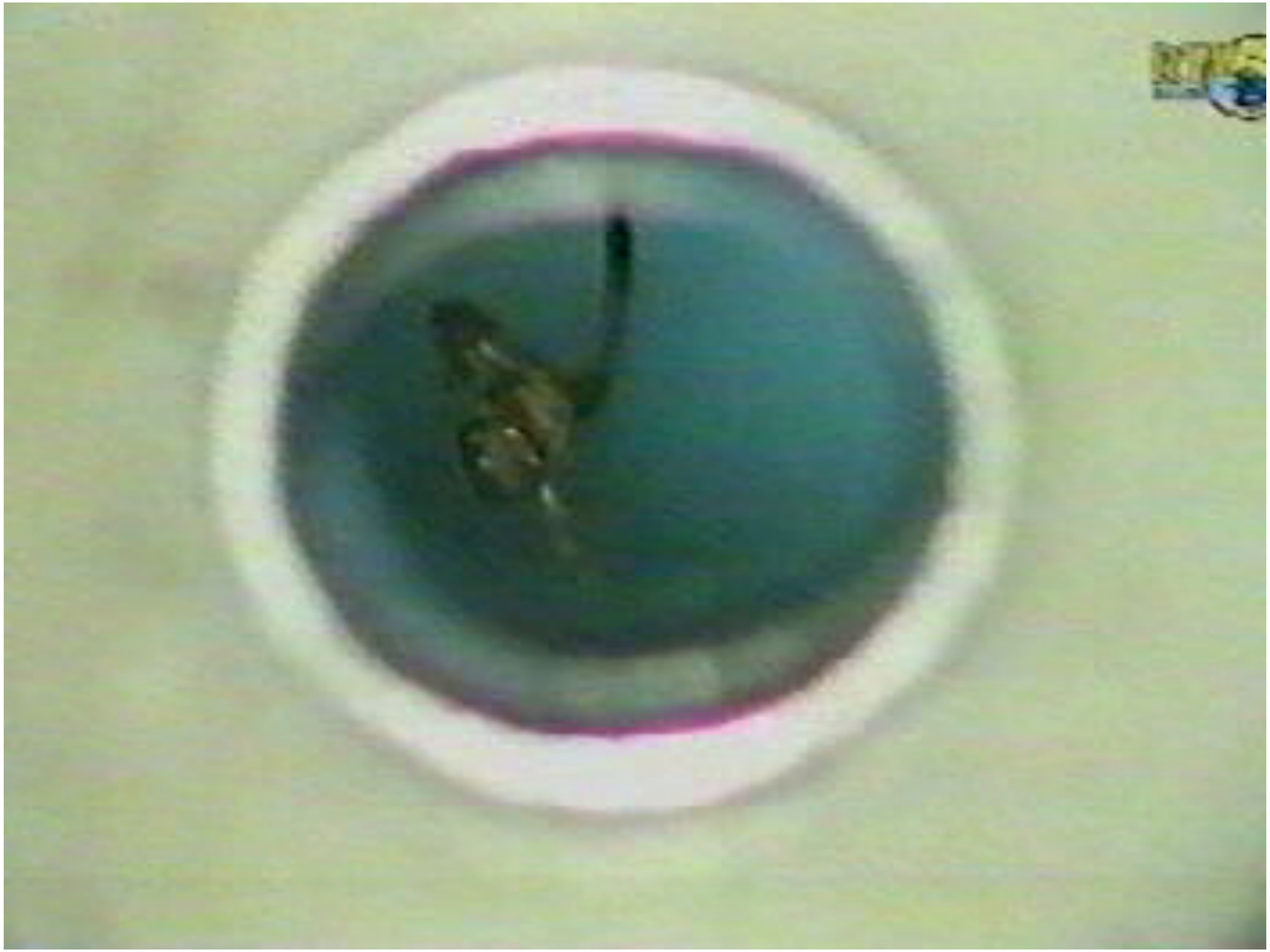
$$\begin{aligned}
\mathbf{T} &= I \int_{\partial\mathscr{A}} (\mathbf{r} \cdot \mathbf{B}) \mathrm{d}\mathbf{r} \\
&= -I \int_{\mathscr{A}} \operatorname{grad}(\mathbf{r} \cdot \mathbf{B}) \times \mathrm{d}\mathbf{a} \\
&= -I \int_{\mathscr{A}} \mathbf{B} \times \mathrm{d}\mathbf{a} = I \int_{\mathscr{A}} \mathrm{d}\mathbf{a} \times \mathbf{B} = \mathbf{m} \times \mathbf{B} = m\,B\,\sin(\varphi)
\end{aligned}$$



$$\mathbf{T} = \mathbf{m} \times \mathbf{B}$$

$$\mathbf{F} = (\mathbf{m} \cdot \operatorname{grad})\mathbf{B} = \operatorname{grad}(\mathbf{m} \cdot \mathbf{B})$$

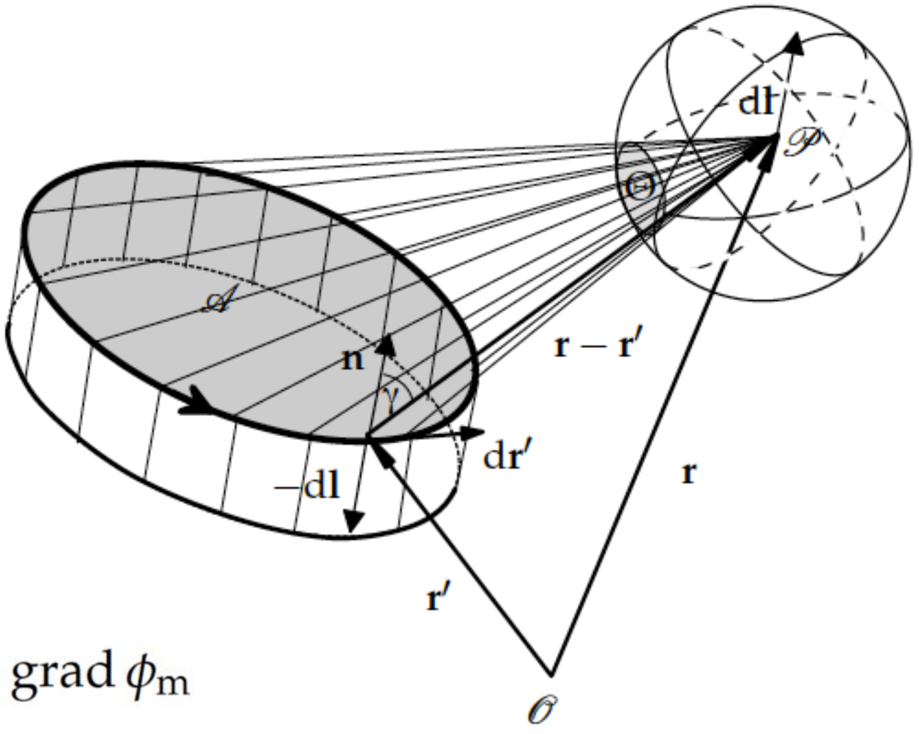
Microgravity



Solid Angle and Magnetic Scalar Potential

$$\mathrm{d}\Theta = -\int_{\partial\mathscr{A}} \frac{1}{|\mathbf{r}-\mathbf{r}'|^2}(\mathrm{d}\mathbf{l}\times\mathrm{d}\mathbf{r}')\cdot\mathbf{e}_R = -\int_{\partial\mathscr{A}} \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\cdot(\mathrm{d}\mathbf{l}\times\mathrm{d}\mathbf{r}')$$

$$= -\mathrm{d}\mathbf{l}\int_{\partial\mathscr{A}} \frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}.$$



Expressing $\mathrm{d}\Theta$ as $\operatorname{grad}\Theta\cdot\mathrm{d}\mathbf{l}$

