EM algorithm for muon scattering tomography

Second MODE Workshop on Differential Programming for Experimental Design

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Outline

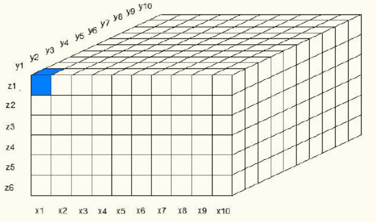
- Considering a volume sized [1 m, 1 m, 0.8 m] in x,y,z.
- It is subdivided in cubes of sides [0.1 m, 0.1 m, 0.1 m].

Data:
Given a batch of muons, for each of them assume knowledge of:

- Measured entry and exit points in x,y,z coordinates (x_{in}, y_{in}, z_{in}), $(x_{out}, y_{out}, z_{out})$.
- Measured entry and exit angles $\theta_{x_{in}}$, $\theta_{y_{in}}$, $\theta_{x_{out}}$, $\theta_{y_{out}}$.
- Momentum.

Problem:
Determining the material composition of each cube:

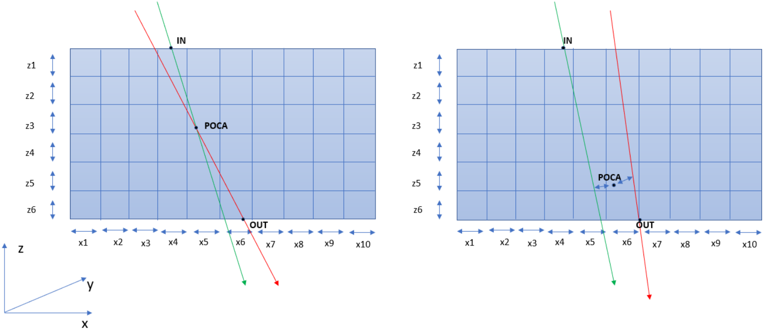
- Inferring the value of X0, radiation length of the material.



POCA method

Assumption:

- the muon scattering occurs in a single point;
- POCA point: *Point Of Closest Approach*, which minimizes the distance between the two reconstructed tracks.

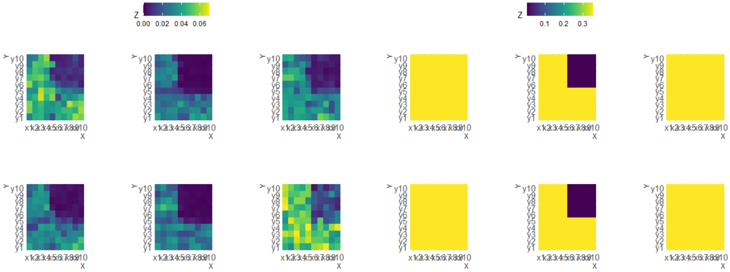


Two possible cases: incident lines (left) and skew lines (right).

Results

DATASET:

- 10000 muons *(the number of muons used is 9981)*;
- 100μ*m* single hit resolution;
- Incoming and outgoing measured point coordinates;
- Incoming and outgoing measured angles.



(a) *POCA* (b) *TRUE*

EM algorithm

Based on:

- **A sample of observed data**: measured scattering and displacement in x and y: $\Delta\theta_x = \theta_{x_{out}} - \theta_{x_{in}}$; $\Delta\theta_y = \theta_{y_{out}} - \theta_{y_{in}}$; $\Delta x = x_{out} - x_{in}$; $\Delta y = y_{out} - y_{in}$.
- **A sample of hidden data**: scattering angles and displacements inside material.
- **An initial assumption on value of X0 for each cube**.

It iteratively alternates between an Expectation step and a Maximization step:

- To obtain the value which maximizes the log-likelihood associated to the sample of *complete* data (including both observed and hidden data).

