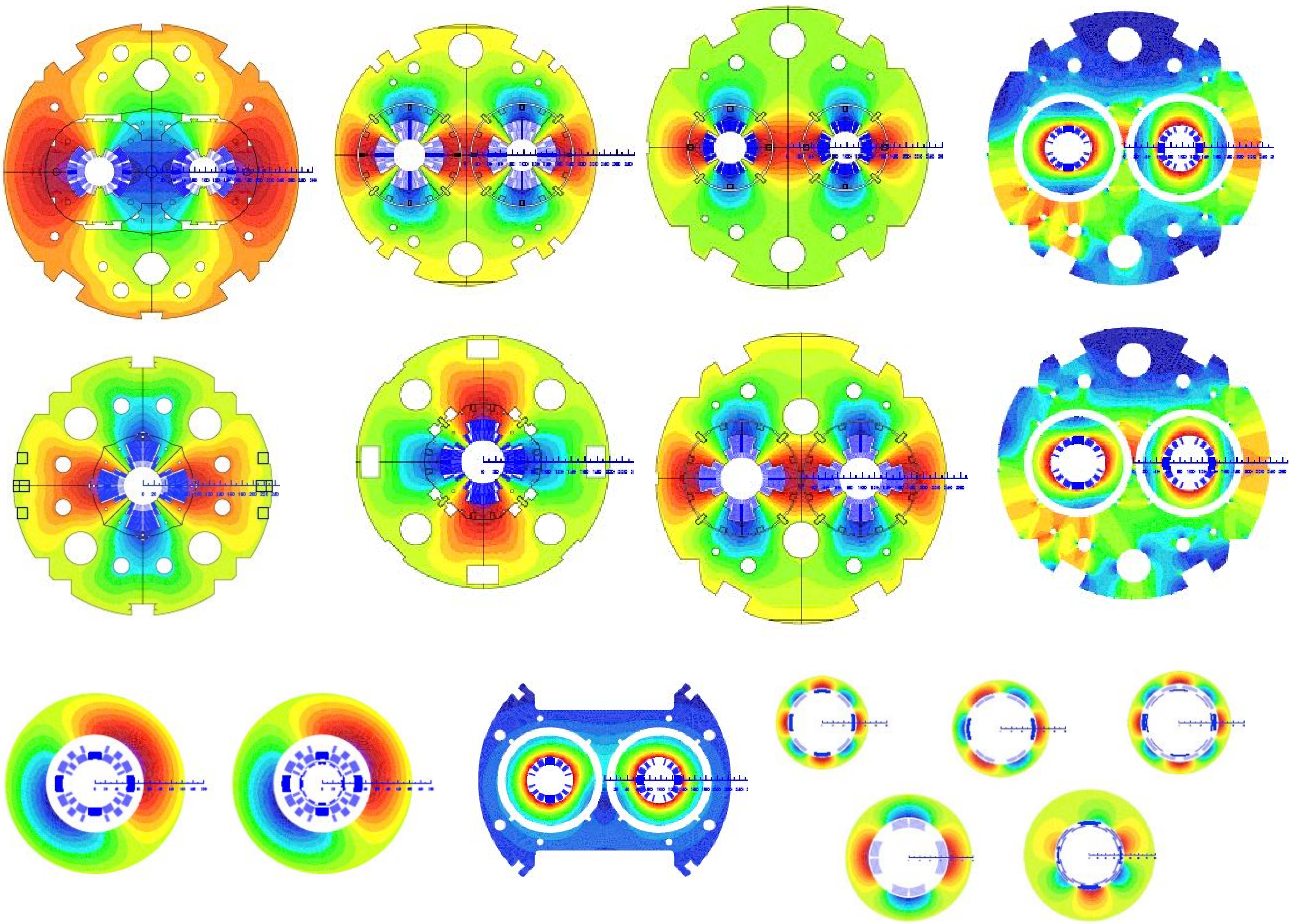
Electromagnetic Design of Accelerator Magnets and ROXIE User's Course