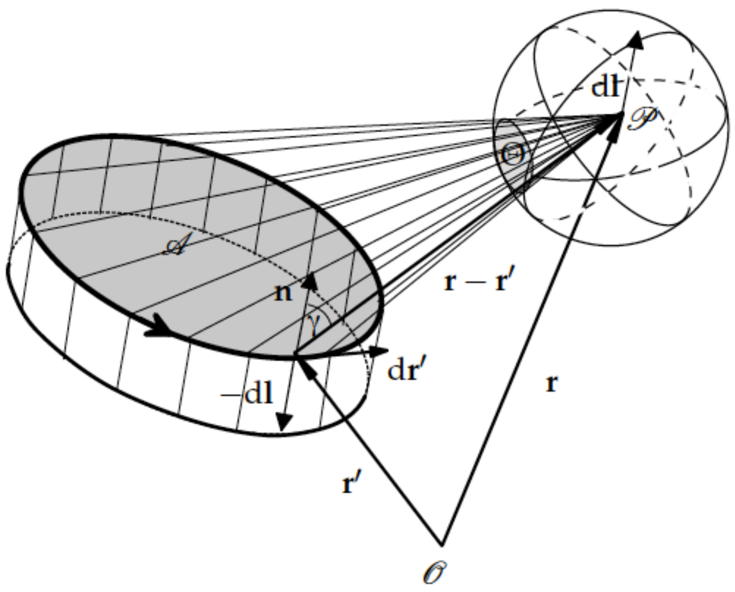
$$\operatorname{grad}\Theta = -\int_{\partial\mathscr{A}} \frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

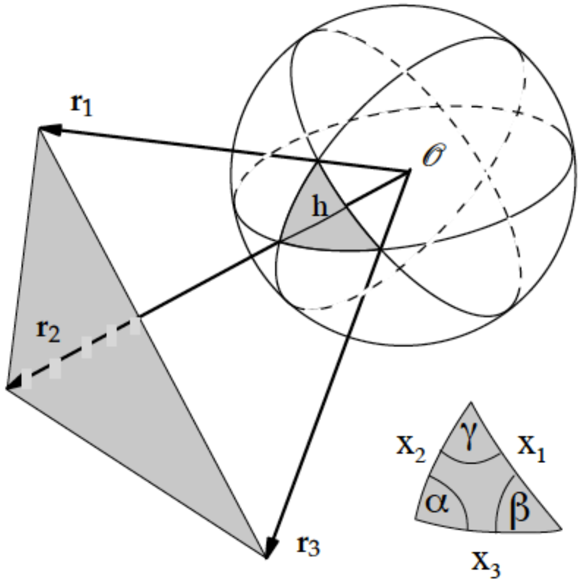
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_{\partial\mathscr{A}_c} \frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \mu_0\mathbf{H} = -\mu_0\operatorname{grad}\phi_m$$

$$\phi_m(\mathbf{r}) = \frac{I}{4\pi}\Theta$$

Solid angle (easy to compute) yields the magnetic scalar potential of a current loop

Solid Angle and Magnetic Scalar Potential

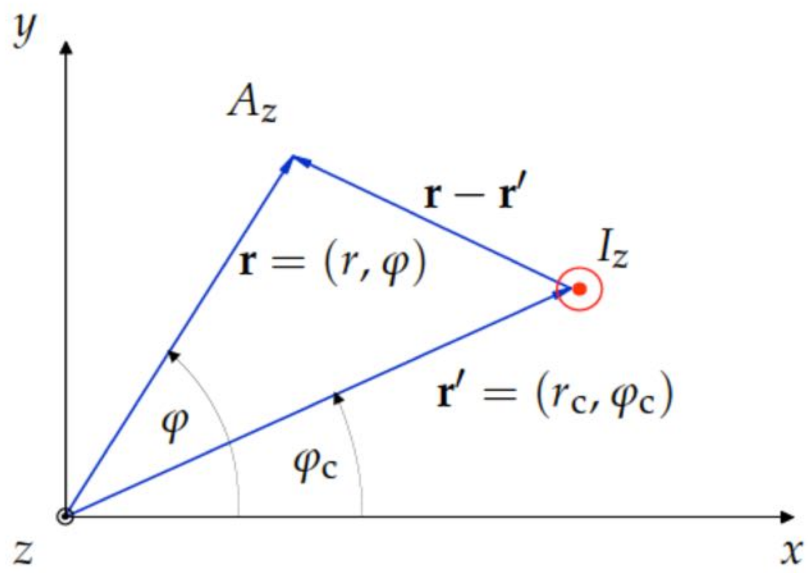




$$\Theta = \int_{\mathscr{A}} \frac{\cos\gamma}{R^2} \mathrm{d}a = \int_{\mathscr{A}} \frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|^3} \mathrm{d}a \,,$$

$$\tan\left(\frac{\Theta}{2}\right) = \frac{\mathbf{r}_1\cdot(\mathbf{r}_2\times\mathbf{r}_3)}{r_1r_2r_3 + (\mathbf{r}_1\cdot\mathbf{r}_2)r_3 + (\mathbf{r}_1\cdot\mathbf{r}_3)r_2 + (\mathbf{r}_2\cdot\mathbf{r}_3)r_1} \,.$$

Expanding the Green Function



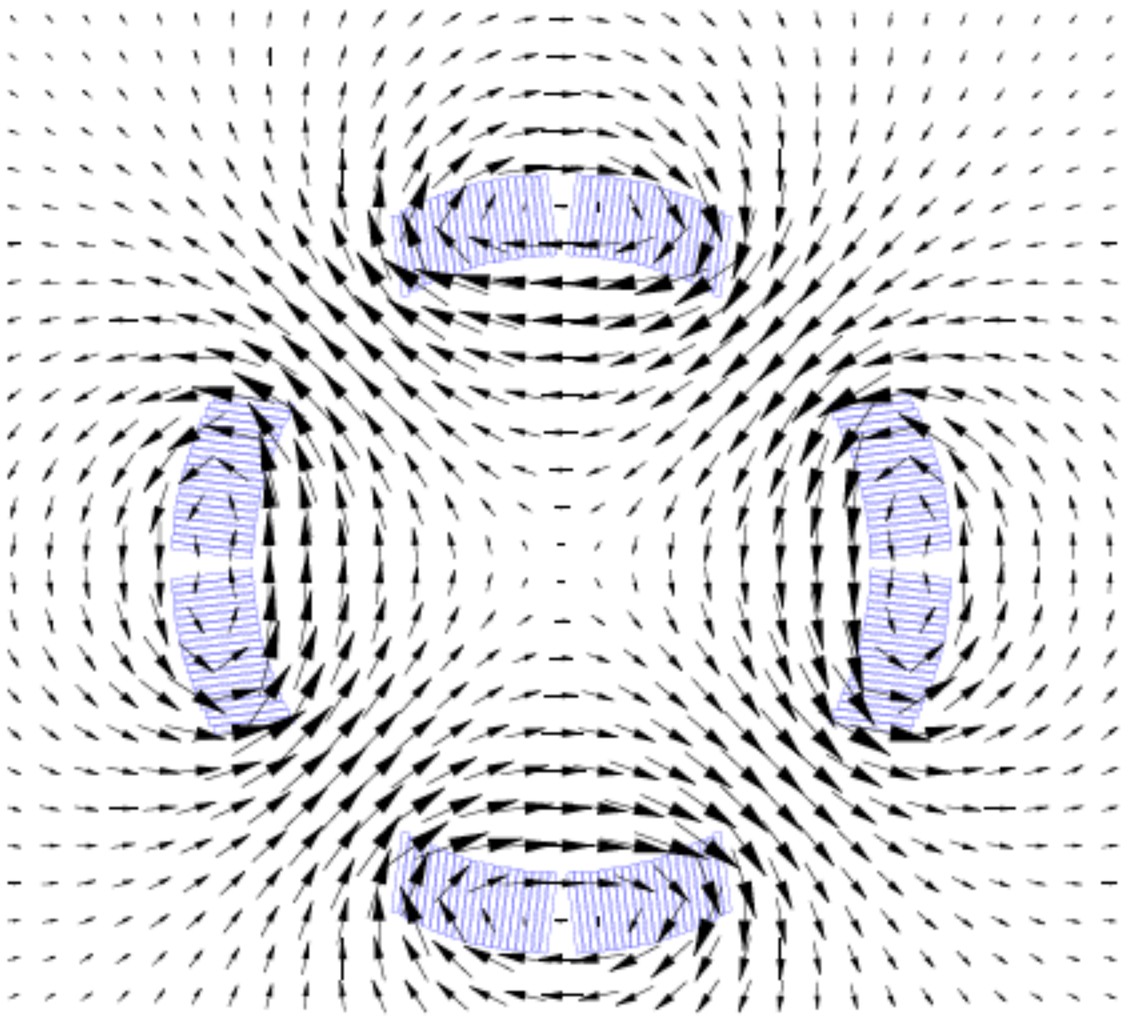
$$A_z(\mathbf{r}) = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{r_{\text{ref}}}\right)$$

$$A_z(r,\varphi) = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{r_c}{r_{\text{ref}}}\right) + \frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{r_c}\right)^n \cos n(\varphi - \varphi_c)$$

$$\mathcal{B}_n = \frac{\mu_0 I}{2\pi} \frac{1}{n\, r_c^n} \cos(n\varphi_c), \qquad \mathcal{A}_n = \frac{\mu_0 I}{2\pi} \frac{1}{n\, r_c^n} \sin(n\varphi_c).$$

$$B_n(r_0) = -\frac{\mu_0 I}{2\pi r_c} \left(\frac{r_0}{r_c}\right)^{n-1} \cos n\varphi_c, \qquad A_n(r_0) = \frac{\mu_0 I}{2\pi r_c} \left(\frac{r_0}{r_c}\right)^{n-1} \sin n\varphi_c.$$