- **The distribution of scattering angles** may be approximated as a **zero-mean Gaussian**: $\Delta\theta \sim N(0, \sigma^2_{\Delta\theta})$.
    - The width of distribution may be expressed in terms of material properties: $\sigma_{\Delta\theta} \simeq \frac{15Mev}{\beta cp}\sqrt{\frac{H}{X_0}}$, with p muon momentum and H depth of the material.
    - Defining the scattering density of a material in terms of $X0$: $\lambda(X_0) \equiv \left(\frac{15}{p_0}\right)^2 \frac{1}{X_0}$, with p_0 nominal muon momentum $\Rightarrow \sigma^2_{\Delta\theta} = \lambda H(\frac{p_0}{p})^2$.
- **The distribution of displacement** may be approximated as a **zero-mean Gaussian**: $\Delta x \sim N(0, \sigma^2_{\Delta x})$.
    - The width of distribution may be expressed in terms of $\sigma_{\Delta\theta}$: $\sigma_{\Delta x} = \frac{H}{\sqrt{3}}\sigma_{\Delta_\theta}$.
- **Scattering angles and displacement in a plane are correlated**:
    - Correlation: $\rho_{\Delta x \Delta\theta} = \frac{\sqrt{3}}{2}$.
    - $\sigma_{\Delta x \Delta\theta} \equiv \rho_{\Delta x \Delta\theta}\ \sigma_{\Delta_x}\ \sigma_{\Delta_\theta} = \frac{\lambda}{2} H^2 (\frac{p_0}{p})^2$
    - The probability density function of scattering angles and displacement may be approximated as a **jointly Gaussian**: $f(\Delta x, \Delta\theta_x | X_0, p) \sim N(0, \Sigma)$, with $\Sigma = \begin{pmatrix} \sigma^2_{\Delta\theta} & \sigma_{\Delta x \Delta_\theta} \\ \sigma_{\Delta x \Delta_\theta} & \sigma^2_{\Delta x} \end{pmatrix} = \lambda \begin{pmatrix} H & \frac{H^2}{2} \\ \frac{H^2}{2} & \frac{H^3}{3} \end{pmatrix} \left(\frac{p_0}{p}\right)^2$.

For each muon $i = 1, \dots, M$, given $j = 1, \dots, N$ voxels:

- Observed data : $D_i = (D_{x_i}, D_{y_i})$, where $D_{x_i} = (\Delta\theta_{x_i}, \Delta x_i)$, $D_{y_i} = (\Delta\theta_{y_i}, \Delta y_i)$
- Hidden data : $H_{ij} = (H_{x_{ij}}, H_{y_{ij}})$, *where* $\mathrm{H}_{x_{ij}} = (\Delta\theta_{x_{ij}}, \Delta x_{ij})$, $D_{y_{ij}} = (\Delta\theta_{y_{ij}}, \Delta y_{ij})$
- Considering scattering occurring in xy plane
  - $P(D_{x_i}|\lambda) = \frac{1}{2\pi(\det\Sigma_i)^{1/2}} exp\left(\frac{-D_{x_i}^T \Sigma_i^{-1} D_{x_i}}{2}\right)$
  - Likelihood: $L(\lambda|D_x) = \prod_{i=1}^M P(D_{x_i}|\lambda) \Rightarrow logL(\lambda|D_x) = \sum_{i=1}^M log(P(D_{x_i}|\lambda))$
  - $P(H_{x_{ij}}|\lambda) = \frac{1}{2\pi(\det\Sigma_{ij})^{1/2}} exp\left(\frac{-H_{x_{ij}}^T \Sigma_{ij}^{-1} H_{x_{ij}}}{2}\right)$
  - Likelihood: $L(\lambda|H_x) = \prod_i \prod_j P(H_{x_{ij}}|\lambda)$
    $\Rightarrow logL(\lambda|H_x) = \sum_{i=1}^M \sum_{j\le N} log(P(H_{x_{ij}}|\lambda))$

E step

Determining the auxiliary function $Q(\lambda;\lambda^{(n)}) = E_{H|D,\lambda^{(n)}}[log(P(H|\lambda))]$

M step

Maximizing the auxiliary function $\lambda^{(n+1)} = argmax_\lambda Q(\lambda;\lambda^{(n)})$

Geometric assumptions

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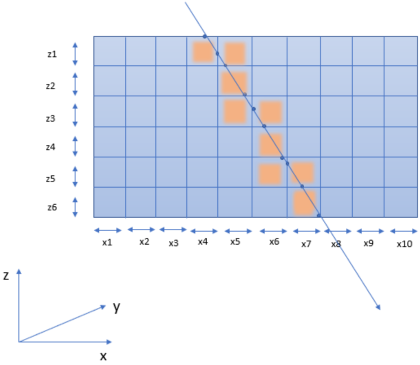
For each muon, needs to know:

- which voxels it hits;
- 3d path length inside each voxel.