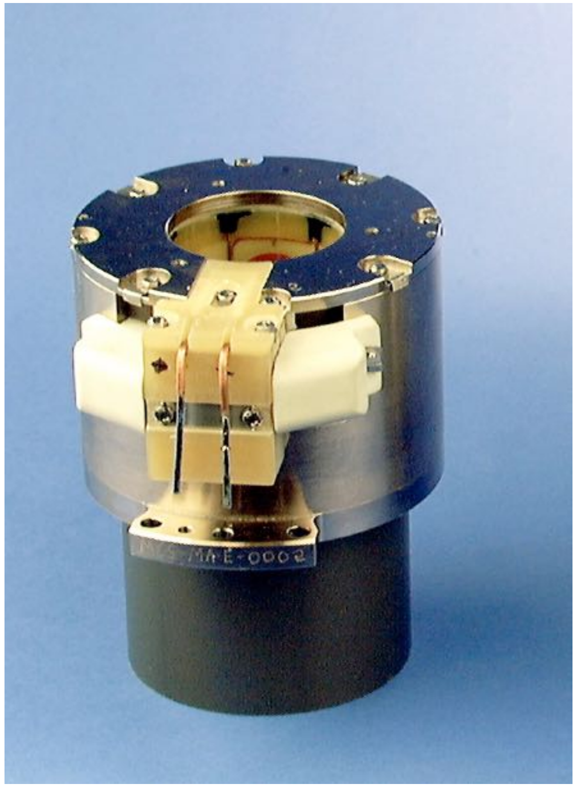
Numerical Field Simulation
FEM / BEM

Stephan Russenschuck, CERN, 2022

The LHC Magnet Zoo

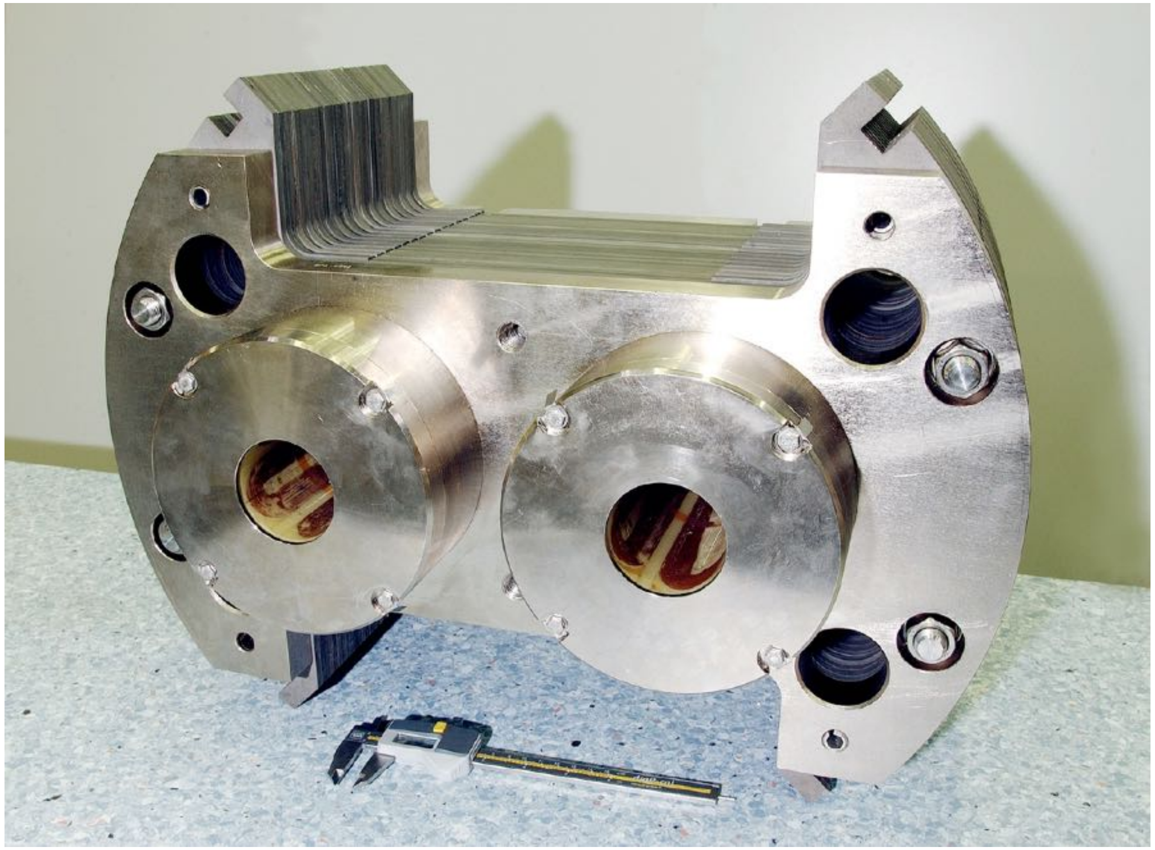


Corrector Magnets

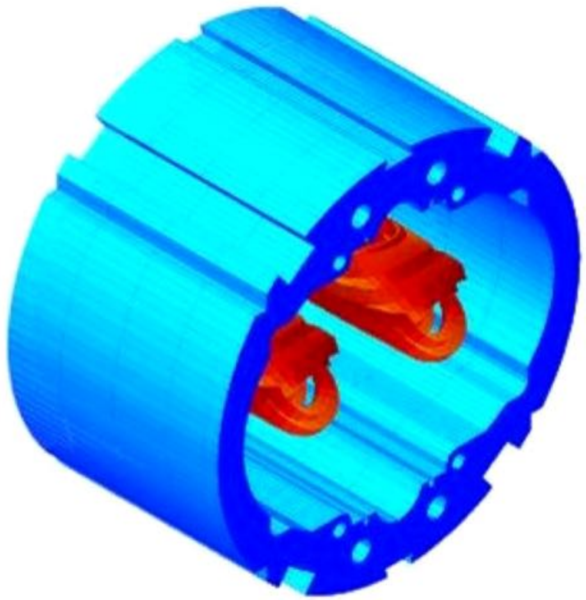


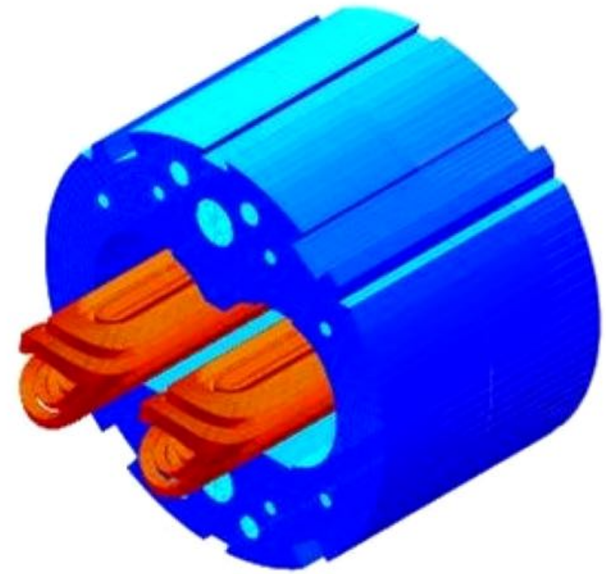
Sextupole-spool pieces

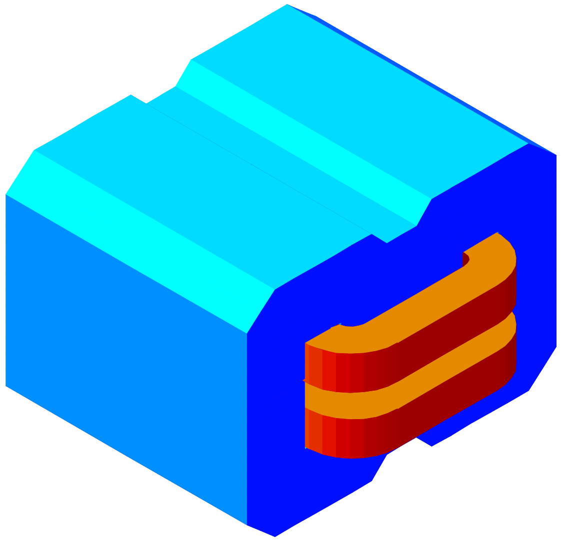
Octupole

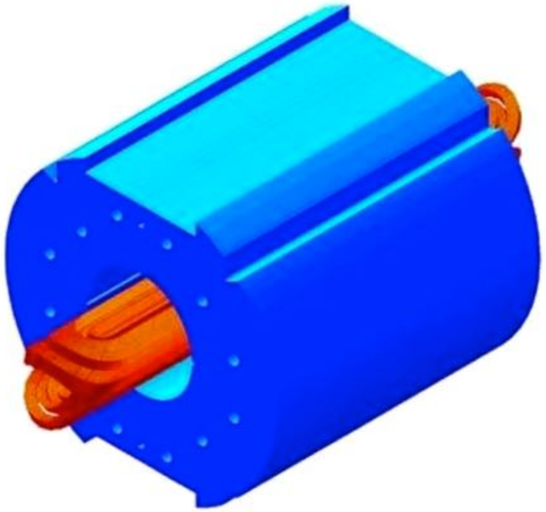


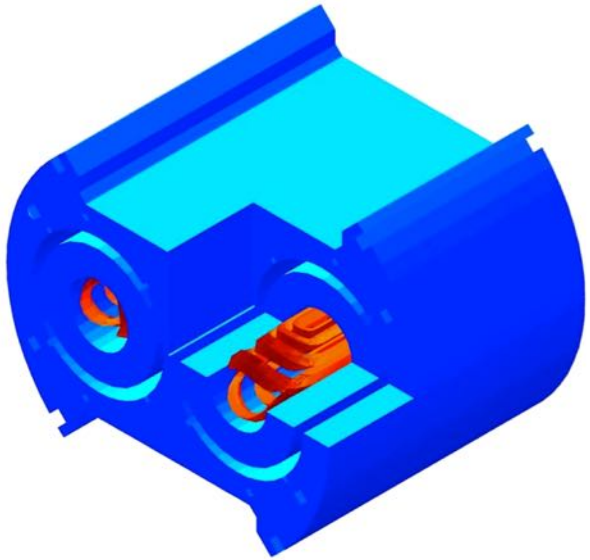
Magnet Extremities



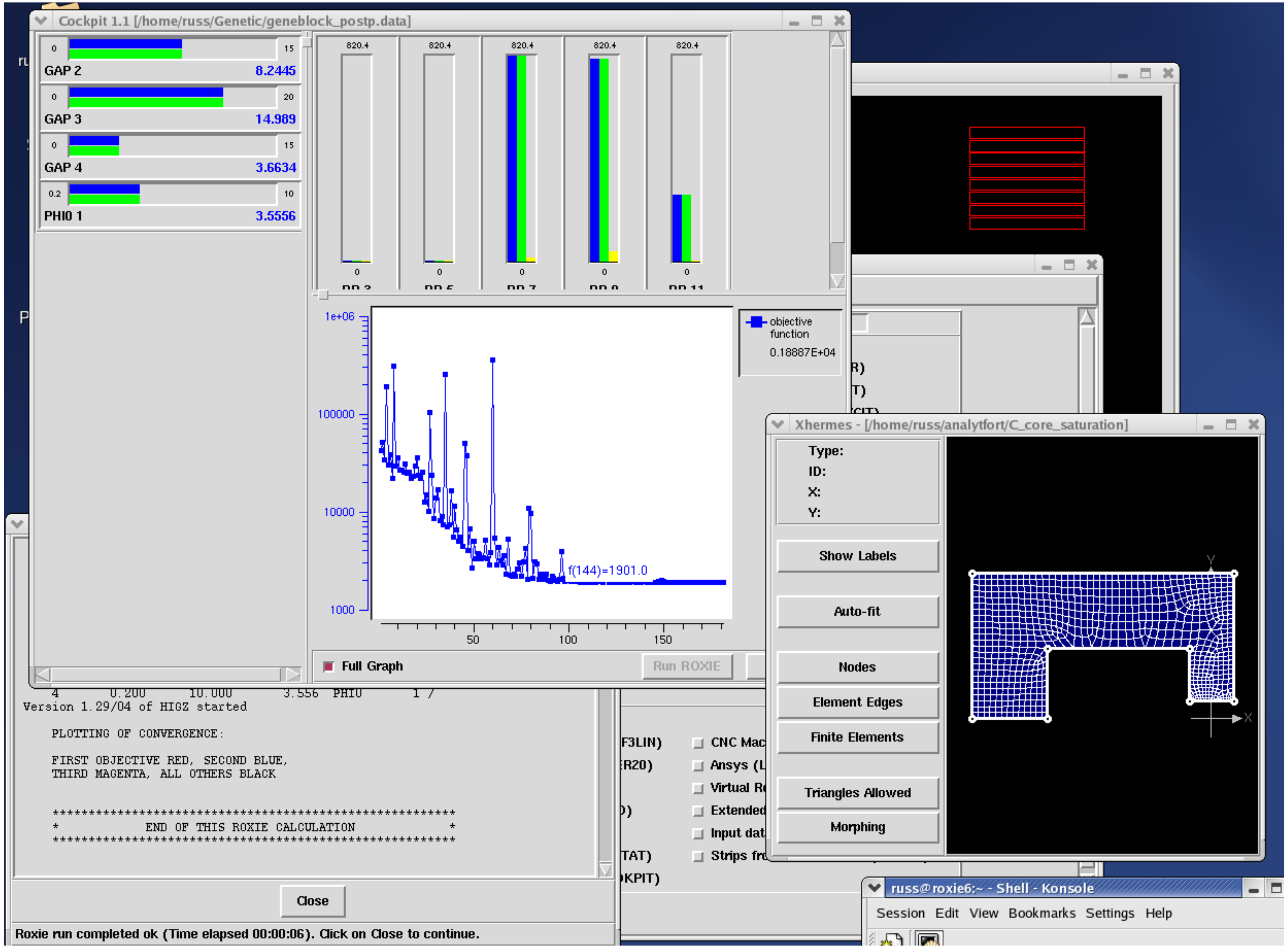




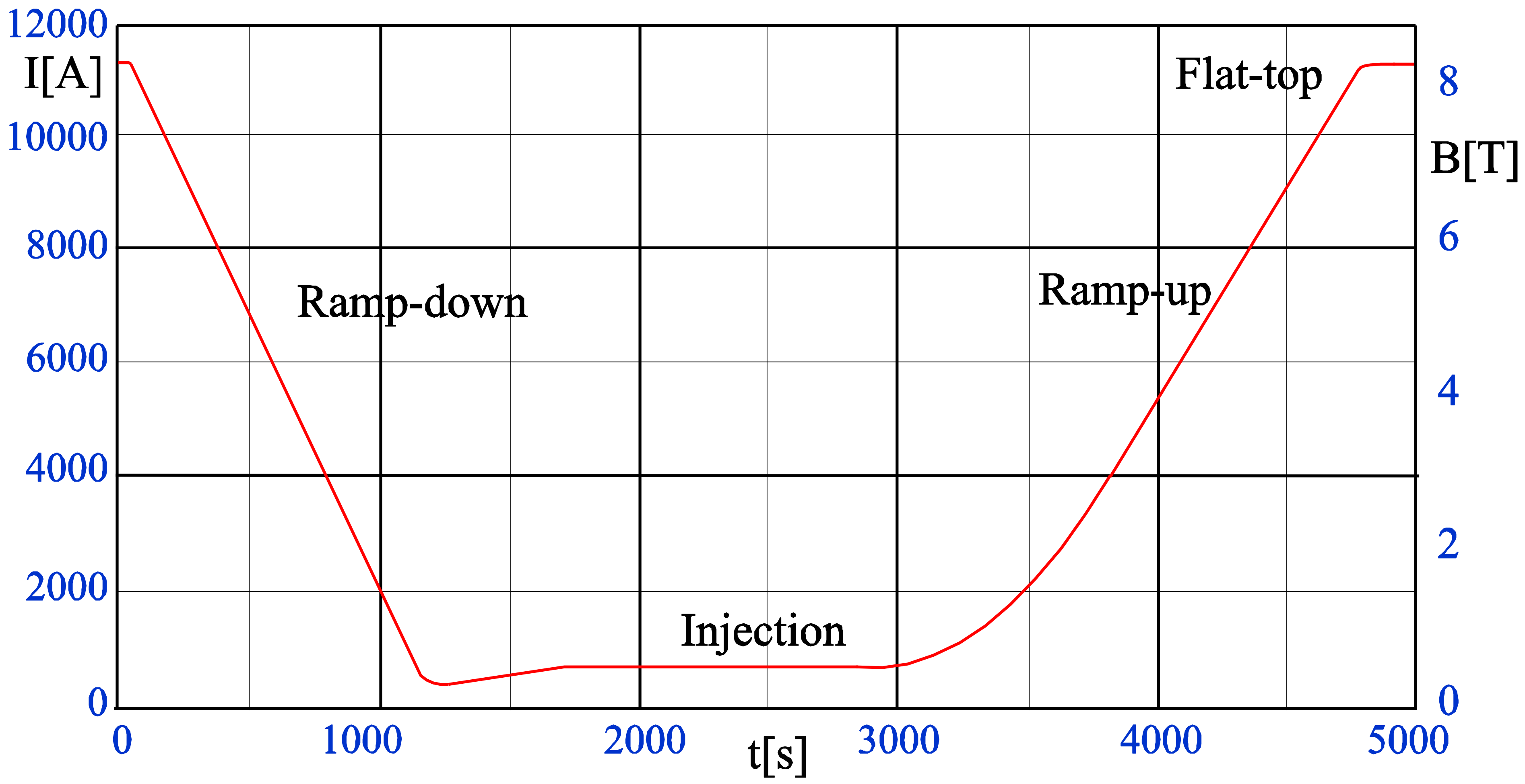




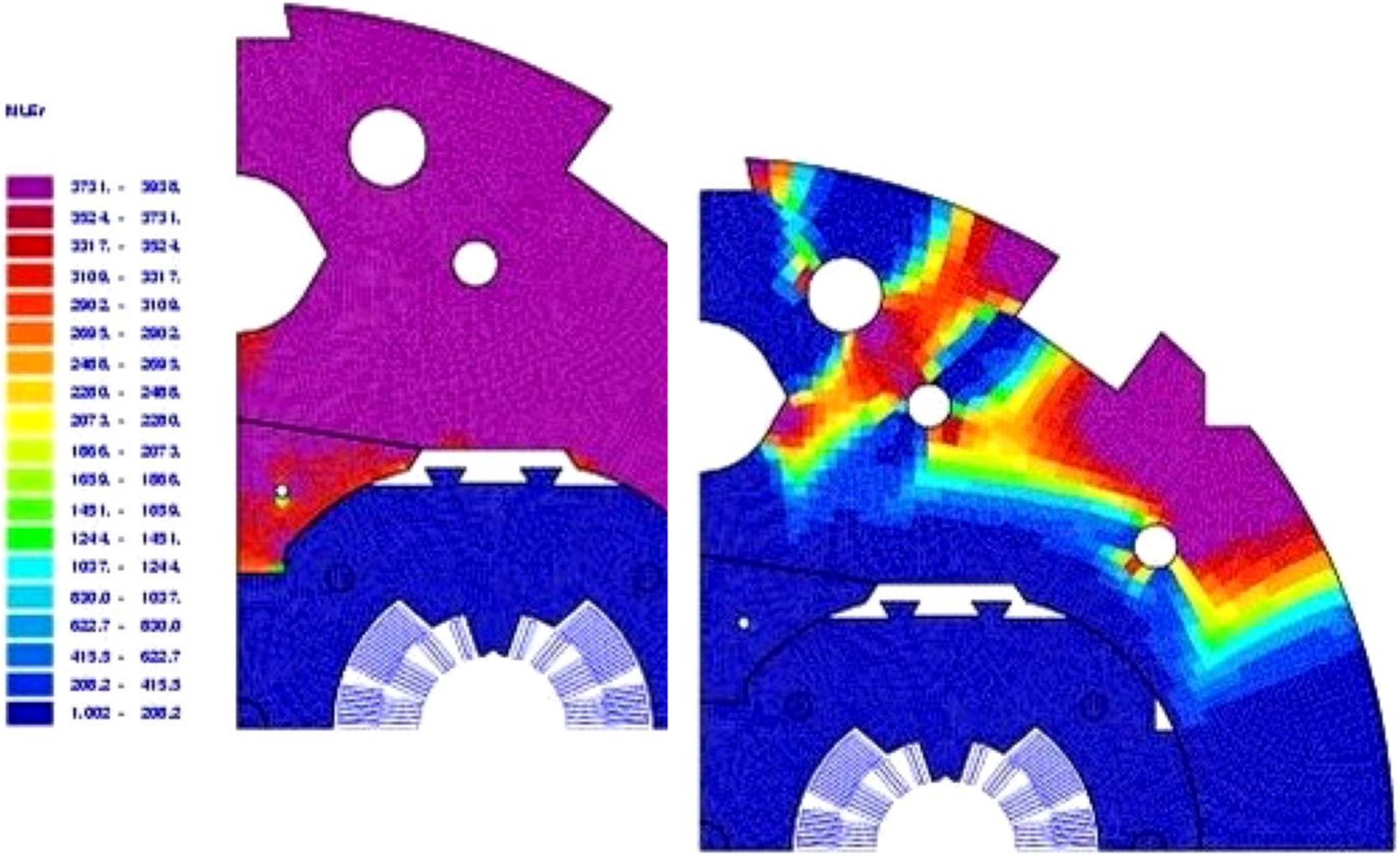
The CERN Field Computation Program ROXIE

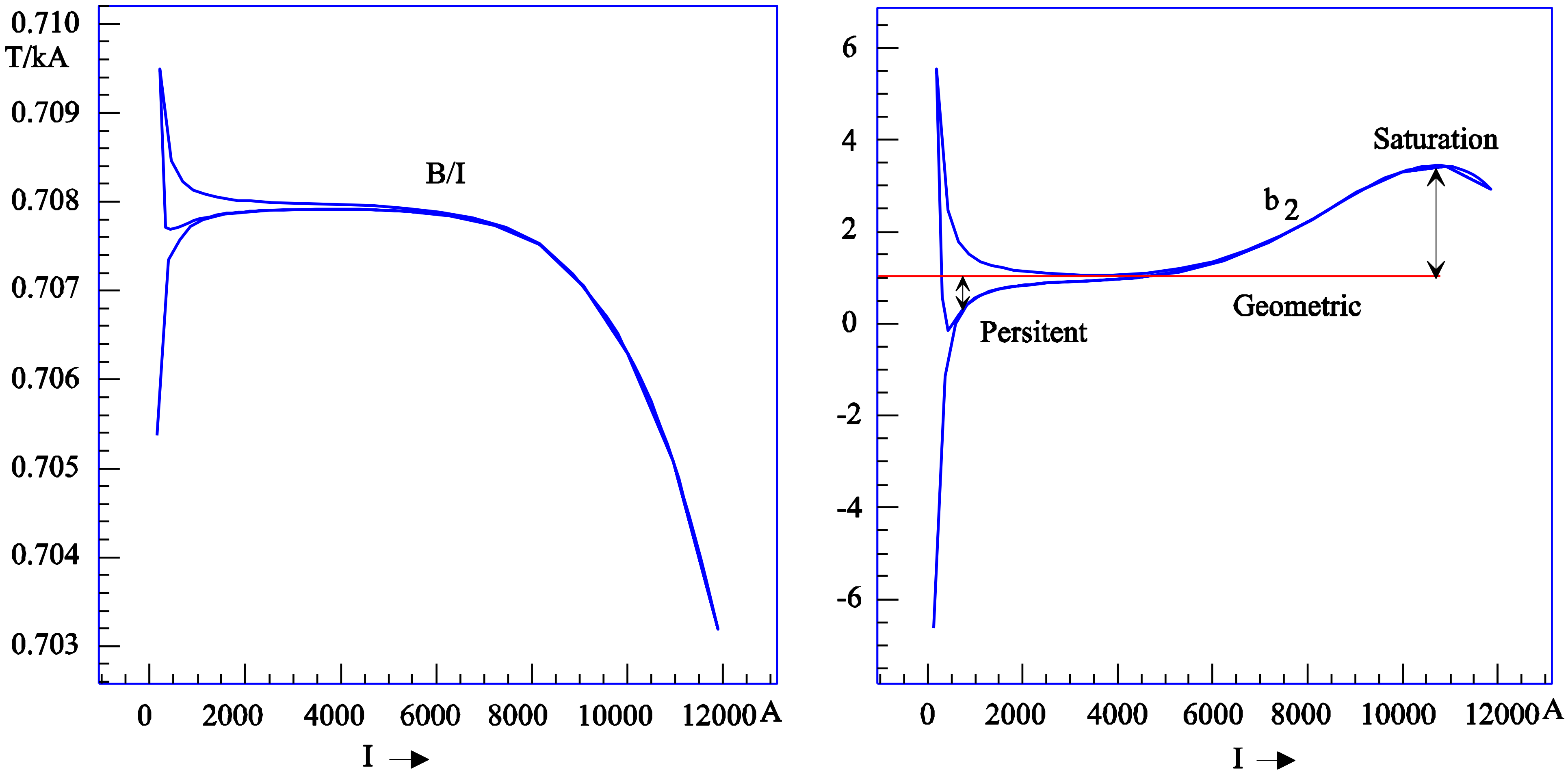


Excitation Cycle



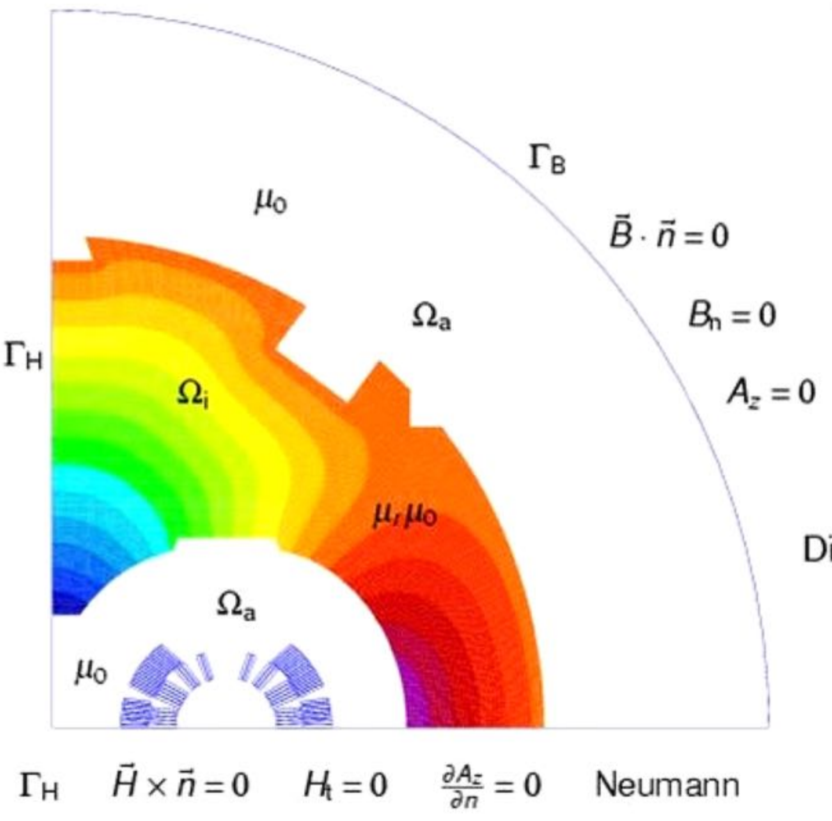
Saturation Effects in the Dipole Iron Yoke

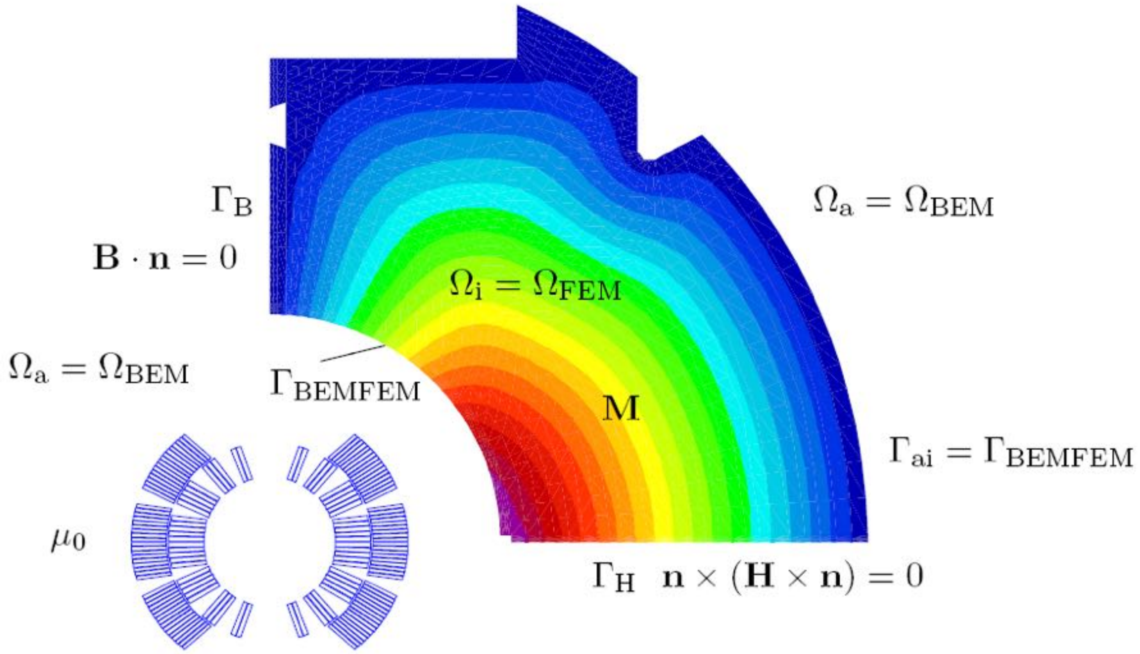



0.710
T/kA
0.709
0.708
0.707
0.706
0.705
0.704
0.703
B/I
0
2000
4000
6000
8000
10000
12000
A
I →
6
4
2
0
-2
-4
-6
Saturation
b_2
Geometric
Persitent
0
2000
4000
6000
8000
10000
12000
A
I →

The Problem Domain

FEM





BEM-FEM

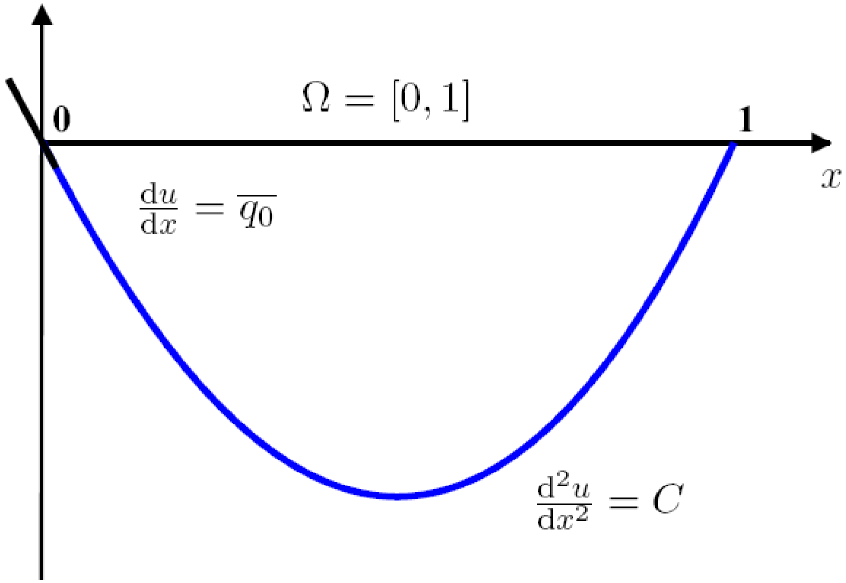
The Model Problem (1-D)