The Imaging Current Method

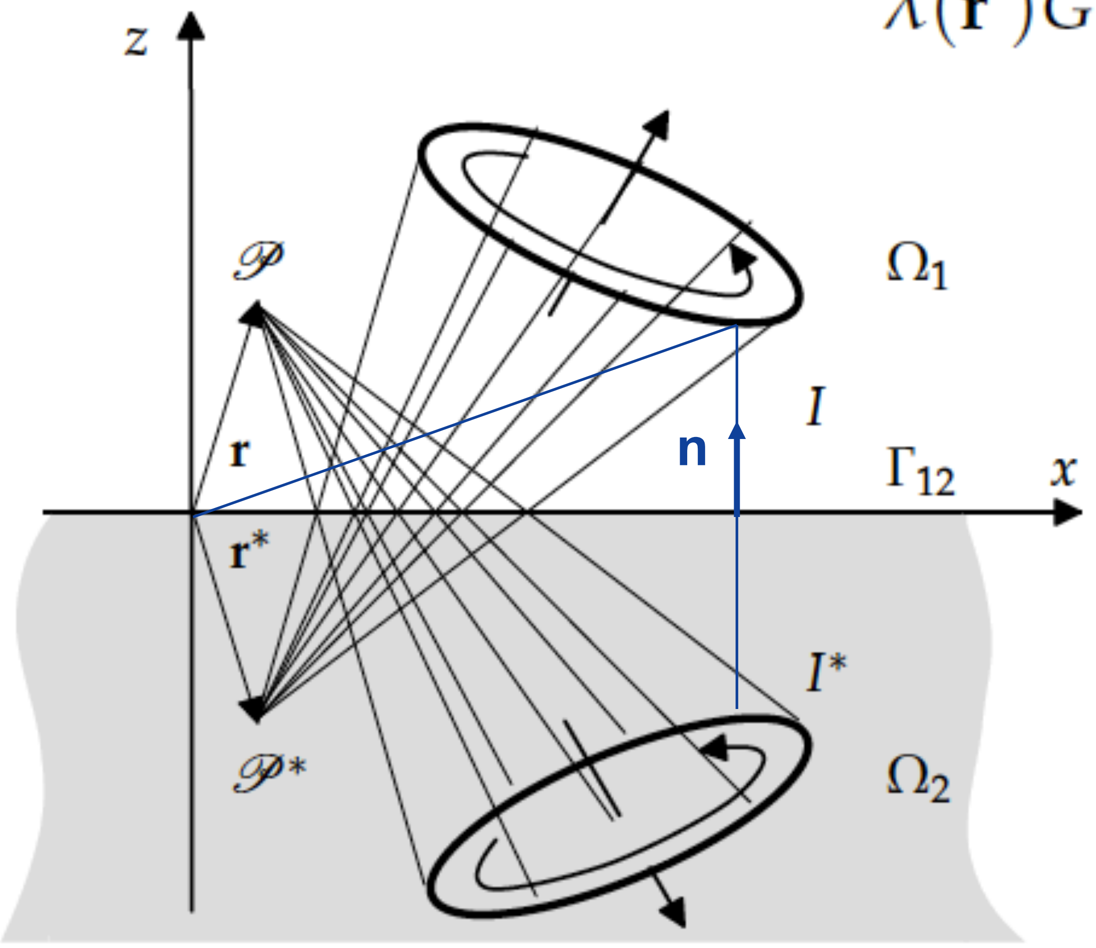


Notes on the Imaging Method

- Domain 1: Domain with current sources
- Domain 2: Highly permeable material
  - All imaging currents must be in domain 2
  - The sources and the images must create a field that satisfies the continuity conditions at the interface between domains 1 and 2
  - The image of the image must be the original source
  - The field generated in domain 1 is identical to the source field plus the field from the (iron) magnetization.
  - The field generated in domain 2 has no physical significance

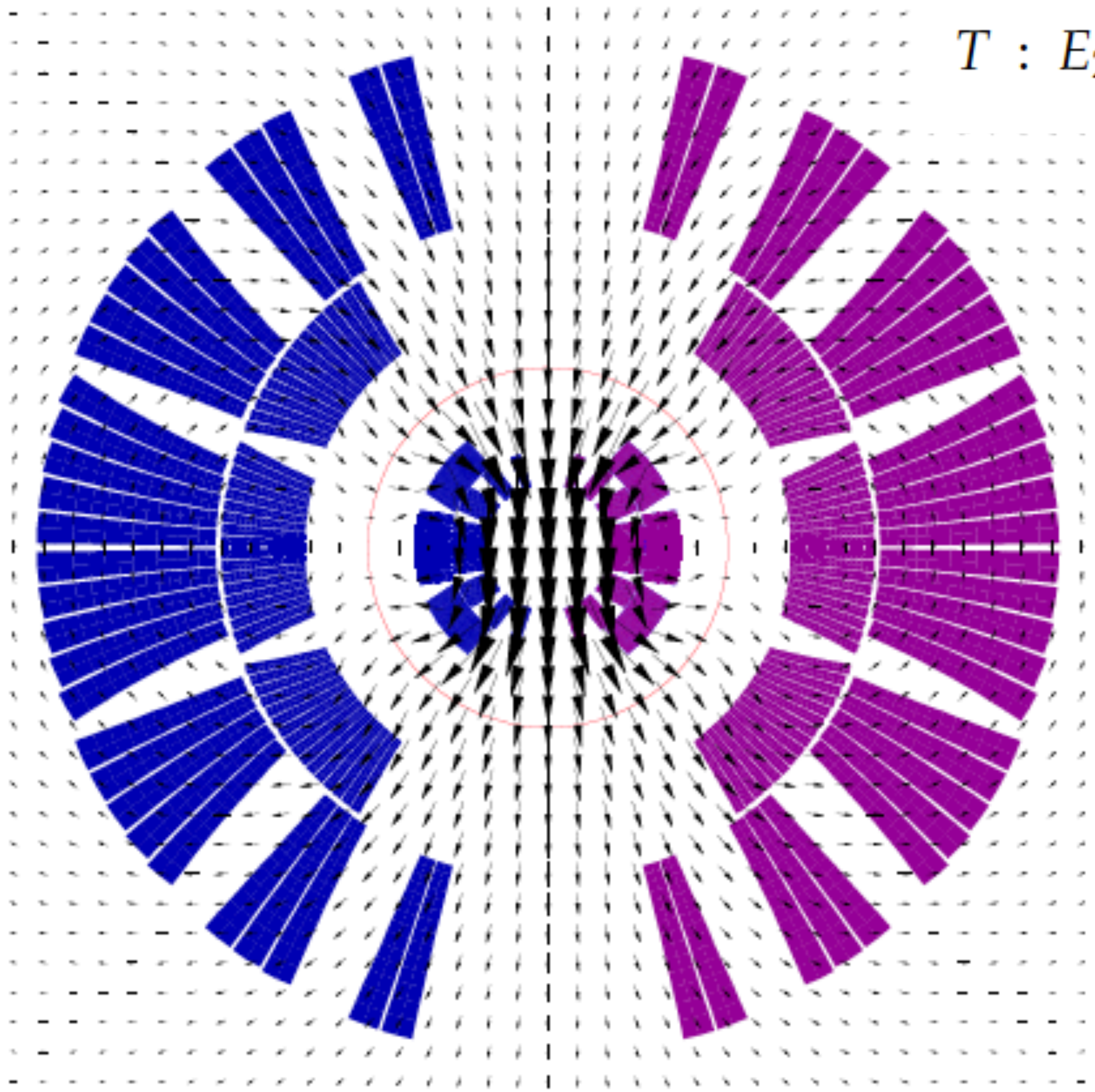
The Imaging Current Method

$$\lambda(\mathbf{r}')G(\mathbf{r}, T\mathbf{r}') = \lambda(\mathbf{r})G(T\mathbf{r}, \mathbf{r}')$$



$$T : E_3 \to E_3 : \mathbf{r}' \mapsto T\mathbf{r}' = \mathbf{r}' - 2\mathbf{n}\,(\mathbf{n}\cdot\mathbf{r}')\,,$$