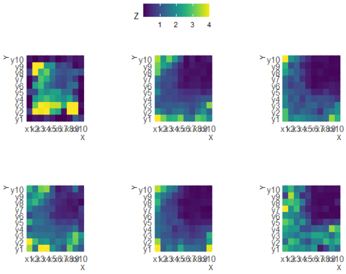
Initially, considering a straight line connecting entry to exit points.



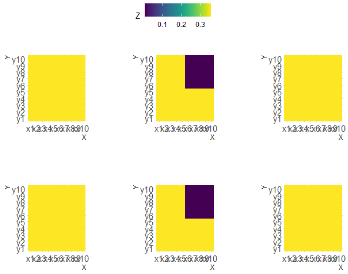
Results

DATASET:

- 10000 muons *(the number of muons used is 9981)*;
- 100µ*m* single hit resolution;
- Incoming and outgoing measured point coordinates;
- Incoming and outgoing measured angles;
- X0 initial guess: **0.005 m** in each voxel *(similar to beryllium)*;
- 55 iterations.

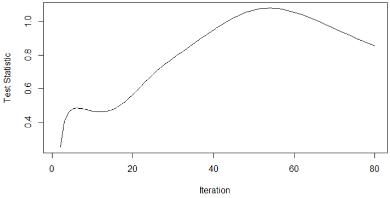


(a) *EM*



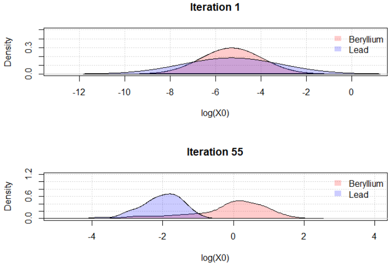
(b) *TRUE*

Results



(a) *Test Statistic:*

$$\frac{E[X0_{beryllium}] - E[X0_{lead}]}{\sqrt{\sigma^2_{X0_{beryllium}} + \sigma^2_{X0_{lead}}}}$$

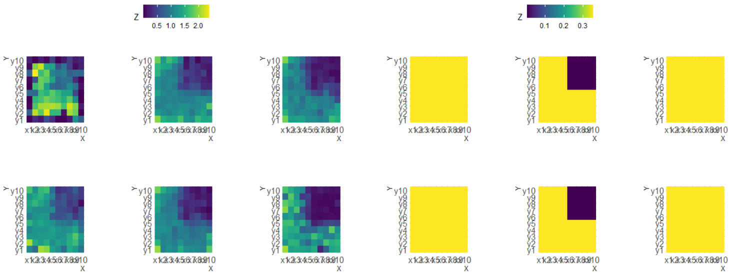


(b) *Densities of estimated log(X0) for voxels known as being of beryllium and lead.*

Results

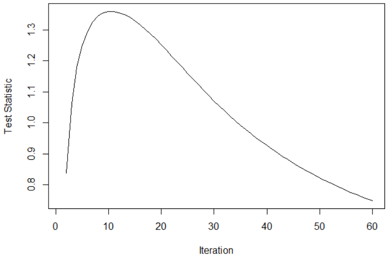
DATASET:

- 10000 muons *(the number of muons used is 9981)*;
- 100μ*m* single hit resolution;
- Incoming and outgoing measured point coordinates;
- Incoming and outgoing measured angles;
- X0 initial guess: **0.5 m** in each voxel *(similar to beryllium)*;
- 12 iterations.