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} = f(x), \qquad x \in \Omega$$

$$u(x)|_{x=0} = u_0, \qquad u(x)|_{x=1} = u_1,$$

or

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=0} = q_0, \qquad \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=1} = q_1.$$

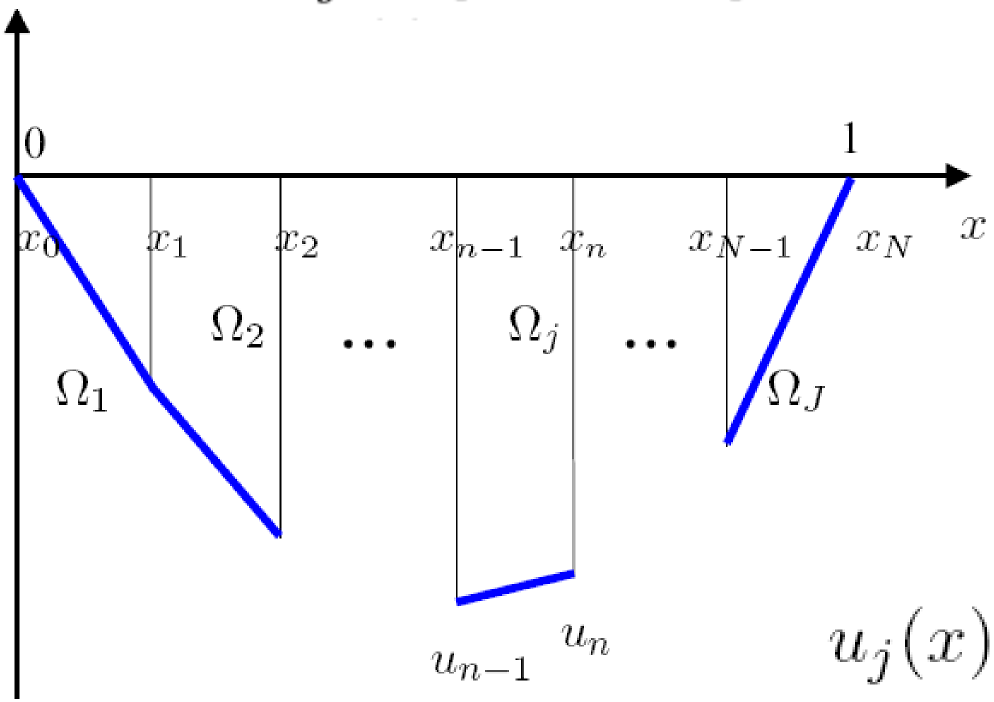


Constant heat source C and
u(x) = 0 at x=0 and x=1

$$u(x) = \frac{C}{2}\left(x^2 - x\right)$$

Shape Functions

$$\Omega_j = [x_{n-1}, x_n]$$

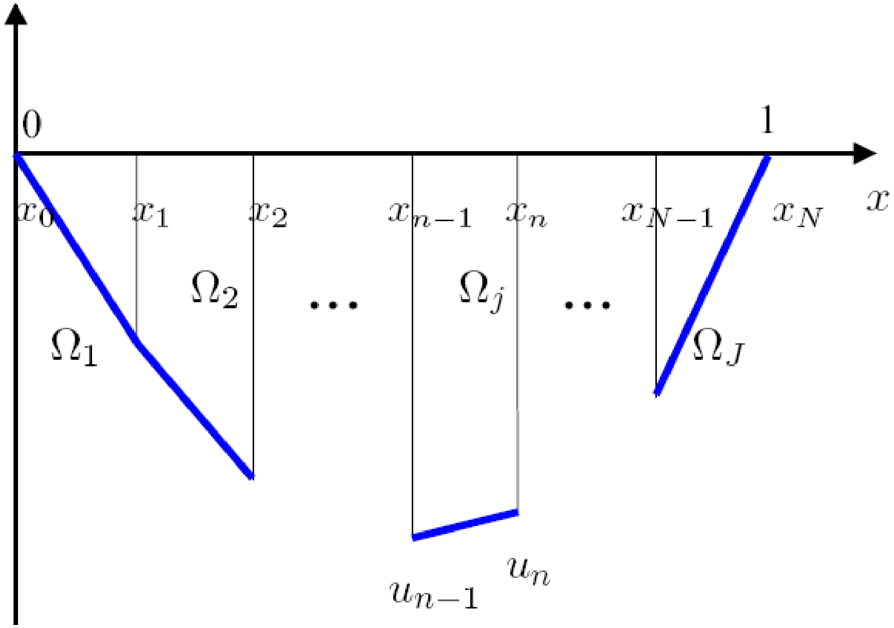


$$\Omega = \bigcup_{j=1}^{J} \Omega_j$$

$$u_j(x) = \alpha_{j1} + \alpha_{j2} x \qquad x \in \Omega_j$$

$$u_{n-1} = \alpha_{j1} + \alpha_{j2} x_{n-1}$$

$$u_n = \alpha_{j1} + \alpha_{j2} x_n$$



$$\alpha_{j1} = \frac{\begin{vmatrix} u_{n-1} & x_{n-1} \\ u_n & x_n \end{vmatrix}}{\begin{vmatrix} 1 & x_{n-1} \\ 1 & x_n \end{vmatrix}}$$

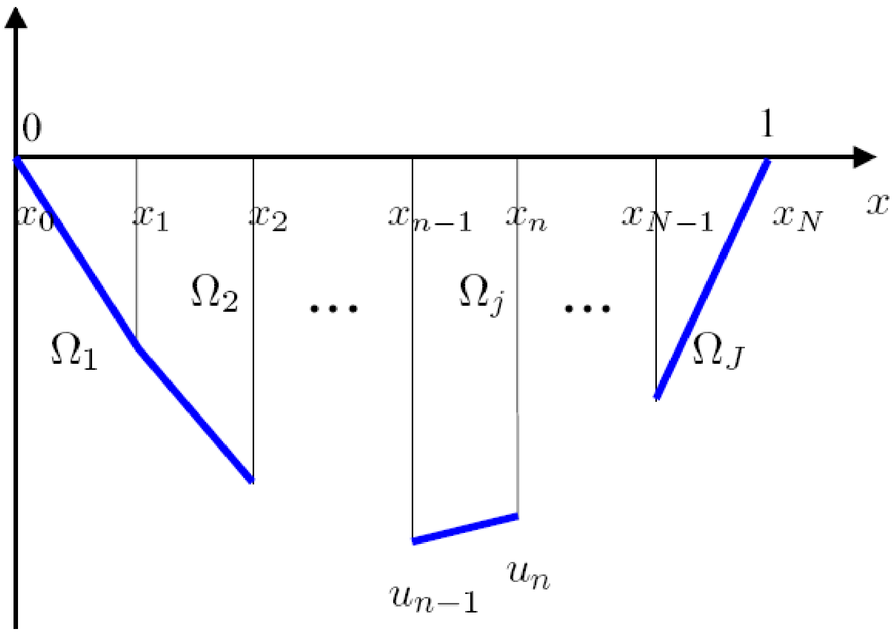
Cramer's rule

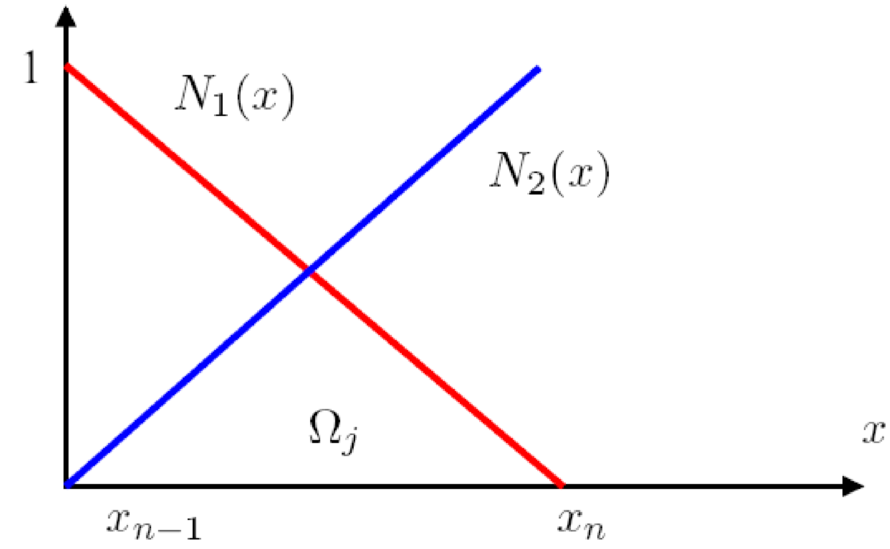
$$\alpha_{j1} = \frac{x_n u_{n-1} - x_{n-1} u_n}{x_n - x_{n-1}}$$

$$\alpha_{j2} = \frac{u_n - u_{n-1}}{x_n - x_{n-1}}$$

$$u_j(x) = \alpha_{j1} + \alpha_{j2} x = \frac{x_n - x}{x_n - x_{n-1}} u_{n-1} + \frac{-x_{n-1} + x}{x_n - x_{n-1}} u_n$$

Shape Functions





$$N_{j1}(x) = \frac{x_n - x}{x_n - x_{n-1}} \qquad N_{j2}(x) = \frac{-x_{n-1} + x}{x_n - x_{n-1}}$$

What have we won? We can express the field in the element as a function of the node potentials using known polynomials in the spatial coordinates

The Weighted Residual

$$R(x) := \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - f(x)$$

Integration by parts

$$\int_\Omega w(x) R(x)\,\mathrm{d}\Omega = \int_\Omega w(x)\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2}\,\mathrm{d}\Omega - \int_\Omega w(x) f(x)\,\mathrm{d}\Omega = 0$$

$$\int_a^b \phi\psi'\,\mathrm{d}x = [\phi\psi]_a^b - \int_a^b \phi'\psi\,\mathrm{d}x \qquad w(x) = \phi \qquad \frac{\mathrm{d}u(x)}{\mathrm{d}x} = \psi$$

$$-\int_\Omega \frac{\mathrm{d}w(x)}{\mathrm{d}x}\frac{\mathrm{d}u(x)}{\mathrm{d}x}\,\mathrm{d}\Omega + \left[w(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right]_0^1 - \int_\Omega w(x) f(x)\,\mathrm{d}\Omega = 0$$

What have we won? Removal of the second derivative, a way to incorporate Neumann boundary conditions

Galerkin's Method

$$\int_\Omega \frac{\mathrm{d}w(x)}{\mathrm{d}x}\frac{\mathrm{d}u(x)}{\mathrm{d}x}\,\mathrm{d}\Omega = -\int_\Omega w(x) f(x)\,\mathrm{d}\Omega$$

$$\int_{\Omega_j} \frac{\mathrm{d}w_l(x)}{\mathrm{d}x}\sum_{k=1,2}\frac{\mathrm{d}N_{jk}(x)}{\mathrm{d}x}u^{(k)}\,\mathrm{d}\Omega_j = -\int_{\Omega_j} w_l(x) f(x)\,\mathrm{d}\Omega_j\,, \qquad l = 1,2.$$

$$\int_{\Omega_j} \frac{\mathrm{d}N_{jl}(x)}{\mathrm{d}x}\sum_{k=1,2}\frac{\mathrm{d}N_{jk}(x)}{\mathrm{d}x}u^{(k)}\,\mathrm{d}\Omega_j = -\int_{\Omega_j} N_{jk}(x) f(x)\,\mathrm{d}\Omega_j\,, \qquad l = 1,2$$

$$\int_{x_{n-1}}^{x_n}\left(\frac{\mathrm{d}N_{j1}}{\mathrm{d}x}\frac{\mathrm{d}N_{j1}}{\mathrm{d}x}u_{n-1} + \frac{\mathrm{d}N_{j1}}{\mathrm{d}x}\frac{\mathrm{d}N_{j2}}{\mathrm{d}x}u_n\right)\mathrm{d}x = -\int_{x_{n-1}}^{x_n} N_{j1} f(x)\,\mathrm{d}x$$

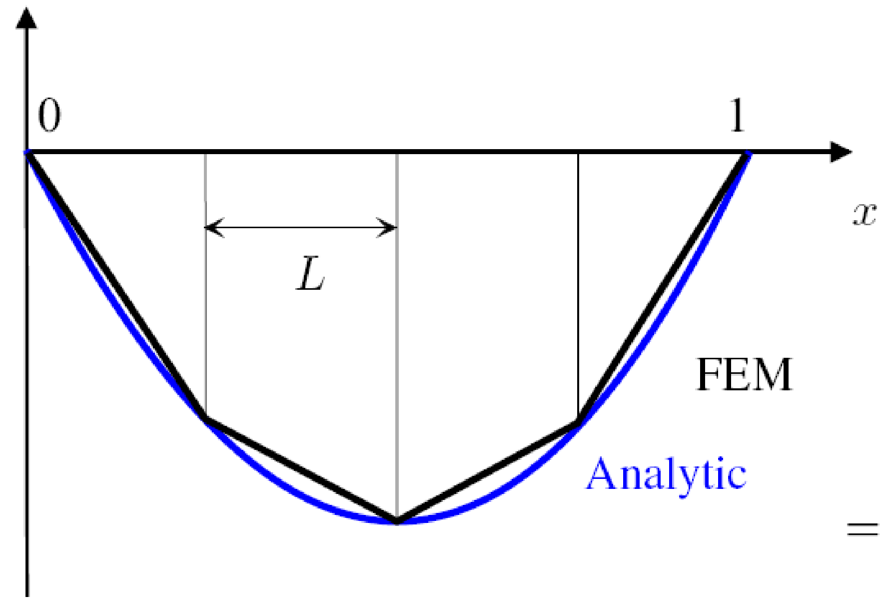
$$\int_{x_{n-1}}^{x_n}\left(\frac{\mathrm{d}N_{j2}}{\mathrm{d}x}\frac{\mathrm{d}N_{j1}}{\mathrm{d}x}u_{n-1} + \frac{\mathrm{d}N_{j2}}{\mathrm{d}x}\frac{\mathrm{d}N_{j2}}{\mathrm{d}x}u_n\right)\mathrm{d}x = -\int_{x_{n-1}}^{x_n} N_{j2} f(x)\,\mathrm{d}x$$

$$\boxed{[k_j]\{u_j\} = \{f_j\}}$$

Linear equation system for the node potentials

Numerical Example

4 finite elements $\Omega_j, j = 1, \ldots, 4$ of equidistant length L



$$[k_j] = \int_{x_{n-1}}^{x_n} \begin{pmatrix} \frac{\mathrm{d}N_{j1}}{\mathrm{d}x}\frac{\mathrm{d}N_{j1}}{\mathrm{d}x} & \frac{\mathrm{d}N_{j1}}{\mathrm{d}x}\frac{\mathrm{d}N_{j2}}{\mathrm{d}x} \\ \frac{\mathrm{d}N_{j2}}{\mathrm{d}x}\frac{\mathrm{d}N_{j1}}{\mathrm{d}x} & \frac{\mathrm{d}N_{j2}}{\mathrm{d}x}\frac{\mathrm{d}N_{j2}}{\mathrm{d}x} \end{pmatrix} \mathrm{d}x$$

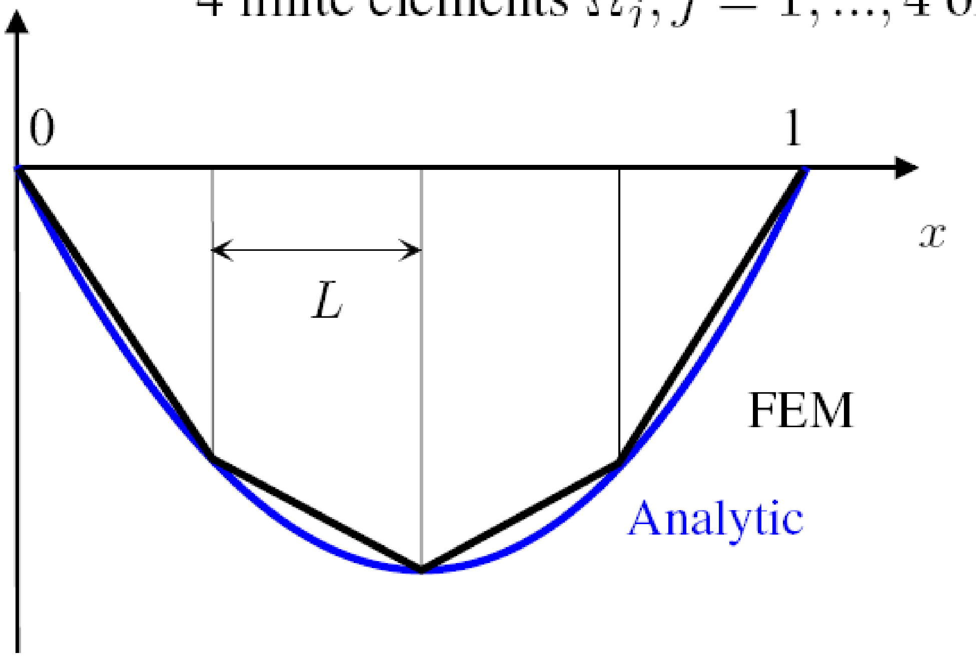
$$= \int_{x_{n-1}}^{x_n} \begin{pmatrix} \frac{1}{(x_n - x_{n-1})^2} & \frac{-1}{(x_n - x_{n-1})^2} \\ \frac{-1}{(x_n - x_{n-1})^2} & \frac{1}{(x_n - x_{n-1})^2} \end{pmatrix} \mathrm{d}x = \begin{pmatrix} \frac{1}{L} & \frac{-1}{L} \\ \frac{-1}{L} & \frac{1}{L} \end{pmatrix}$$

$$\{f_j\} = -\int_{x_{n-1}}^{x_n} \begin{pmatrix} N_{j1} \\ N_{j2} \end{pmatrix} C \mathrm{d}x = -C \int_{x_{n-1}}^{x_n} \begin{pmatrix} \frac{x_n - x}{x_n - x_{n-1}} \\ \frac{-x_{n-1} + x}{x_n - x_{n-1}} \end{pmatrix} \mathrm{d}x$$

$$= -\frac{C}{2L} \begin{pmatrix} 2x_n x - x^2 \\ -2x_{n-1}x + x^2 \end{pmatrix} \Bigg|_{x_{n-1}}^{x_n} = -\frac{C}{2L} \begin{pmatrix} (x_n - x_{n-1})^2 \\ (x_{n-1} - x_n)^2 \end{pmatrix} = -\begin{pmatrix} 0.5\,CL \\ 0.5\,CL \end{pmatrix}$$