The Imaging Current Method

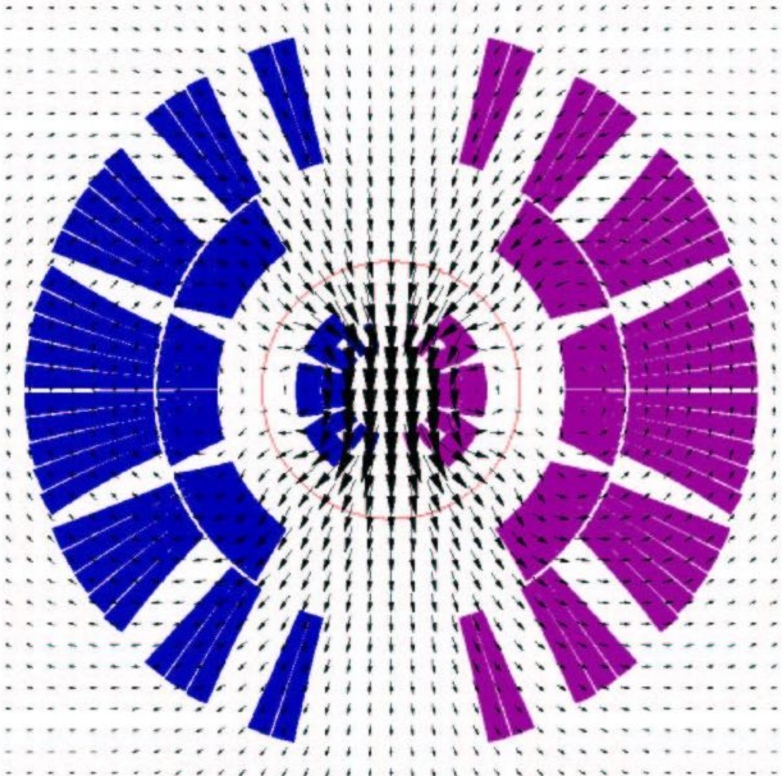


$$T : E_2 \to E_2 : \mathbf{r}' \mapsto T\mathbf{r}' = \frac{r_y^2}{|\mathbf{r}'|^2}\mathbf{r}' ,$$

$$I^* = \lambda_\mu I := \frac{\mu_r - 1}{\mu_r + 1} I .$$

Imaging Method

$$B_n(r_0) = -\sum_{k=1}^{K} \frac{\mu_0 I_k}{2\pi} \frac{r_0^{n-1}}{r_{c,k}^n} \left(1 + \lambda_\mu \left(\frac{r_{c,k}}{r_y}\right)^{2n}\right) \cos n\varphi_{c,k} \,,$$



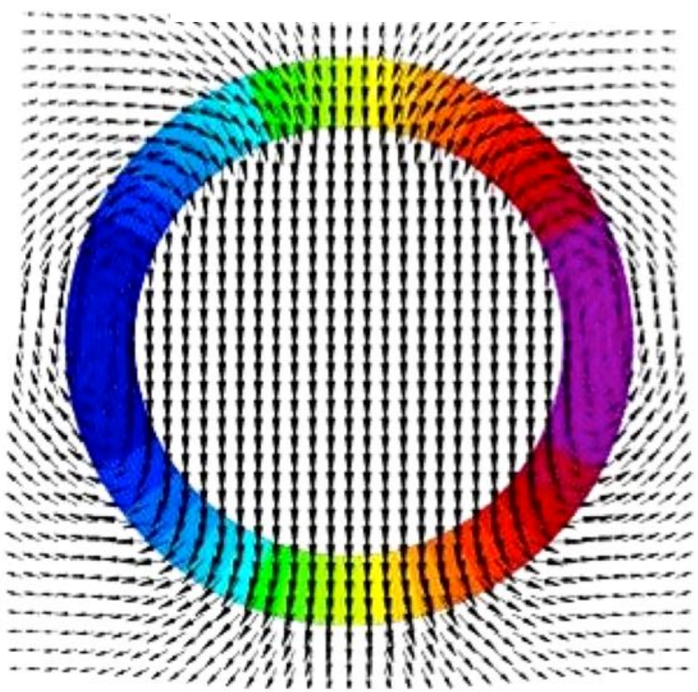
$$\lambda_\mu I := \frac{\mu_r - 1}{\mu_r + 1} I \,.$$

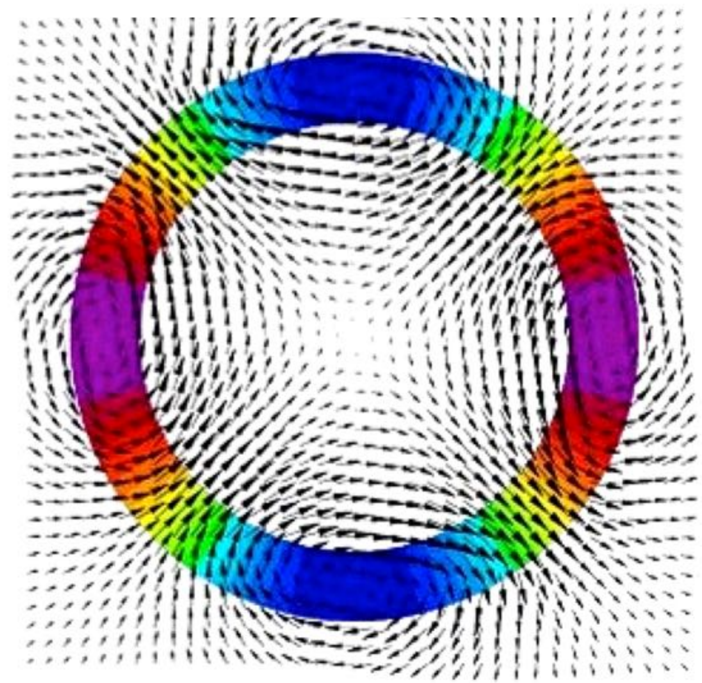
$$\frac{B_N^{\text{imag}}}{B_N + B_N^{\text{imag}}} \approx \left(1 + \left(\frac{r_y}{r}\right)^{2N}\right)^{-1} .$$