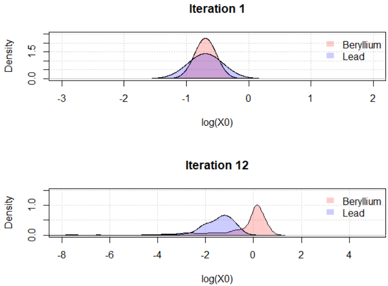
(a) *EM* (b) *TRUE*

Results



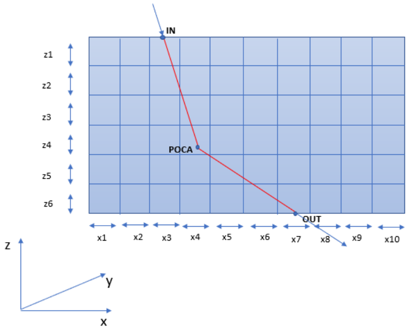
(a) *Test Statistic:*

$$\frac{E[X0_{beryllium}] - E[X0_{lead}]}{\sqrt{\sigma^2_{X0_{beryllium}} + \sigma^2_{X0_{lead}}}}$$



(b) *Densities of estimated log(X0) for known voxels of beryllium and lead.*

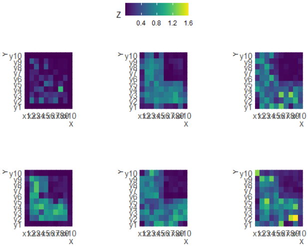
Improving the model considering muon path length as a straight line connecting entry to exit points, passing for the POCA location.



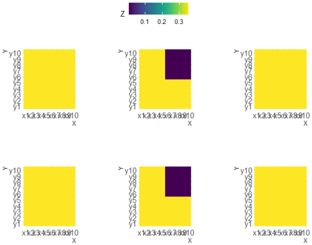
Results

DATASET:

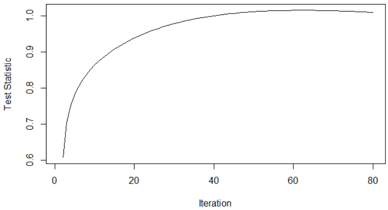
- 10000 muons *(the number of muons used is 9981)*;
- 100μ*m* single hit resolution;
- POCA point coordinates;
- Incoming and outgoing measured point coordinates;
- Incoming and outgoing measured angles;
- X0 initial guess: **0.5 m** in each voxel *(similar to beryllium)*;
- 60 iterations.



(a) *EM*

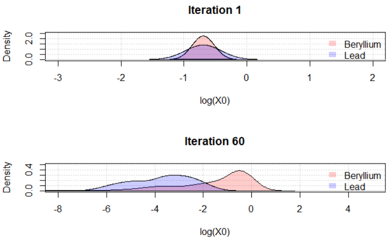


(b) *TRUE*

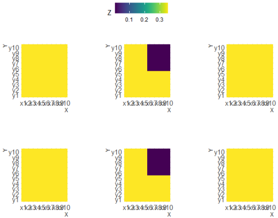


(a) *Test Statistic:*

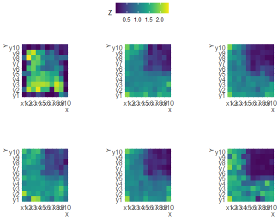
$$\frac{E[X0_{beryllium}] - E[X0_{lead}]}{\sqrt{\sigma^2_{X0_{beryllium}} + \sigma^2_{X0_{lead}}}}$$



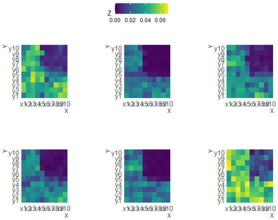
(b) *Densities of estimated log(X0) for known voxels of beryllium and lead.*



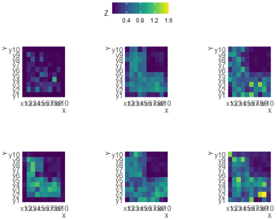
(a) *TRUE.*



(b) *EM without POCA coordinates.*



(c) *POCA method.*



(d) *EM with POCA coordinates.*

References

[1] L. J. Schultz et al. *Statistical Reconstruction for Cosmic Ray Muon Tomography*. IEEE transactions on image processing, Vol. 16, No. 8, 1985-1993. 2007.
[2] R. L. Siddon. *Fast calculation of the exact radiological path for a three-dimensional CT array*. Medical Physics, Vol. 12, No. 2, 252-255. 1985.
[3] T. Dorigo, A. Giammanco, P. Vischia (editors) MODE Collaboration. *Toward the End-to-End Optimization of Particle Physics Instruments with Differentiable Programming: a White Paper*. 2022. arXiv:2203.13818v1.

Thanks for your attention