Numerical Example

4 finite elements $\Omega_j, j = 1, ..., 4$ of equidistant length L



S (Stiffness matrix)

$$\begin{pmatrix} \frac{1}{L} & \frac{-1}{L} & 0 & 0 & 0 \\ \frac{1}{L} & \frac{2}{L} & \frac{-1}{L} & 0 & 0 \\ 0 & \frac{-1}{L} & \frac{2}{L} & \frac{-1}{L} & 0 \\ 0 & 0 & \frac{-1}{L} & \frac{2}{L} & \frac{-1}{L} \\ 0 & 0 & 0 & \frac{-1}{L} & \frac{1}{L} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = - \begin{pmatrix} 0.5CL \\ CL \\ CL \\ CL \\ 0.5CL \end{pmatrix}$$

Essential boundary conditions (Dirichlet)

S^{-1}

$$\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} \frac{3L}{4} & \frac{L}{2} & \frac{L}{4} \\ \frac{L}{2} & L & \frac{L}{2} \\ \frac{L}{4} & \frac{L}{2} & \frac{2L}{4} \end{pmatrix} \begin{pmatrix} CL \\ CL \\ CL \end{pmatrix} = \begin{pmatrix} -0.375 \\ -0.5 \\ -0.375 \end{pmatrix}$$

Higher order elements

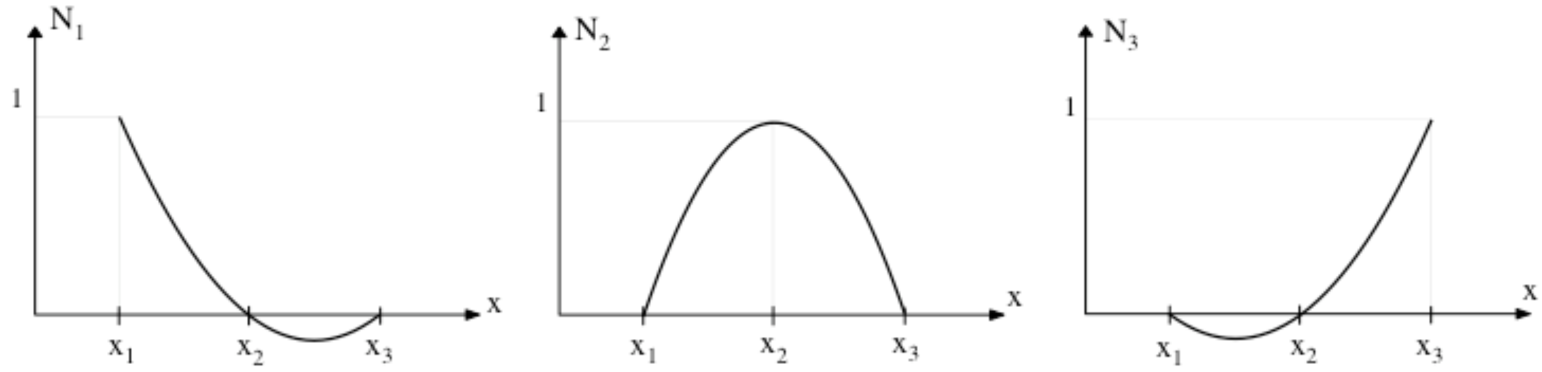
$$u^{(1)} = \alpha_{j1} + \alpha_{j2}x_1 + \alpha_{j3}x_1^2$$
$$u^{(2)} = \alpha_{j1} + \alpha_{j2}x_2 + \alpha_{j3}x_2^2$$
$$u^{(3)} = \alpha_{j1} + \alpha_{j2}x_3 + \alpha_{j3}x_3^2$$

$$u_j(x) = \sum_{k=1}^{3} N_{jk}(x)u^{(k)}$$

$$N_{j1}(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}, \qquad N_{j2}(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$N_{j3}(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}.$$

h-refinement
Versus
p-refinement



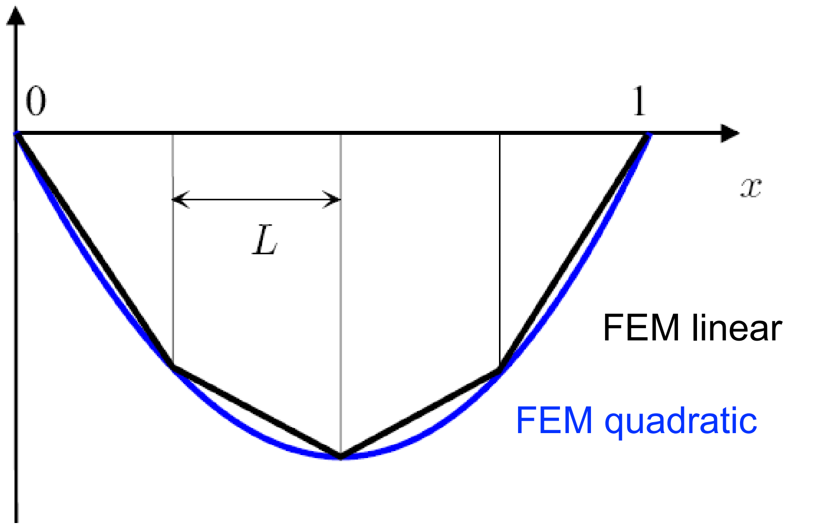
Two Quadratic Elements

$$[k_j] = \int_{x_1}^{x_3} \begin{pmatrix} \frac{dN_{j1}}{dx}\frac{dN_{j1}}{dx} & \frac{dN_{j1}}{dx}\frac{dN_{j2}}{dx} & \frac{dN_{j1}}{dx}\frac{dN_{j3}}{dx} \\ \frac{dN_{j2}}{dx}\frac{dN_{j1}}{dx} & \frac{dN_{j2}}{dx}\frac{dN_{j2}}{dx} & \frac{dN_{j2}}{dx}\frac{dN_{j3}}{dx} \\ \frac{dN_{j3}}{dx}\frac{dN_{j1}}{dx} & \frac{dN_{j3}}{dx}\frac{dN_{j2}}{dx} & \frac{dN_{j3}}{dx}\frac{dN_{j3}}{dx} \end{pmatrix} dx \qquad [k_j] = \begin{pmatrix} \frac{7}{6l} & \frac{-8}{6l} & \frac{1}{6l} \\ \frac{-8}{6l} & \frac{16}{6l} & \frac{-8}{6l} \\ \frac{1}{6l} & \frac{-8}{6l} & \frac{7}{6l} \end{pmatrix}$$

$$\{f_j\} = -\int_{x_1}^{x_3} \begin{pmatrix} N_{j1} \\ N_{j2} \\ N_{j3} \end{pmatrix} f(x)\, dx \qquad \{f_j\} = -\frac{1}{3}c \begin{pmatrix} l \\ 4l \\ l \end{pmatrix}$$

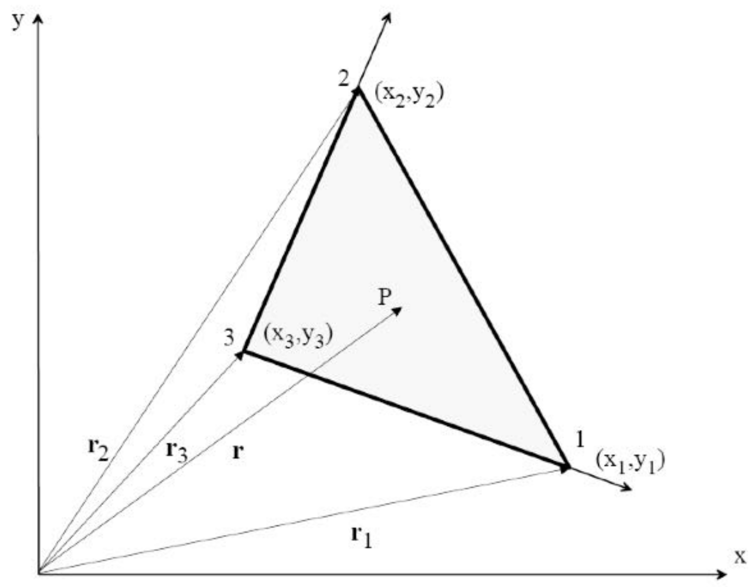
$$\begin{pmatrix} \frac{16}{6l} & \frac{-8}{6l} & 0 \\ \frac{-8}{6l} & \frac{14}{6l} & \frac{-8}{6l} \\ 0 & \frac{-8}{6l} & \frac{16}{6l} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} 4cl \\ 2cl \\ 4cl \end{pmatrix}$$

$$\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = -\begin{pmatrix} \frac{3l}{4} & \frac{l}{2} & \frac{l}{4} \\ \frac{l}{2} & l & \frac{l}{2} \\ \frac{l}{4} & \frac{l}{2} & \frac{3l}{4} \end{pmatrix} \begin{pmatrix} cl \\ cl \\ cl \end{pmatrix} = \begin{pmatrix} -0.375 \\ -0.5 \\ -0.375 \end{pmatrix}$$



Shape Functions

$$A_j(\mathbf{x}) = \alpha_1 + \alpha_2 x + \alpha_3 y, \qquad \mathbf{x} \in \Omega_j \qquad A_j(\mathbf{x}) = A_{z_j}(x,y)$$



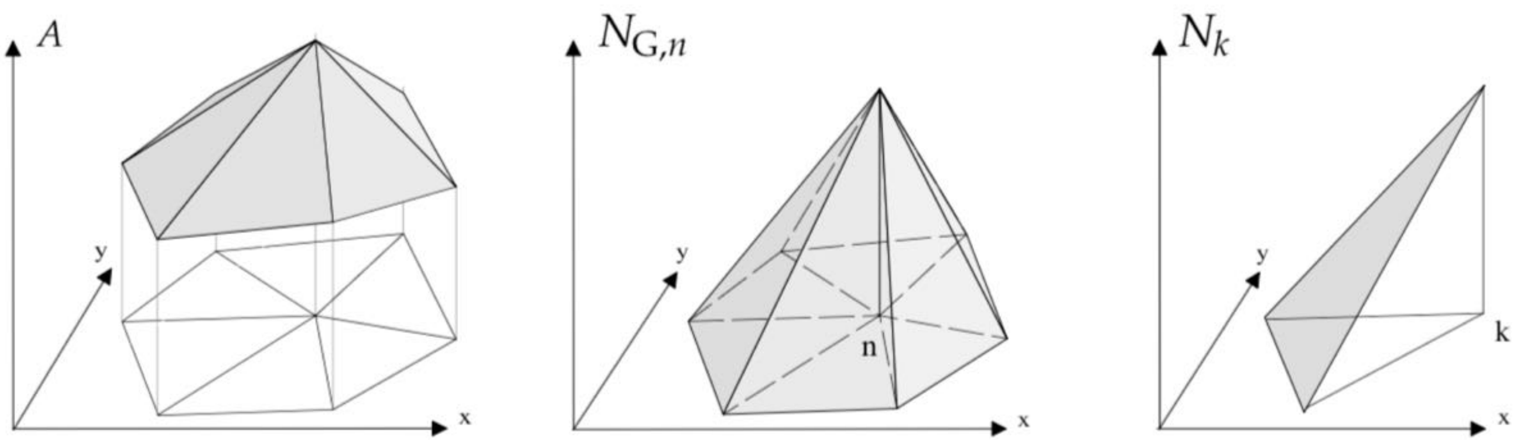
$$A^{(1)} = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$
$$A^{(2)} = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$
$$A^{(3)} = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

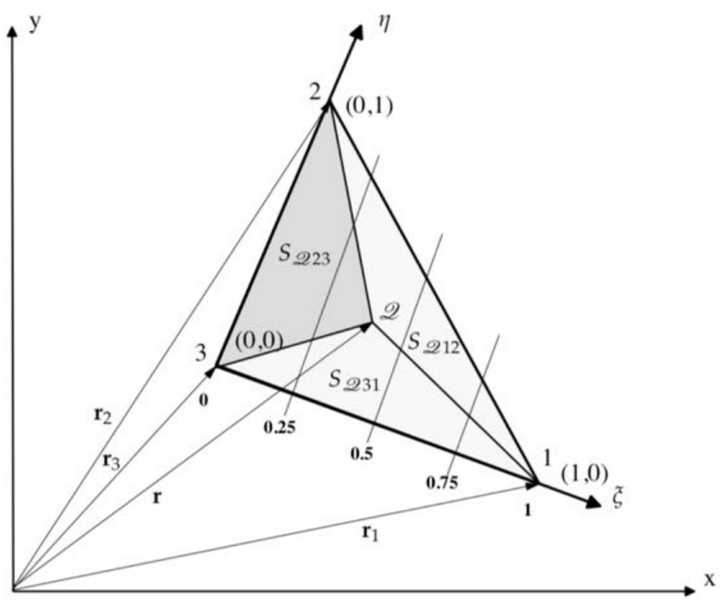
$$\begin{pmatrix} A^{(1)} \\ A^{(2)} \\ A^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\{A\} = [C]\{\alpha\}$$
$$\{\alpha\} = [C]^{-1}\{A\}$$

$$N_k(\mathbf{r}) = \frac{a_k + b_k x + c_k y}{a_k + b_k x_k + c_k y_k}, \qquad k = 1,2,3.$$

Finite Element Shape Functions and Barycentric Coordinates

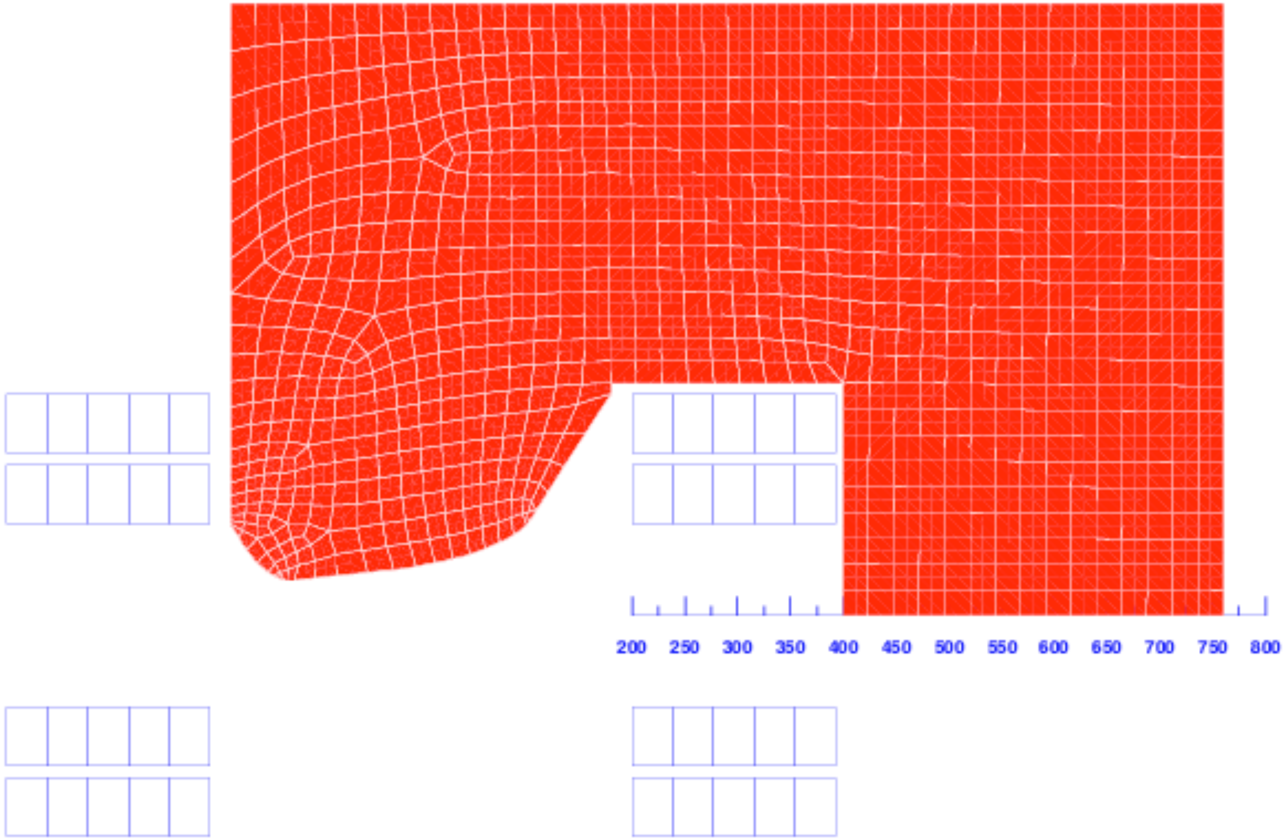




$$x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \,,$$
$$y = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 \,,$$
$$1 = \lambda_1 + \lambda_2 + \lambda_3 .$$