Ideal Current Distributions

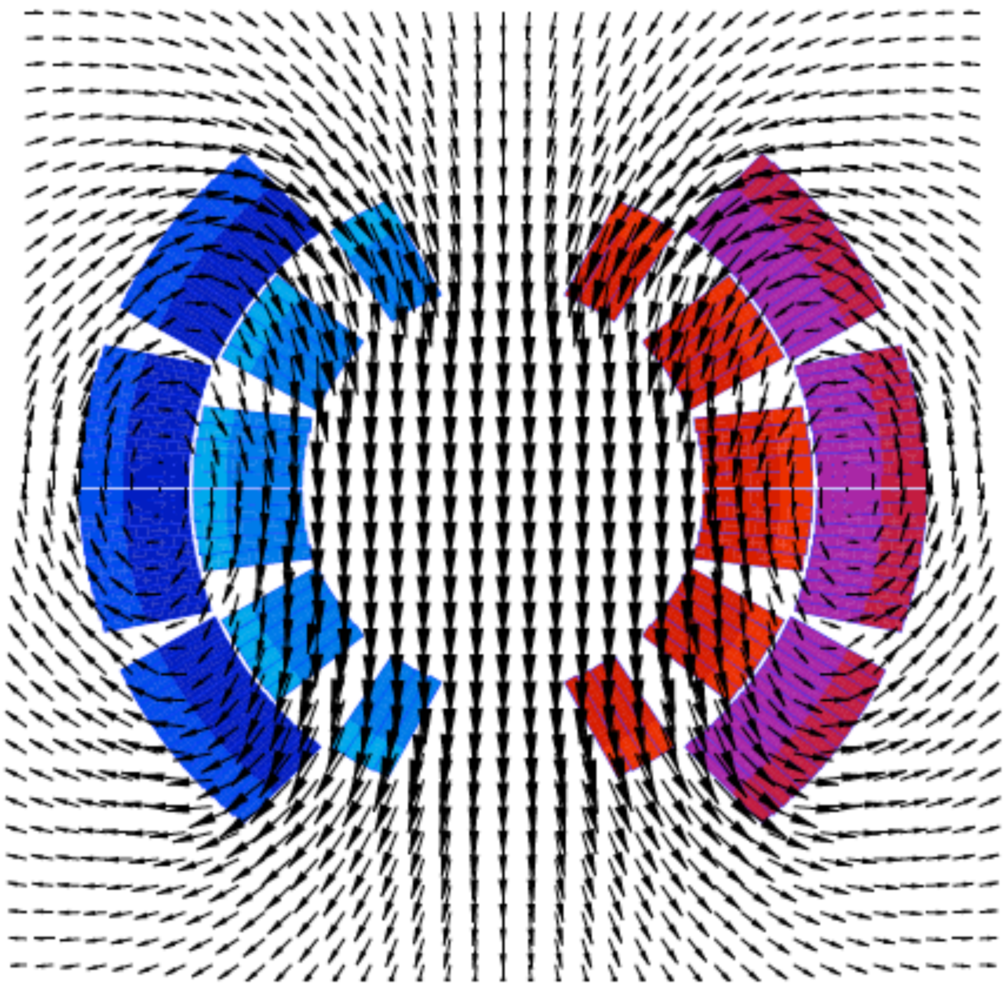
$$B_n(r_0) = \int_{r_a}^{r_e} \int_0^{2\pi} - \frac{\mu_0 J_E}{2\pi} \quad J_c(B) = d\,(B_{c2} - B) \quad \cos n\varphi_c\, r_c \mathrm{d}\varphi_c\, \mathrm{d}r_c$$

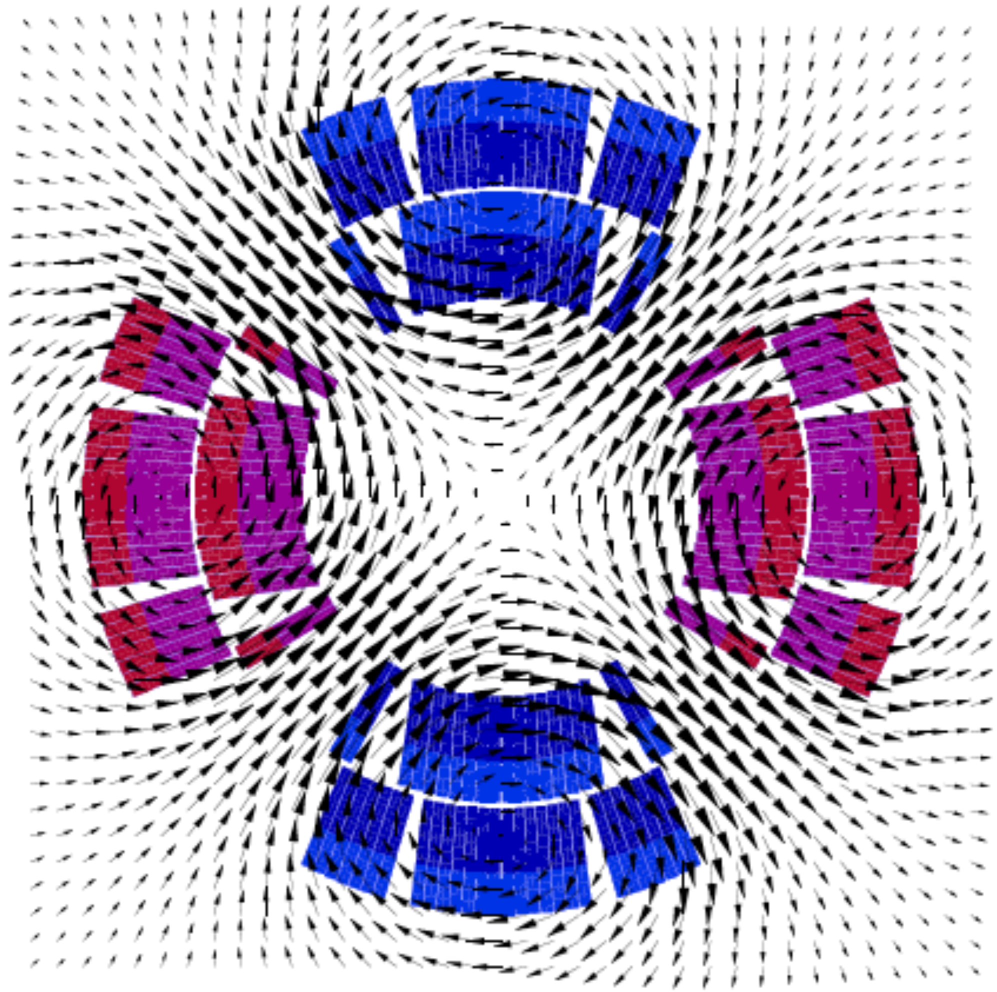
$$B = \frac{\mu_0}{2} \lambda_{\mathrm{tot}} J_c\,(r_e - r_a) = \frac{\mu_0}{2} \lambda_{\mathrm{tot}}\, d\,(B_{c2} - B)\,(r_e - r_a)\,,$$





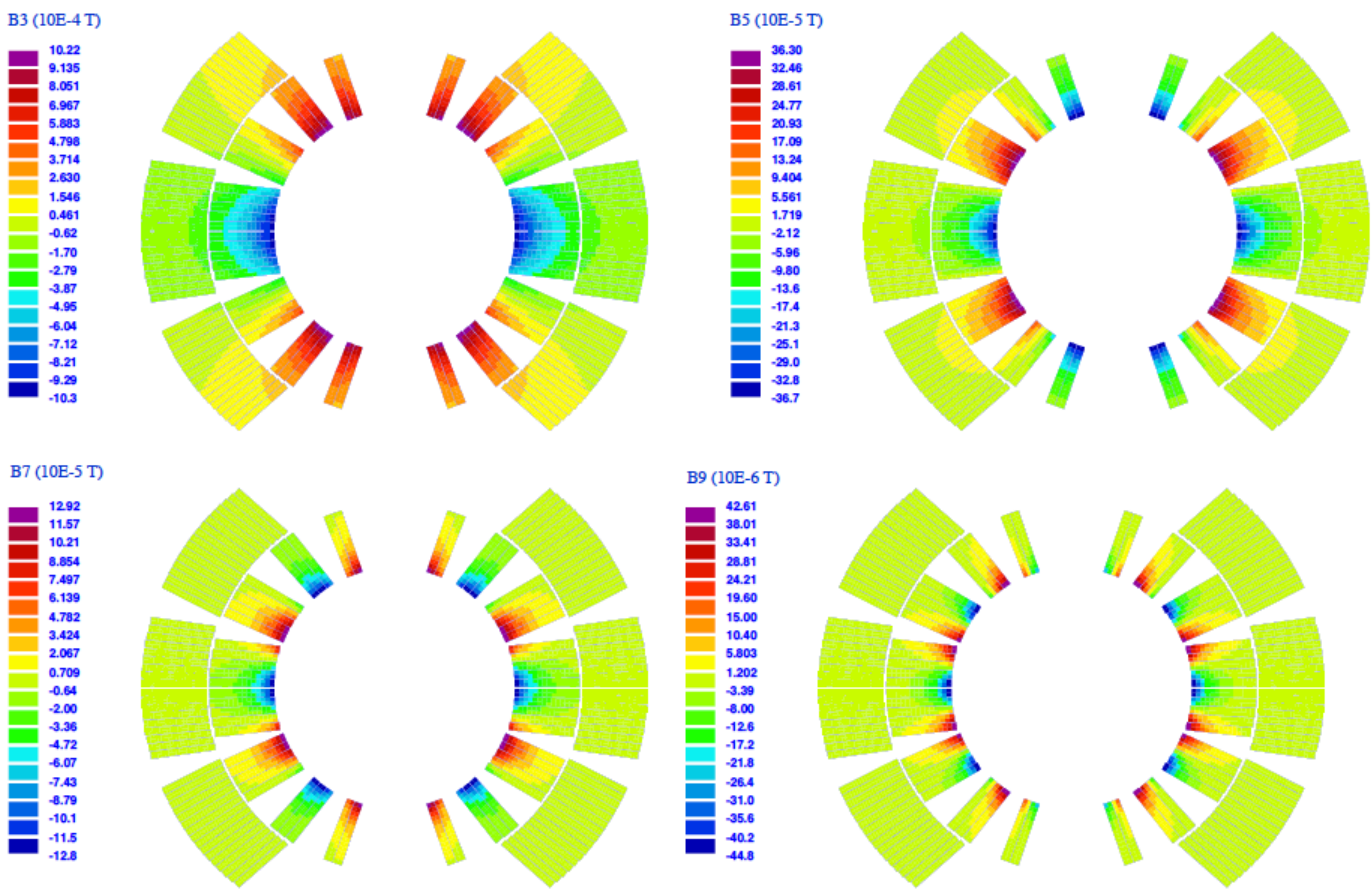
Coil-Block Approximations





Generation of Multipole Field Errors

$$B_n(r_0) = -\sum_{k=1}^{K} \frac{\mu_0 I_k}{2\pi} \frac{r_0^{n-1}}{r_{c,k}^{n}} \left(1 + \lambda_\mu \left(\frac{r_{c,k}}{r_y}\right)^{2n}\right) \cos n\varphi_{c,k} \,,$$

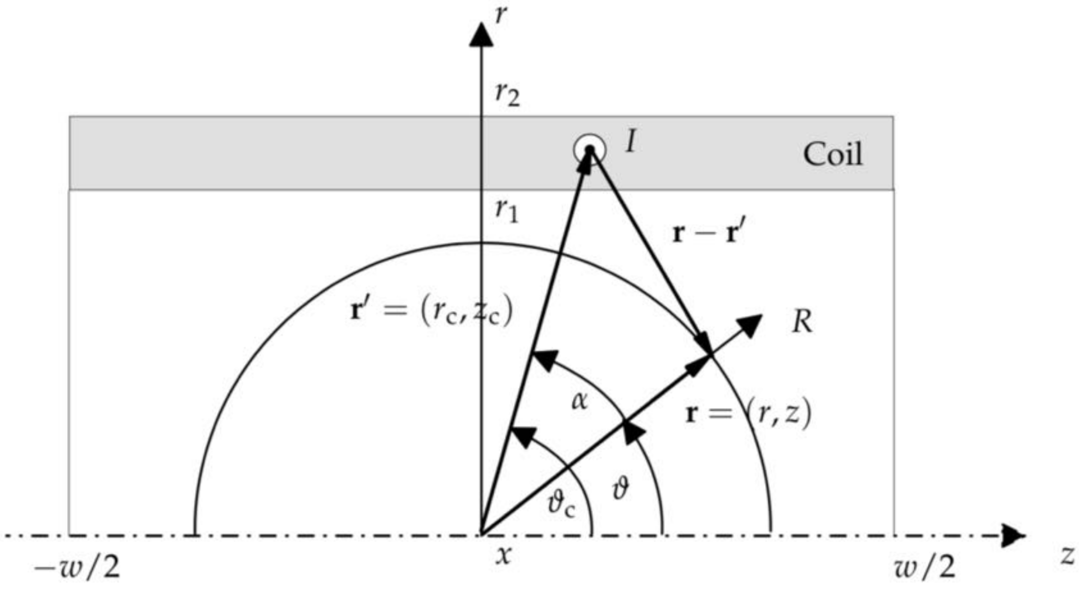


Expanding the Green Function

$$A_\varphi = \frac{\mu_0 I r_c}{2\pi} \int_0^\pi \frac{\cos(\varphi_c)\,\mathrm{d}\varphi_c}{\sqrt{r^2 + r_c^2 + (z - z_c)^2 - 2 r r_c \cos(\varphi_c)}}$$

$$= \frac{\mu_0 I r_c}{2\pi} \int_0^\pi \frac{\cos(\varphi_c)\,\mathrm{d}\varphi_c}{\sqrt{|\mathbf{r}|^2 + |\mathbf{r}'|^2 - 2|\mathbf{r}||\mathbf{r}'|(\cos(\vartheta)\cos(\vartheta_c) + \sin(\vartheta)\sin(\vartheta_c)\cos(\varphi_c))}}$$

$$= \frac{\mu_0 I r_c}{2} \frac{1}{|\mathbf{r}'|} \sum_{n=1}^{\infty} \left(\frac{|\mathbf{r}|}{|\mathbf{r}'|}\right)^n \frac{(n-1)!}{(n+1)!} P_n^1(\cos(\vartheta)) P_n^1(\cos(\vartheta_c)) .$$



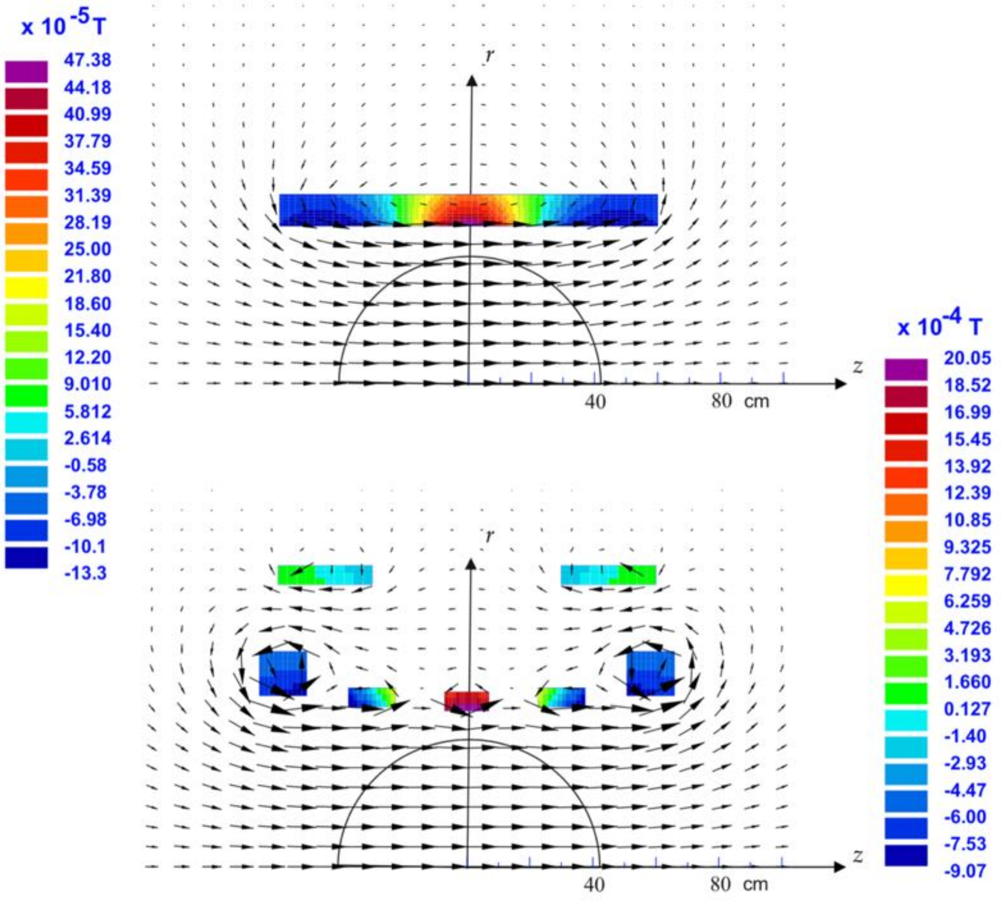
$$A_\varphi(R, \vartheta) = \sum_{n=1}^{\infty} \mu_0 \mathcal{A}_n R^n P_n^1(\cos(\vartheta)) .$$

$$\mathcal{A}_n = \frac{I r_c}{2} \frac{1}{R_c^{n+1}} \frac{1}{n(n+1)} P_n^1(\cos(\vartheta_c))$$

$$A_n(R_0) = -\mu_0 I \frac{r_c}{2} \frac{R_0^{n-1}}{R_c^{n+1}} \frac{1}{(n+1)} P_n^1(\cos(\vartheta_c))$$