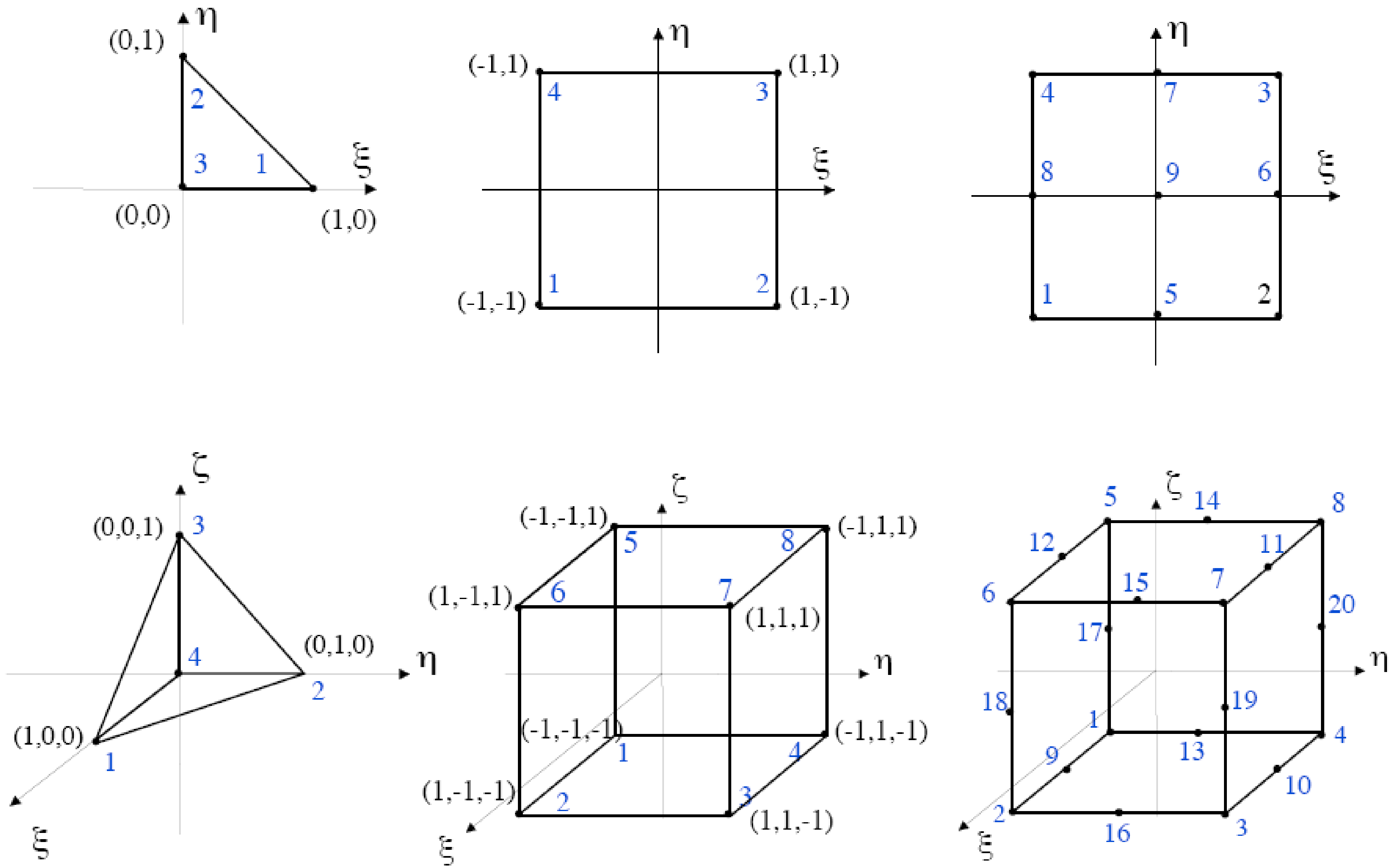
$$\lambda_1 = \frac{S_{\mathscr{Q}23}}{S_{123}} \,, \qquad \lambda_2 = \frac{S_{\mathscr{Q}31}}{S_{123}} \,, \qquad \lambda_3 = \frac{S_{\mathscr{Q}12}}{S_{123}}$$

Higher Order Elements

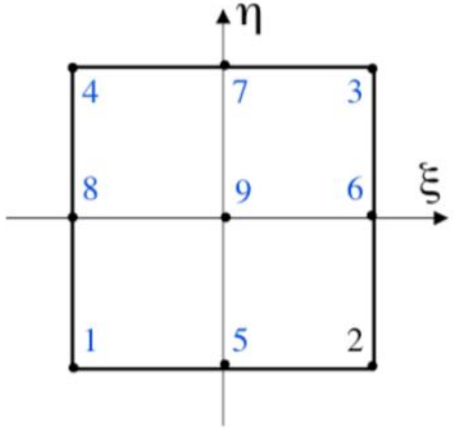


Higher accuracy of the field solution, but also better modeling of the iron contour

Parent Elements



Pascal Triangle



$$N_{1,2,3,4} = \frac{1}{4}(1+\xi\xi_k)(1+\eta\eta_k)(\xi\xi_k+\eta\eta_k-1),$$

$$N_{5,7} = \frac{1}{2}(1-\xi^2)(1+\eta\eta_k),$$

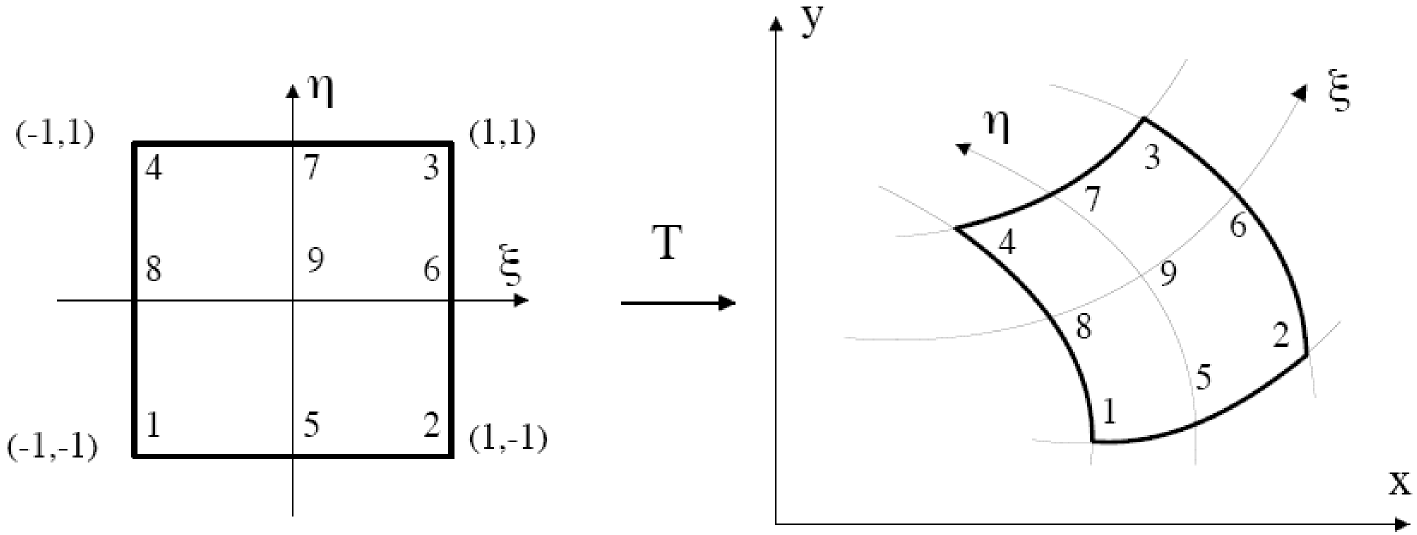
$$N_{6,8} = \frac{1}{2}(1+\xi\xi_k)(1-\eta^2).$$

$$N_9 = (1-\xi^2)(1-\eta^2)\,.$$

$$1$$
$$\xi \qquad \eta$$
$$\xi^2 \qquad 2\xi\eta \qquad \eta^2$$
$$\xi^3 \qquad 3\xi^2\eta \qquad 3\xi\eta^2 \qquad \eta^3$$
$$\xi^4 \qquad 4\xi^3\eta \qquad 6\xi^2\eta^2 \qquad 4\xi\eta^3 \qquad \eta^4$$
$$\xi^5 \qquad 5\xi^4\eta \qquad 10\xi^3\eta^2 \qquad 10\xi^2\eta^3 \qquad 5\xi\eta^4 \qquad \eta^5$$

Mapped Elements

$$x = x(\xi, \eta, \zeta), \qquad y = y(\xi, \eta, \zeta), \qquad z = z(\xi, \eta, \zeta)$$

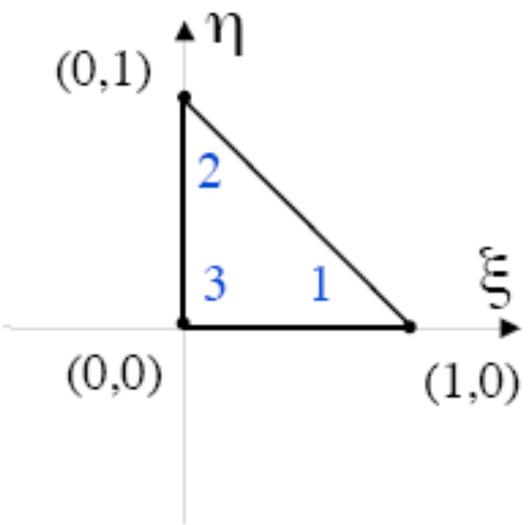


$$A_j(\boldsymbol{\xi}) = \sum_{k=1}^{K} N_k(\boldsymbol{\xi}) A^{(k)} \qquad x_j(\boldsymbol{\xi}) = \sum_{k=1}^{K} N_k(\boldsymbol{\xi}) x^{(k)} \qquad y_j(\boldsymbol{\xi}) = \sum_{k=1}^{K} N_k(\boldsymbol{\xi}) y^{(k)}$$

Use of the same shape functions for the transformation of the elements

Transformation of Differential Operators

$$\frac{\partial N_k}{\partial x} = \frac{\partial N_k}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_k}{\partial \eta}\frac{\partial \eta}{\partial x}$$



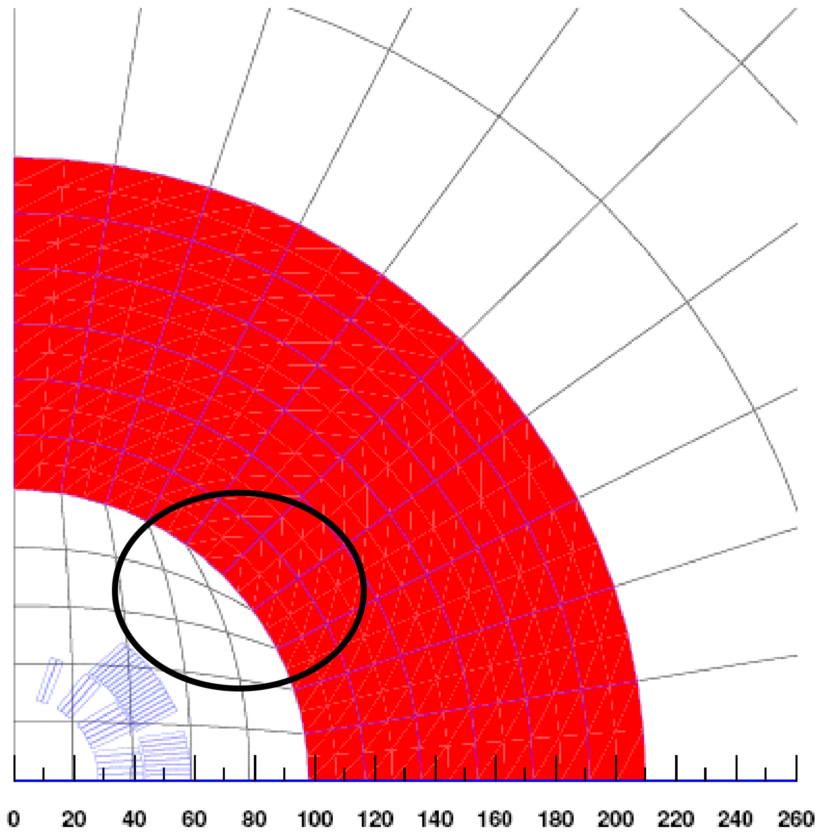
Complicated **Easy**

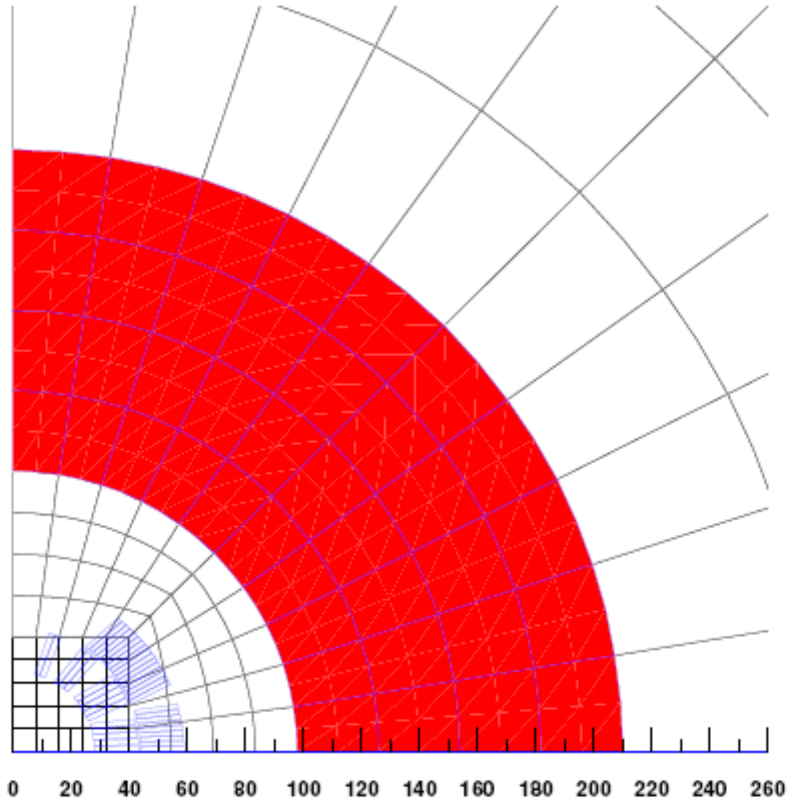
$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} N_k = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} N_k = [J]_{T^{-1}} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} N_k$$

$$[J]_{T^{-1}} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}^{-1} = [J]_T^{-1}$$

$$[J]_T = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{K} \frac{\partial N_k}{\partial \xi} x^{(k)} & \sum_{k=1}^{K} \frac{\partial N_k}{\partial \xi} y^{(k)} \\ \sum_{k=1}^{K} \frac{\partial N_k}{\partial \eta} x^{(k)} & \sum_{k=1}^{K} \frac{\partial N_k}{\partial \eta} y^{(k)} \end{pmatrix} = \begin{pmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \cdots & \frac{\partial N_K}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \cdots & \frac{\partial N_K}{\partial \eta} \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_k & y_k \end{pmatrix}$$