Magnetic Field in the CMS Return Yoke

$$A_n(R_0) = \sum_{k=1}^{K} A_{n,k}(R_0) = \frac{-\mu_0}{n+1} \sum_{k=1}^{K} I_k \frac{r_{c,k}}{2} \frac{R_0^{n-1}}{R_{c,k}^{n+1}} P_n^1(\cos(\vartheta_{c,k}))$$



Magnetic Energy

$$W = \frac{1}{2}\int_V \mathbf{H}\cdot\mathbf{B}\,\mathrm{d}V$$

$$\mathrm{div}\,(\mathbf{A}\times\mathbf{H}) = \mathbf{H}\cdot\ \mathrm{curl}\,\mathbf{A} - \mathbf{A}\cdot(\,\mathrm{curl}\,\mathbf{H})$$

$$\begin{aligned} W &= \frac{1}{2}\int_V \mathbf{H}\cdot\ \mathrm{curl}\,\mathbf{A}\,\mathrm{d}V \\ &= \frac{1}{2}\int_V \mathrm{div}\,(\mathbf{A}\times\mathbf{H})\,\mathrm{d}V + \frac{1}{2}\int_V \mathbf{A}\cdot\ \mathrm{curl}\,\mathbf{H}\,\mathrm{d}V \\ &= \frac{1}{2}\int_{\partial V}(\mathbf{A}\times\mathbf{H})\cdot\mathrm{d}\mathbf{a} + \frac{1}{2}\int_V \mathbf{A}\cdot\ \mathrm{curl}\,\mathbf{H}\,\mathrm{d}V \end{aligned}$$

$$W = \frac{1}{2}\int_V \mathbf{A}\cdot\ \mathrm{curl}\,\mathbf{H}\,\mathrm{d}V = \frac{1}{2}\int_V \mathbf{A}\cdot\mathbf{J}\,\mathrm{d}V$$

Inner Energy

$$\int_{\partial a} \mathbf{H} \cdot \mathrm{d}\mathbf{s} = \int_a \mathbf{J} \cdot \mathrm{d}\mathbf{a}$$

$$H\, 2\pi r = \frac{I}{\pi r_0^2}\, \pi r^2$$

$$H = \frac{Ir}{2\pi r_0^2}$$

$$\mathrm{d}W = \frac{1}{2} BH\, 2\pi r l \mathrm{d}r = \mu_0 H^2\, \pi r l \mathrm{d}r = \frac{\mu_0 l I^2}{4\pi r_0^4} r^3 \mathrm{d}r$$

Therefore the total energy in the wire is

$$\frac{W}{l} = \frac{\mu_0 I^2}{4\pi r_0^4} \int_0^{r_0} r^3 \mathrm{d}r = \frac{\mu_0 I^2}{16\pi}$$

For one aperture of the LHC main dipole the stored energy at 8.33 T is 237 kJ/m. The energy stored in the strands is 4.3 J/m.

Inductance

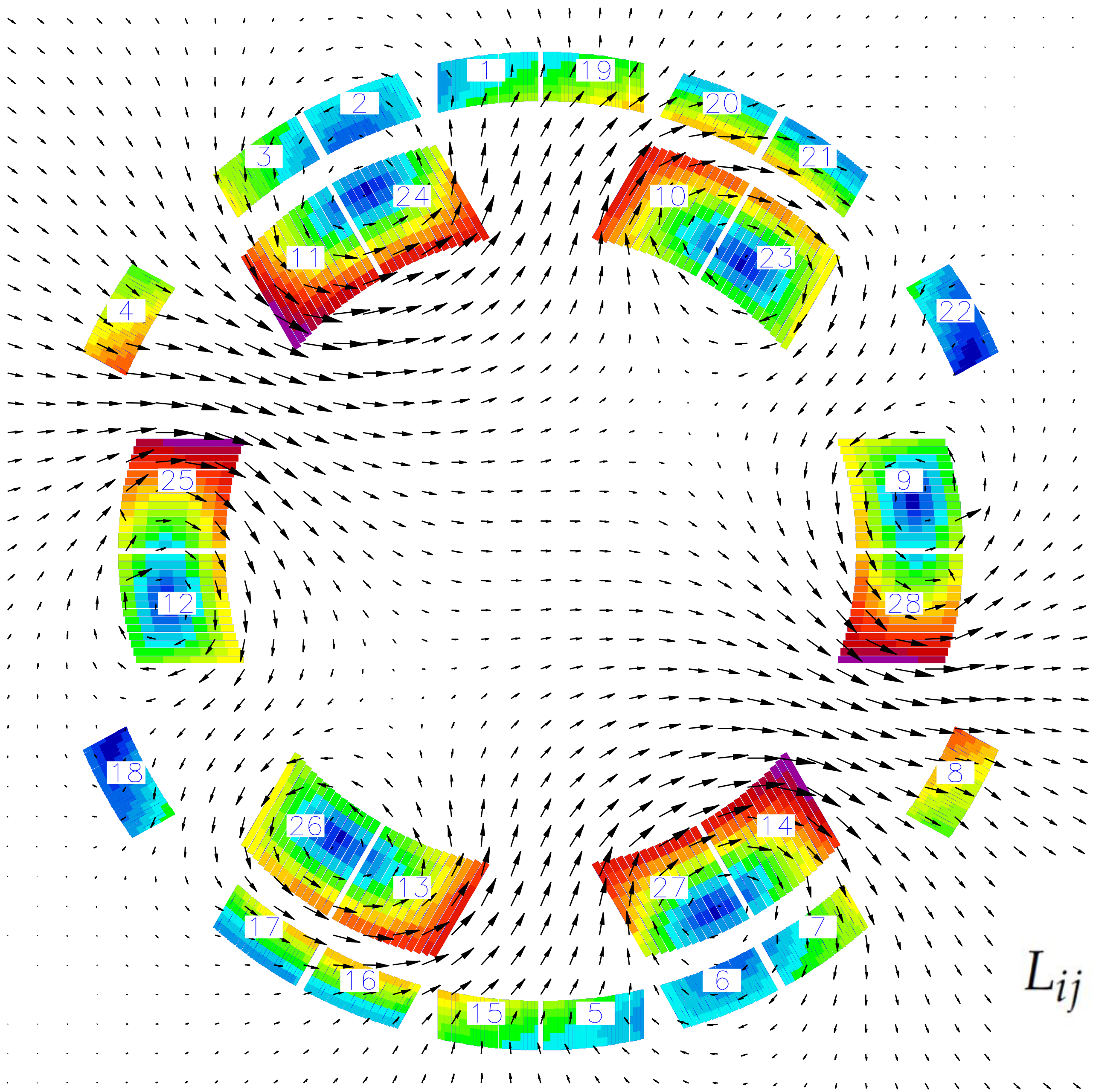
$$W = \frac{\mu_0}{8\pi} \int_V \int_{V'} \frac{\mathbf{J}(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V' \mathrm{d}V$$

$$W = \sum_{i=1}^{n} \sum_{j=1}^{n} W_{ij} = \frac{\mu_0}{8\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_V \int_{V'} \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V' \mathrm{d}V$$

$$= \frac{\mu_0}{8\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} I_i I_j \int_V \int_{V'} \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}')}{I_i I_j |\mathbf{r} - \mathbf{r}'|} \mathrm{d}V' \mathrm{d}V$$

$$L_{ij} = \frac{\mu_0}{4\pi I_i I_j} \int_V \int_{V'} \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V' \mathrm{d}V$$