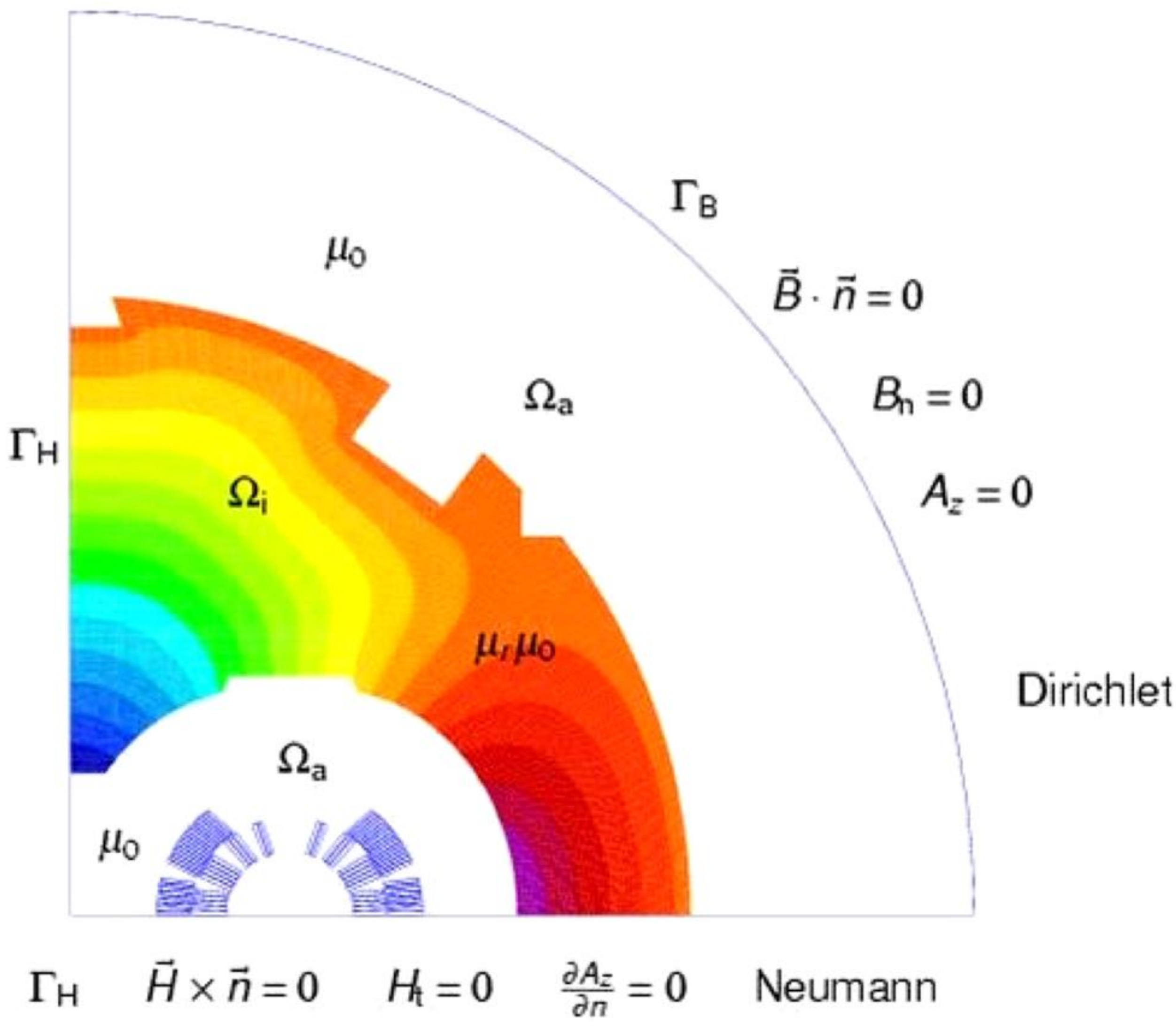
Collinear Sides yield Singular Jacobi Matrices





Note: Bad meshing is not a trivial offence

The Problem Domain



Curl-Curl Equation

$$\mathbf{B} = \text{curl}\,\mathbf{A} \qquad \text{in}\,\Omega$$

$$\text{curl}\,\frac{1}{\mu}\,\text{curl}\,\mathbf{A} = \mathbf{J} \qquad \text{in}\,\Omega$$

$$\mathbf{H}_\text{t} = \mathbf{0} \quad \rightarrow \quad \frac{1}{\mu}(\text{curl}\,\mathbf{A}) \times \mathbf{n} \;=\; \mathbf{0} \;\; \text{on}\,\Gamma_H$$

$$B_\text{n} = 0 \quad \rightarrow \quad \mathbf{B}\cdot\mathbf{n} = \text{curl}\,\mathbf{A}\cdot\mathbf{n} \;=\; 0 \;\; \text{on}\,\Gamma_B$$

$$\left[\frac{1}{\mu}\,(\text{curl}\,\mathbf{A})\times\mathbf{n}\right]_\text{ai} \;=\; \mathbf{0} \;\; \text{on}\,\Gamma_\text{ai}$$

$$[\mathbf{A}]_\text{ai} \;=\; \mathbf{0} \;\; \text{on}\,\Gamma_\text{ai}$$

Problem in 3-D: Gauging

$$\mathbf{A} \rightarrow \mathbf{A}' : \mathbf{A}' = \mathbf{A} + \text{grad}\,\psi$$

$$\text{div}\,\mathbf{A}' = q$$

$$q = \text{div}\,\mathbf{A} + \nabla^2\psi$$

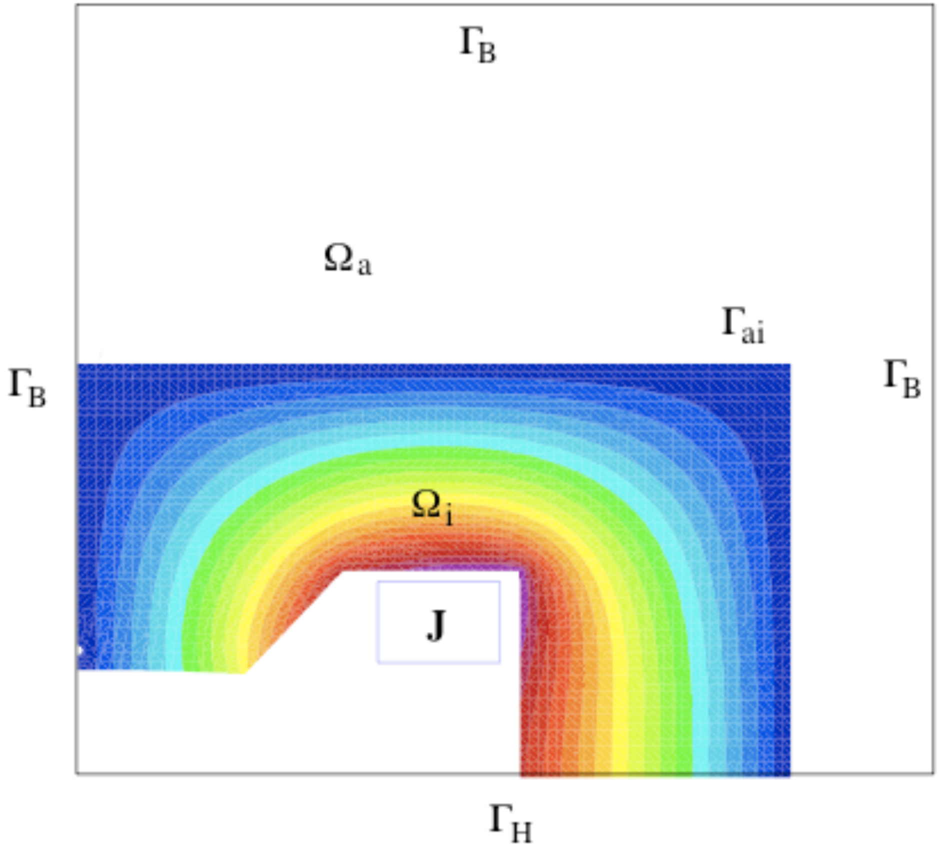
$$\frac{1}{\mu}\,\text{div}\,\mathbf{A} = 0 \;\; \text{in}\,\Omega$$

$$\mathbf{A}\cdot\mathbf{n} = 0 \;\; \text{on}\,\Gamma_H$$

$$\boxed{\text{curl}\,\frac{1}{\mu}\,\text{curl}\,\mathbf{A} - \text{grad}\,\frac{1}{\mu}\,\text{div}\,\mathbf{A} = \mathbf{J} \;\; \text{in}\,\Omega}$$

Weak Form in the FEM Problem

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A} = \mathbf{J} \quad \text{in } \Omega$$



$$\begin{aligned}
\mathbf{A} \cdot \mathbf{n} &= 0 && \text{on } \Gamma_H, \\
\frac{1}{\mu} \operatorname{div} \mathbf{A} &= 0 && \text{on } \Gamma_B, \\
\mathbf{n} \times (\mathbf{A} \times \mathbf{n}) &= \mathbf{0} && \text{on } \Gamma_B, \\
\mathbf{n} \times \left(\frac{1}{\mu} (\operatorname{curl} \mathbf{A}) \times \mathbf{n} \right) &= \mathbf{0} && \text{on } \Gamma_H, \\
\left[\frac{1}{\mu} \operatorname{div} \mathbf{A} \right]_{\text{ai}} &= 0 && \text{on } \Gamma_{\text{ai}}, \\
\left[\frac{1}{\mu} (\operatorname{curl} \mathbf{A}) \times \mathbf{n} \right]_{\text{ai}} &= \mathbf{0} && \text{on } \Gamma_{\text{ai}}, \\
[\mathbf{A}]_{\text{ai}} &= \mathbf{0} && \text{on } \Gamma_{\text{ai}}.
\end{aligned}$$

Weak Form in the FEM Problem

$$\mathrm{curl}\,\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A} - \mathrm{grad}\,\frac{1}{\mu}\,\mathrm{div}\,\mathbf{A} = \mathbf{J} \quad \mathrm{in}\,\Omega$$

$$\mathrm{curl}\,\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A} - \mathrm{grad}\,\frac{1}{\mu}\,\mathrm{div}\,\mathbf{A} - \mathbf{J} = \mathbf{R}$$

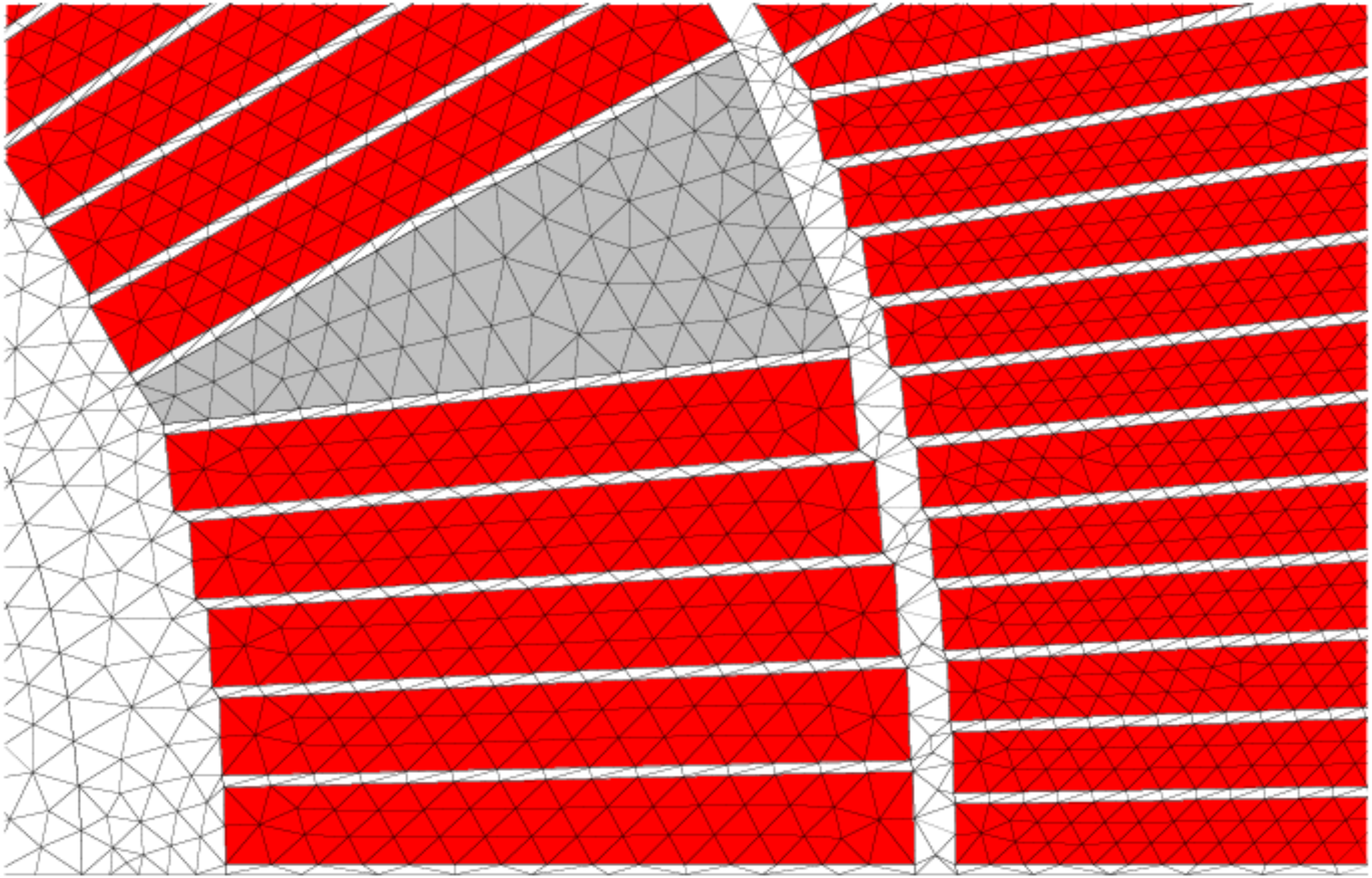
$$\int_\Omega \mathbf{w}_a \cdot \left(\mathrm{curl}\,\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A} - \mathrm{grad}\,\frac{1}{\mu}\,\mathrm{div}\,\mathbf{A}\right) \mathrm{d}\Omega = \int_\Omega \mathbf{w}_a \cdot \mathbf{J}\,\mathrm{d}\Omega\,, \qquad a = 1,2,3.$$

Vector form of Greens theorem and integration by parts rule

$$\int_\Omega \frac{1}{\mu}\mathrm{curl}\,\mathbf{A} \cdot \mathrm{curl}\,\mathbf{w}_a \mathrm{d}\Omega - \int_{\Gamma_\mathrm{H}} \frac{1}{\mu}(\mathrm{curl}\,\mathbf{A} \times \mathbf{n}) \cdot \mathbf{w}_a \mathrm{d}\Gamma_\mathrm{H} + \int_\Omega \frac{1}{\mu}\mathrm{div}\,\mathbf{A}\;\mathrm{div}\,\mathbf{w}_a \mathrm{d}\Omega -$$
$$\int_{\Gamma_\mathrm{B}} \frac{1}{\mu}\mathrm{div}\,\mathbf{A}(\mathbf{n} \cdot \mathbf{w}_a)\mathrm{d}\Gamma_\mathrm{B} - \int_{\Gamma_\mathrm{ai}} \left(\frac{1}{\mu}\mathrm{div}\,\mathbf{A}_\mathrm{i}(\mathbf{n}_\mathrm{i} \cdot \mathbf{w}_a) + \frac{1}{\mu_0}\mathrm{div}\,\mathbf{A}_\mathrm{a}(\mathbf{n}_\mathrm{a} \cdot \mathbf{w}_a)\right) \mathrm{d}\Gamma_\mathrm{ai} -$$
$$\int_{\Gamma_\mathrm{ai}} \left(\frac{1}{\mu}(\mathrm{curl}\,\mathbf{A}_\mathrm{i} \times \mathbf{n}_\mathrm{i}) + \frac{1}{\mu_0}(\mathrm{curl}\,\mathbf{A}_\mathrm{a} \times \mathbf{n}_\mathrm{a})\right) \cdot \mathbf{w}_a \mathrm{d}\Gamma_\mathrm{ai} = \int_\Omega \mathbf{w}_a \cdot \mathbf{J}\,\mathrm{d}\Omega,$$

$$\boxed{\int_\Omega \frac{1}{\mu}\mathrm{curl}\,\mathbf{w}_a \cdot \mathrm{curl}\,\mathbf{A}\,\mathrm{d}\Omega + \int_\Omega \frac{1}{\mu}\mathrm{div}\,\mathbf{w}_a\,\mathrm{div}\,\mathbf{A}\,\mathrm{d}\Omega = \int_\Omega \mathbf{w}_a \cdot \mathbf{J}\,\mathrm{d}\Omega}$$

Meshing the Coil



Reduced Vector Potential Formulation

$$\mathbf{A} = \mathbf{A}_\mathrm{s} + \mathbf{A}_\mathrm{r} \qquad \mathbf{B} = \mu_0 \mathbf{H}_\mathrm{s} + \operatorname{curl} \mathbf{A}_\mathrm{r}$$

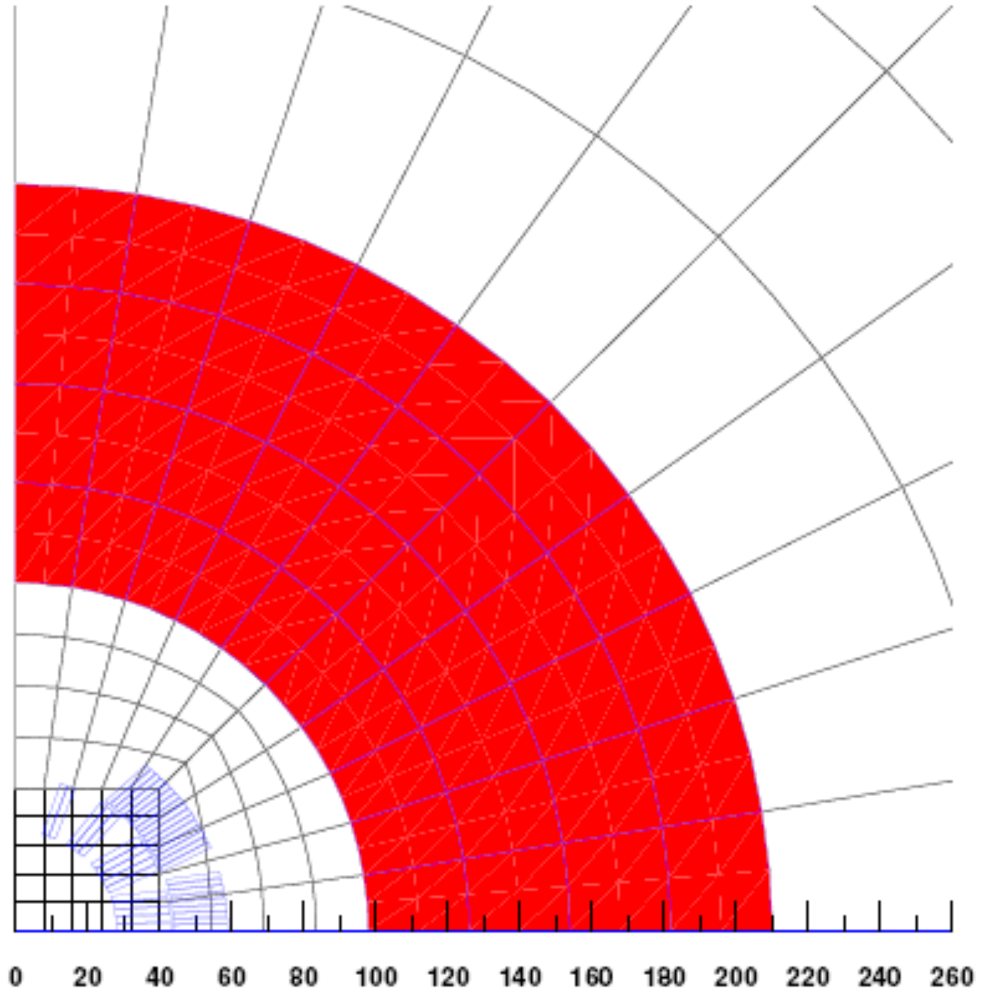
$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} (\mathbf{A}_\mathrm{r} + \mathbf{A}_\mathrm{s}) - \operatorname{grad} \frac{1}{\mu} \operatorname{div} (\mathbf{A}_\mathrm{r} + \mathbf{A}_\mathrm{s}) = \mathbf{J}$$

$$\begin{aligned} \operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A}_\mathrm{r} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A}_\mathrm{r} &= \mathbf{J} - \operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A}_\mathrm{s} \\ &= \operatorname{curl} \mathbf{H}_\mathrm{s} - \operatorname{curl} \frac{\mu_0}{\mu} \mathbf{H}_\mathrm{s} \\ &= \operatorname{curl} \left(\mathbf{H}_\mathrm{s} - \frac{\mu_0}{\mu} \mathbf{H}_\mathrm{s} \right) \end{aligned}$$

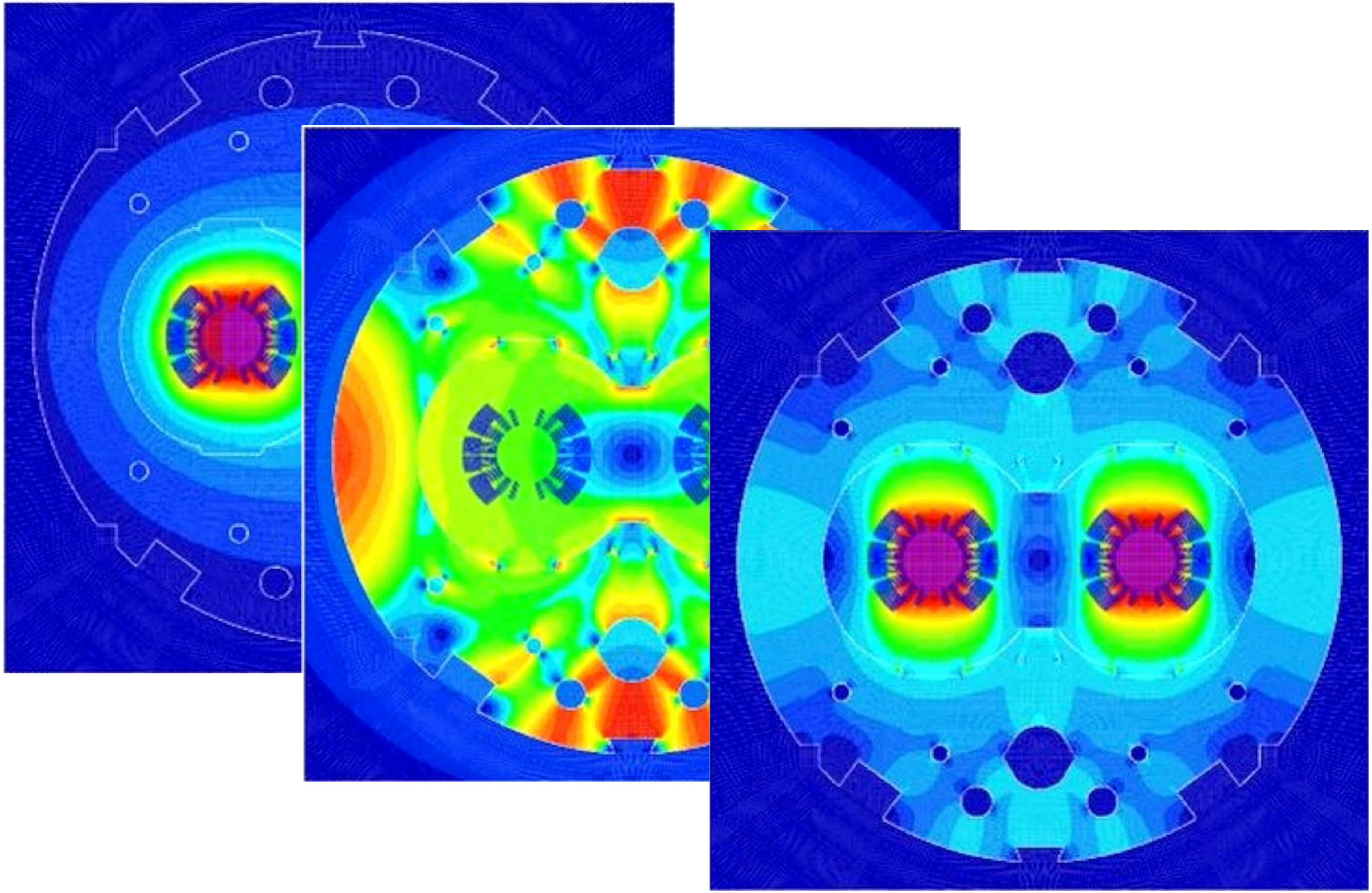
Weak form

$$\int_\Omega \operatorname{curl} \mathbf{w}_a \cdot \frac{1}{\mu} \operatorname{curl} \mathbf{A}_\mathrm{r} \, \mathrm{d}\Omega + \int_\Omega \operatorname{div} \mathbf{w}_a \cdot \frac{1}{\mu} \operatorname{div} \mathbf{A}_\mathrm{r} \, \mathrm{d}\Omega = \int_\Omega \operatorname{curl} \mathbf{w}_a \cdot (\mathbf{H}_\mathrm{s} - \frac{\mu_0}{\mu} \mathbf{H}_\mathrm{s}) \, \mathrm{d}\Omega$$

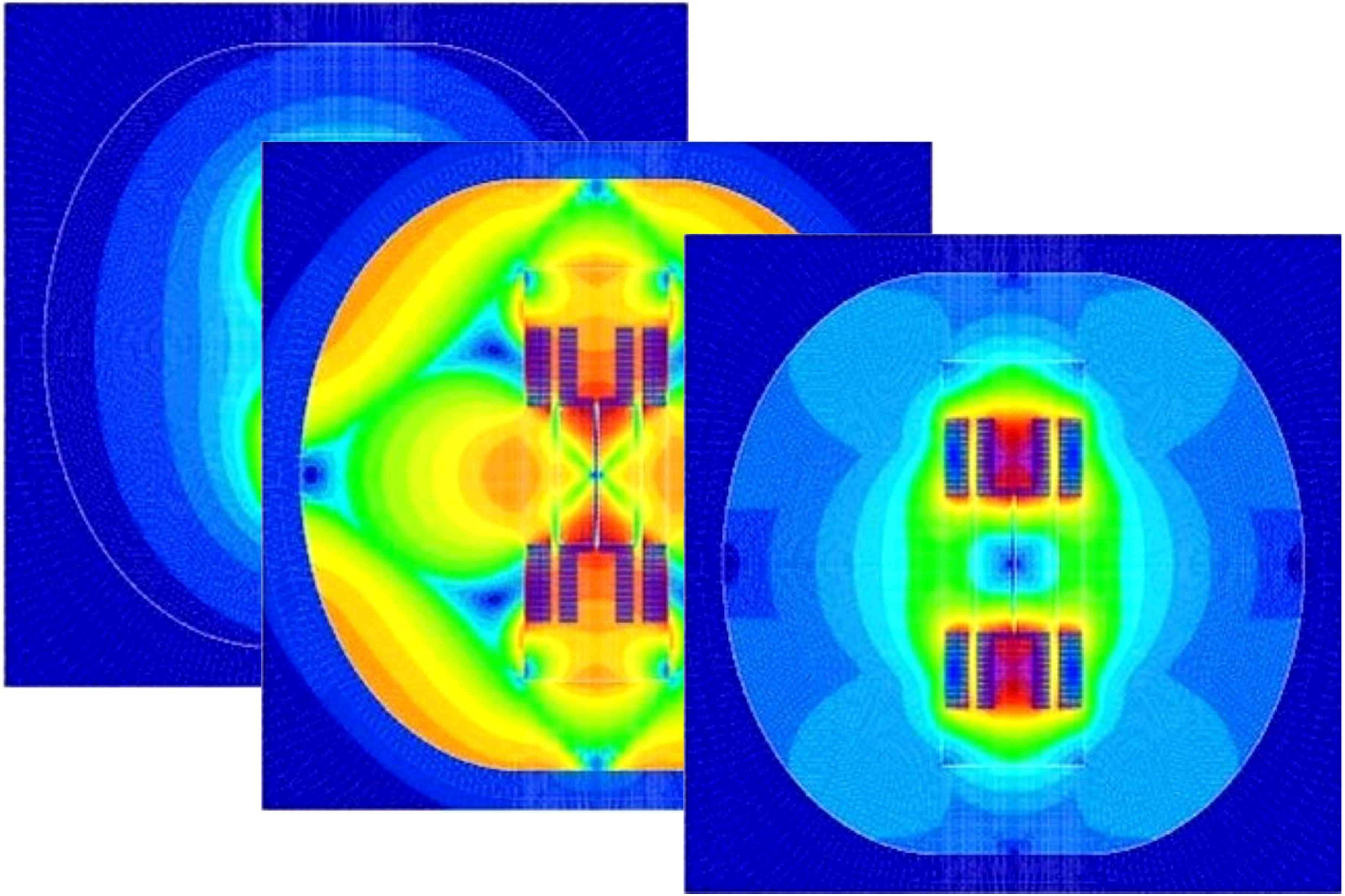
FEM Mesh for Reduced Vector Potential Formulation



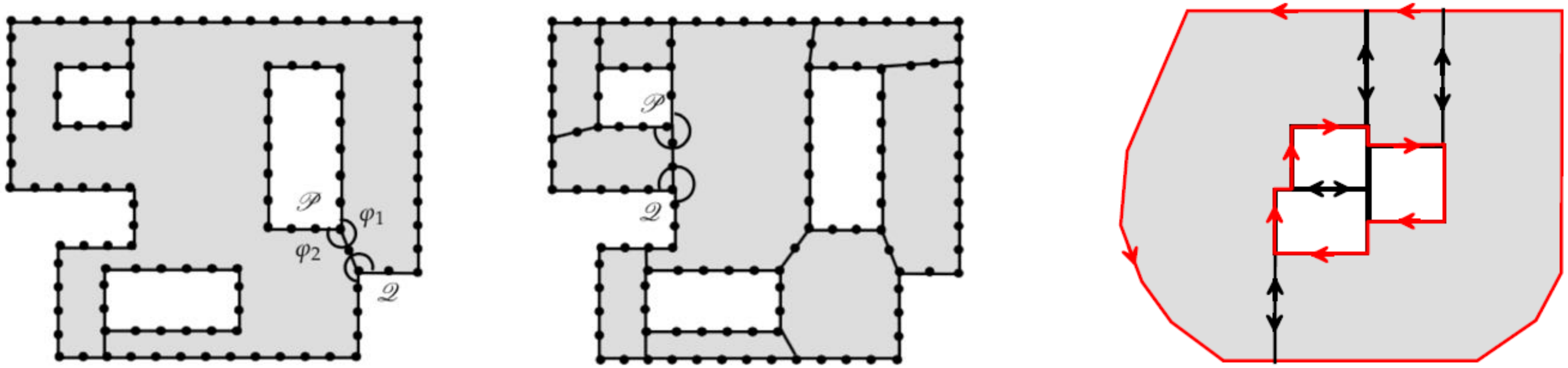
Source, Reduced, Total Field



Source, Reduced, Total Field



Topology Decomposition



$$\Gamma := \mathscr{P} + t(\mathscr{Q} - \mathscr{P})$$

$$\min\{f(\Gamma)\} = \min\{\lambda_1 g_1(\Gamma) + \lambda_2 g_2(\Gamma) + \lambda_3 g_3(\Gamma)\}$$

$$g_1(\Gamma) := 4\pi - \varphi_1(\mathscr{P}) - \varphi_2(\mathscr{P}) - \varphi_1(\mathscr{Q}) - \varphi_2(\mathscr{Q}),$$

Max inner angle

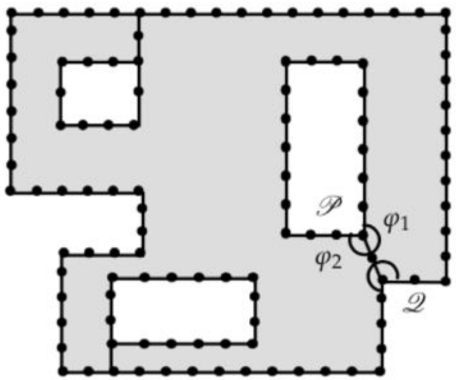
$$g_2(\Gamma) := \sum_{i=1}^{2} \min\left\{\varphi_i(\mathscr{P}) - \frac{\pi}{2}, \varphi_i(\mathscr{P}) - \pi, \varphi_i(\mathscr{P}) - \frac{3\pi}{2}\right\} + \sum_{i=1}^{2} \min\left\{\varphi_i(\mathscr{Q}) - \frac{\pi}{2}, \varphi_i(\mathscr{Q}) - \pi, \varphi_i(\mathscr{Q}) - \frac{3\pi}{2}\right\}$$

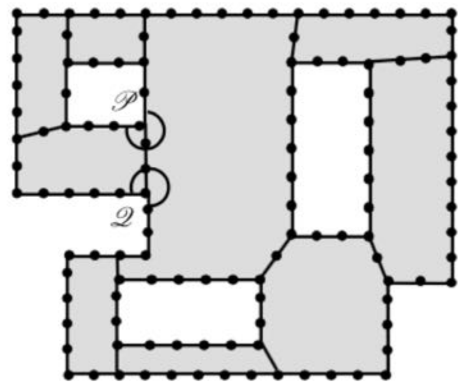
Right angles between cutting edge and boundary

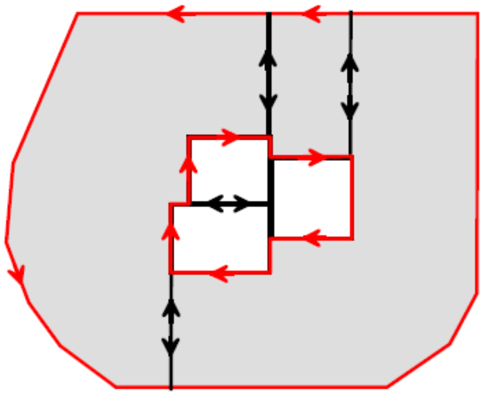
$$g_3(\Gamma) := L,$$

Minimum length

Topology Decomposition







$$\Gamma := \mathscr{P} + t(\mathscr{Q} - \mathscr{P})$$

$$\min\{f(\Gamma)\} = \min\{\lambda_1 g_1(\Gamma) + \lambda_2 g_2(\Gamma) + \lambda_3 g_3(\Gamma) + \lambda_4 g_4(\Gamma)\}$$

$$g_1(\Gamma) := 4\pi - \varphi_1(\mathscr{P}) - \varphi_2(\mathscr{P}) - \varphi_1(\mathscr{Q}) - \varphi_2(\mathscr{Q}),$$

Max inner angle

$$g_2(\Gamma) := \sum_{i=1}^{2} \min\left\{\varphi_i(\mathscr{P}) - \frac{\pi}{2}, \varphi_i(\mathscr{P}) - \pi, \varphi_i(\mathscr{P}) - \frac{3\pi}{2}\right\} + \sum_{i=1}^{2} \min\left\{\varphi_i(\mathscr{Q}) - \frac{\pi}{2}, \varphi_i(\mathscr{Q}) - \pi, \varphi_i(\mathscr{Q}) - \frac{3\pi}{2}\right\}$$