Combined Dipole Sextupole Corrector



$$L_{ii} = \frac{2W_{ii}}{I^2}$$

$$L_{ij} = \frac{1}{2}\left(\frac{2W_{ij}}{I^2} - L_{ii} - L_{jj}\right)$$

Mutual Inductance Matrix

Coil	1	2	3	4	5	6	7	8
1	12.601	6.517	-0.245	0.252	0.478	-0.478	-0.252	0.245
2	6.517	12.601	-0.478	-0.252	0.245	-0.245	0.252	0.478
3	-0.245	-0.478	0.136	0.027	-0.010	0.009	-0.010	0.027
4	0.252	-0.252	0.027	0.136	0.027	-0.010	0.009	-0.010
5	0.478	0.245	-0.010	0.027	0.136	0.027	-0.010	0.009
6	-0.478	-0.245	0.009	-0.010	0.027	0.136	0.027	-0.010
7	-0.252	0.252	-0.010	0.009	-0.010	0.027	0.136	0.027
8	0.245	0.478	0.027	-0.010	0.009	-0.010	0.027	0.136

A coil of multipole order N does not couple into one of order K

Nonlinear Circuits (Differential Inductance)

$$U(t) = \frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}(L\,I)}{\mathrm{d}t} = L\frac{\mathrm{d}I}{\mathrm{d}t} + I\frac{\mathrm{d}L}{\mathrm{d}t}$$

$$U = L^{\mathrm{d}}\,\frac{\mathrm{d}I}{\mathrm{d}t}$$

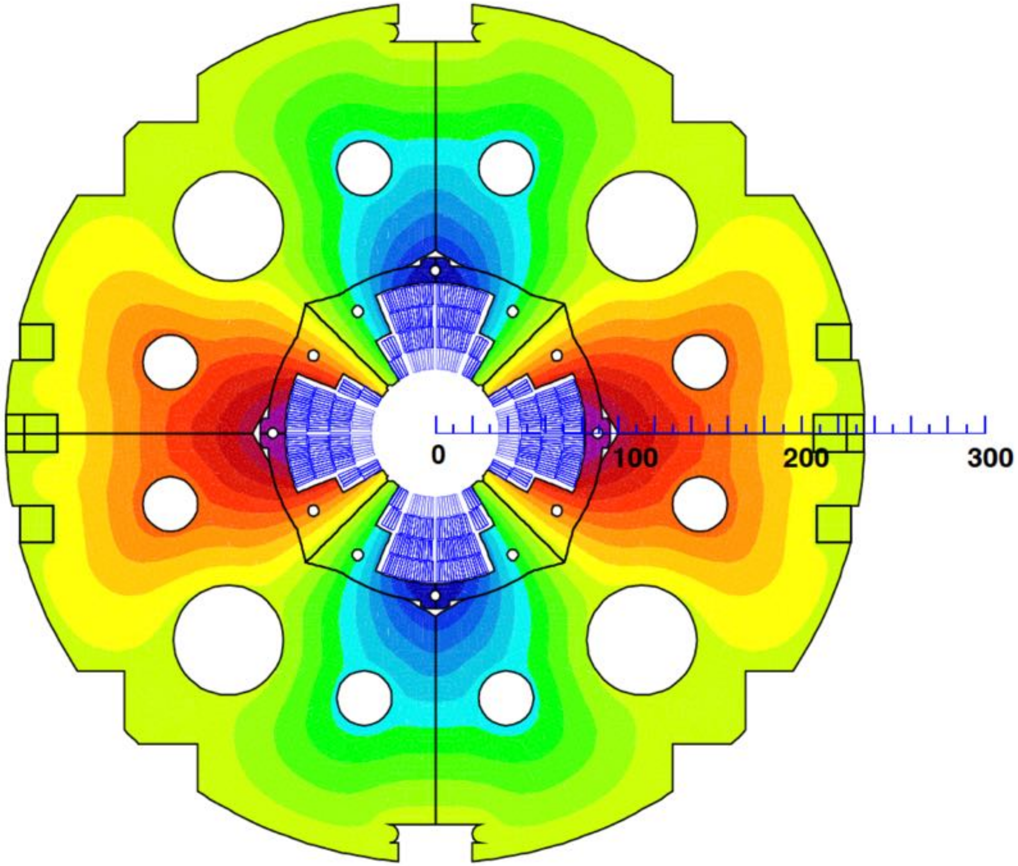
$$\mathrm{d}L = \frac{\partial L}{\partial I}\mathrm{d}I + \frac{\partial L}{\partial t}\mathrm{d}t$$

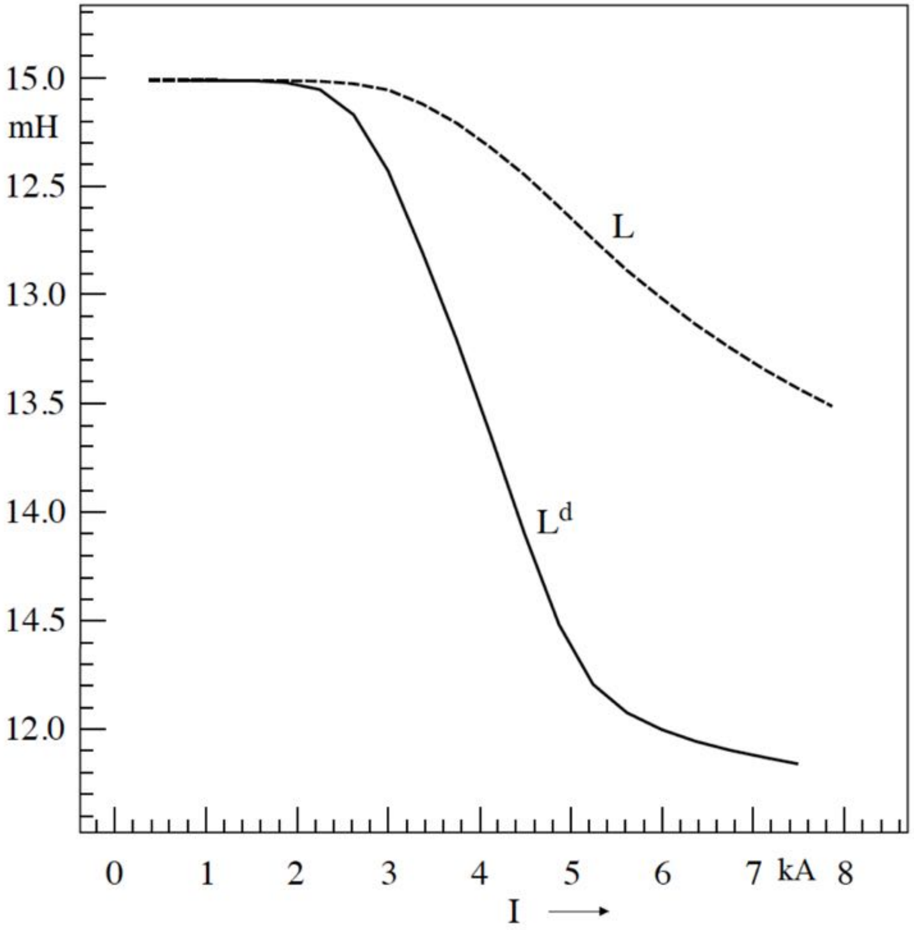
$$U(t) = \left(\frac{\partial L}{\partial I}\,I + L\right)\frac{\mathrm{d}I}{\mathrm{d}t} + I\frac{\partial L}{\partial t}$$

$$L^{\mathrm{d}} = L + I\frac{\partial L}{\partial I} = \frac{\mathrm{d}\Phi}{\mathrm{d}I}$$

For example,
machine rotor motion

Differential Inductance for the MQXY





Nonlinear Circuits

$$W = \frac{1}{2} L^{\mathrm{W}} I^2$$

$$W = \int_0^t U\, I\, \mathrm{d}\tau = \int_0^t \frac{\mathrm{d}\Phi(I(\tau))}{\mathrm{d}\tau} I(\tau)\, \mathrm{d}\tau = \int_0^t \frac{\mathrm{d}\Phi(I(\tau))}{\mathrm{d}I} \frac{\mathrm{d}I(\tau)}{\mathrm{d}\tau} I(\tau)\, \mathrm{d}\tau$$
$$= \int_0^{I(t)} \frac{\mathrm{d}\Phi}{\mathrm{d}I} I\, \mathrm{d}I = \int_0^{I(t)} L^{\mathrm{d}}\, I\, \mathrm{d}I\,,$$

$$L^{\mathrm{W}} = \frac{2}{I(t)^2} \int_0^{I(t)} L^{\mathrm{d}}\, I\, \mathrm{d}I = \int_0^1 2\lambda L^{\mathrm{d}}(\lambda I)\, \mathrm{d}\lambda$$

No hyseresis !