Right angles between cutting edge and boundary

$$g_3(\Gamma) := \frac{L^2}{\min\{a_1, a_2\}},$$

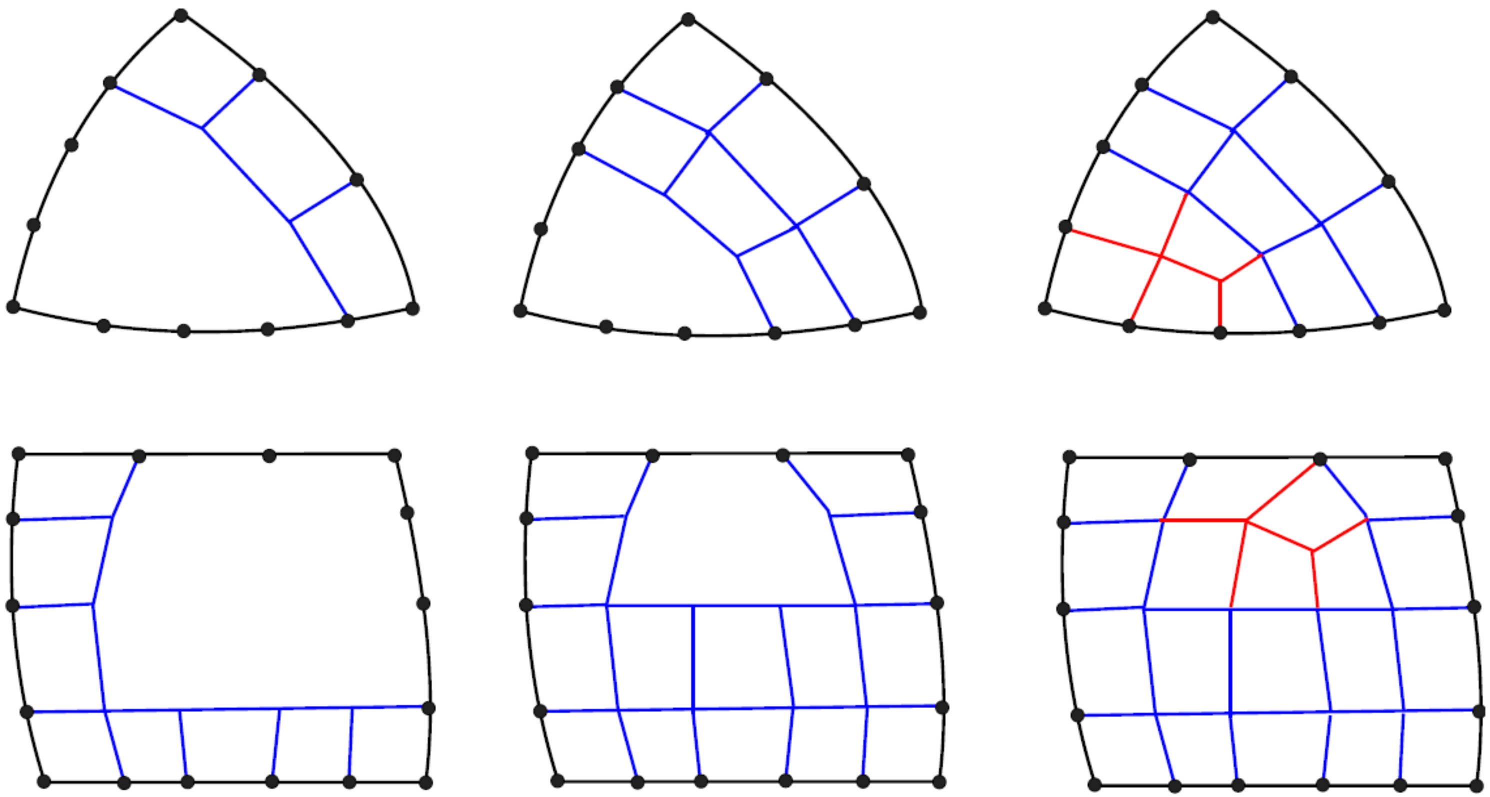
Short edge / large surfaces

$$g_4(\Gamma) := \frac{L}{\min\{C_1, C_2\}}$$

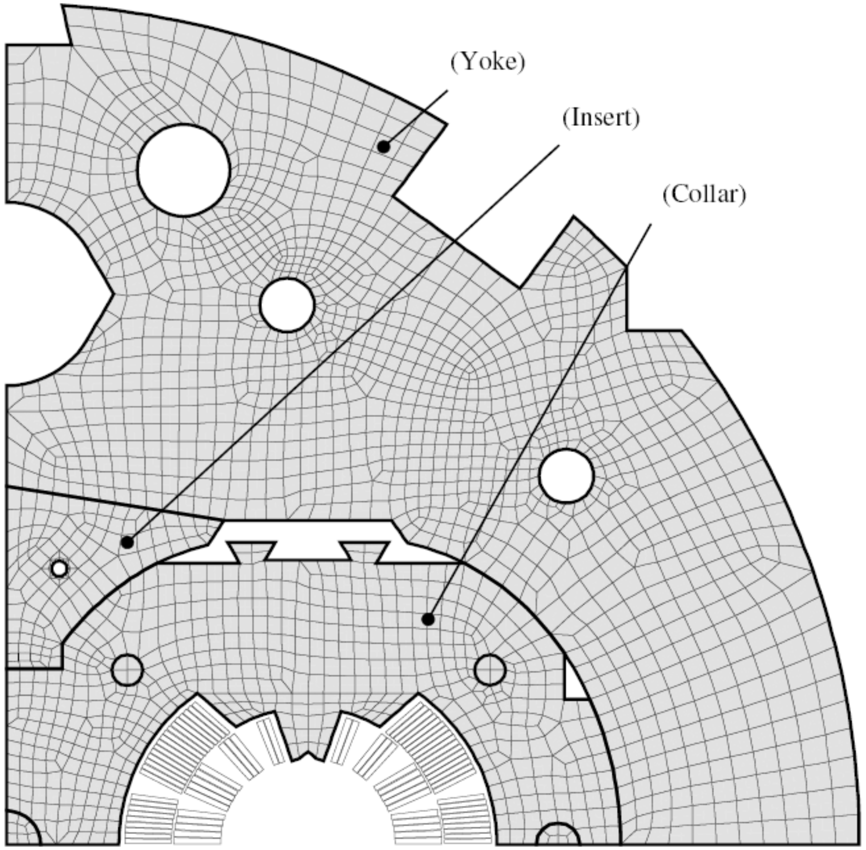
Short edge / large surface boundary

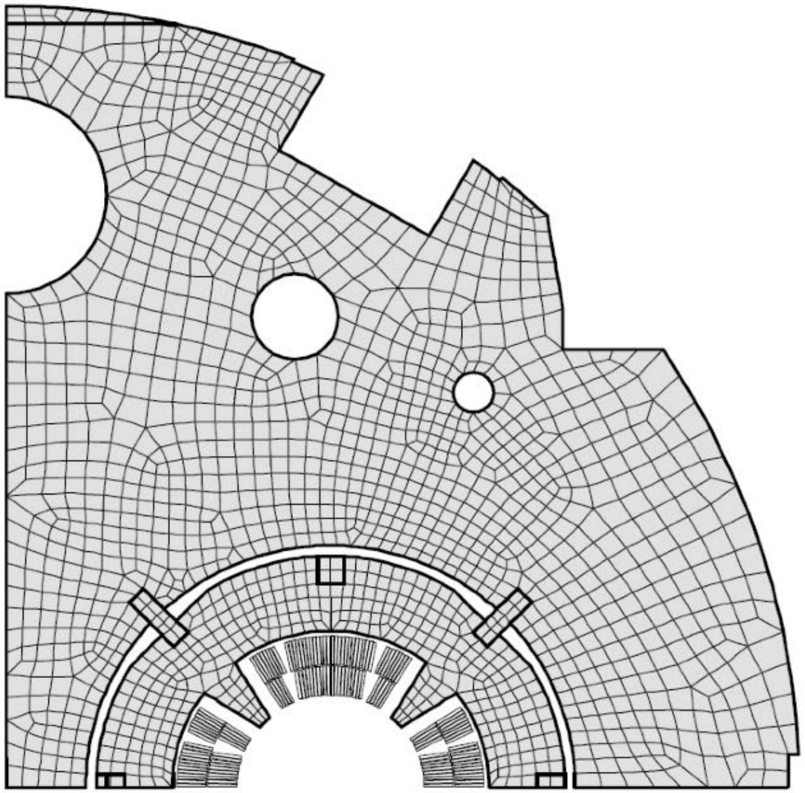
Paving and Mesh Closing in Simple Domains

- The number of nodes is less than 6
- The domian does not contain “bottlenecks” , i.e., C^2/a approaches 4π
- The biggest inner angle is less then π
- For triangles: $a+b < c$



Examples for FEM Meshes

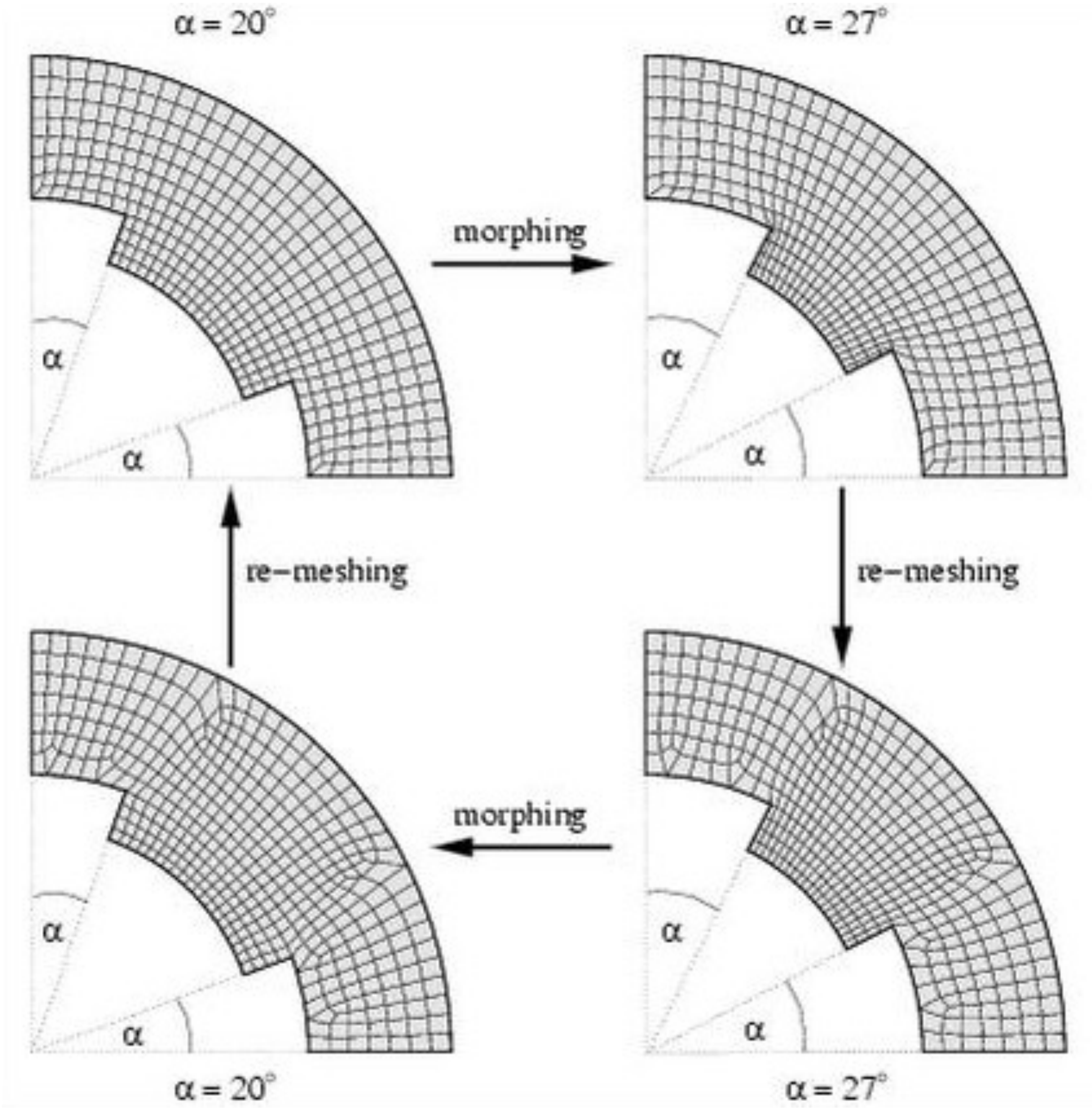




Point-Based Morphing

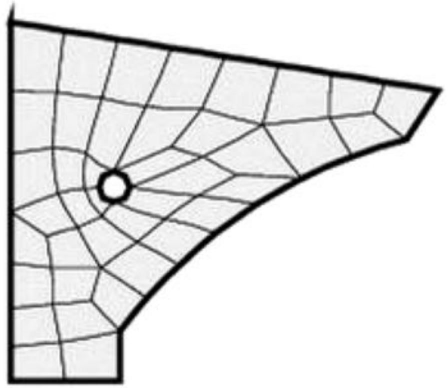
Source: steven.codes / Face Morphing

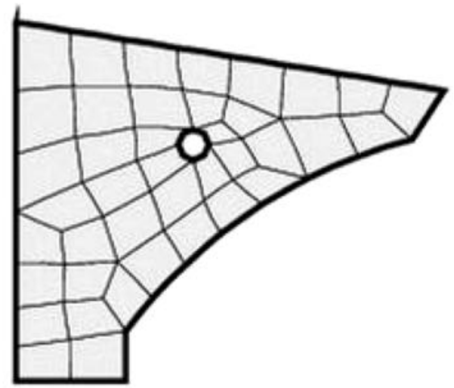
Point Based Morphing

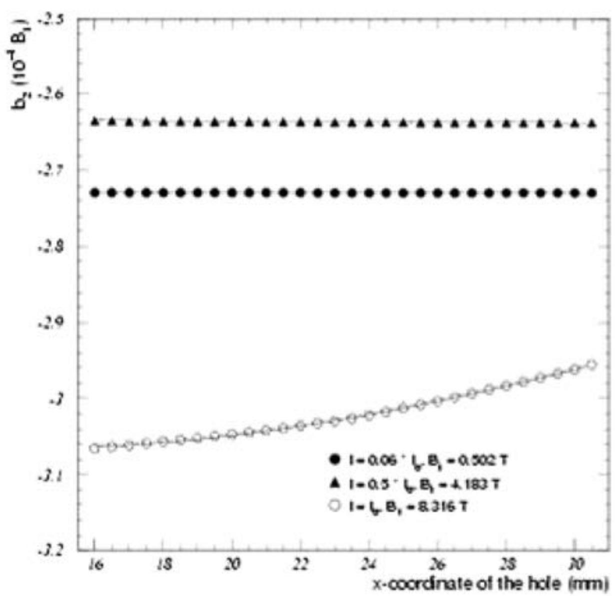


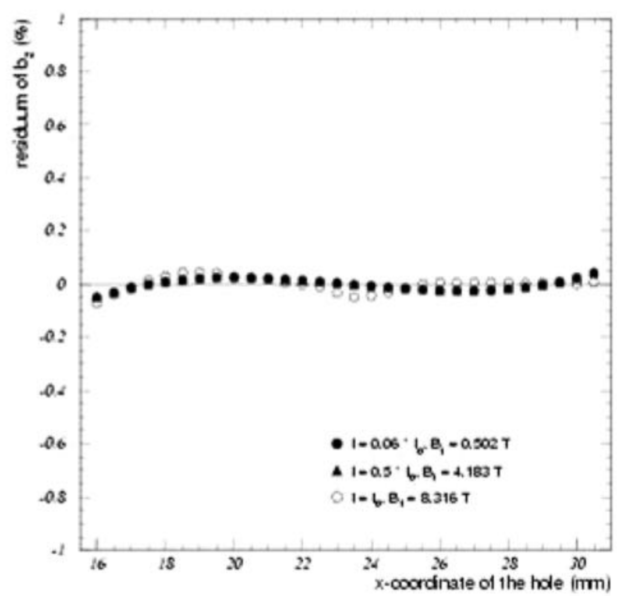
Point Based Morphing

$$\vec{s}(Q) = \sum_{i=1}^{n} \frac{\vec{s}(P_i) \cdot w_i}{|\vec{r}(Q) - \vec{r}(P_i)|} \left(\sum_{i=1}^{n} \frac{w_i}{|\vec{r}(Q) - \vec{r}(P_i)|} \right)^{-1}$$

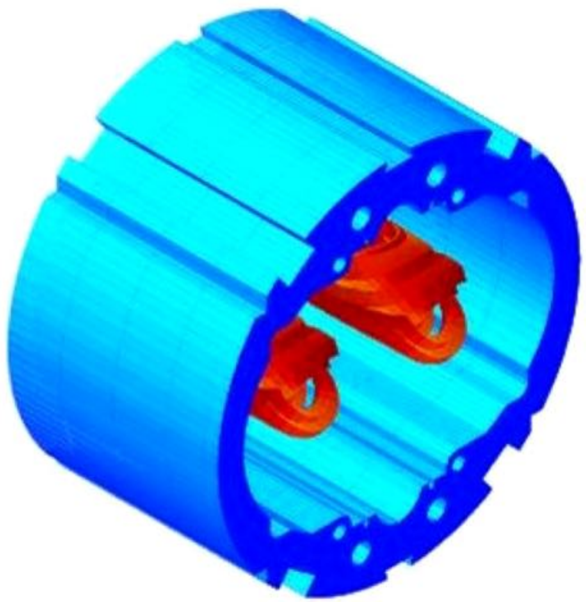


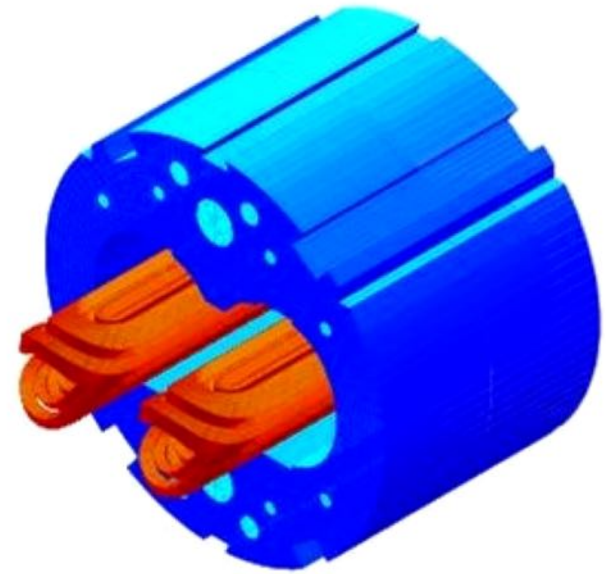


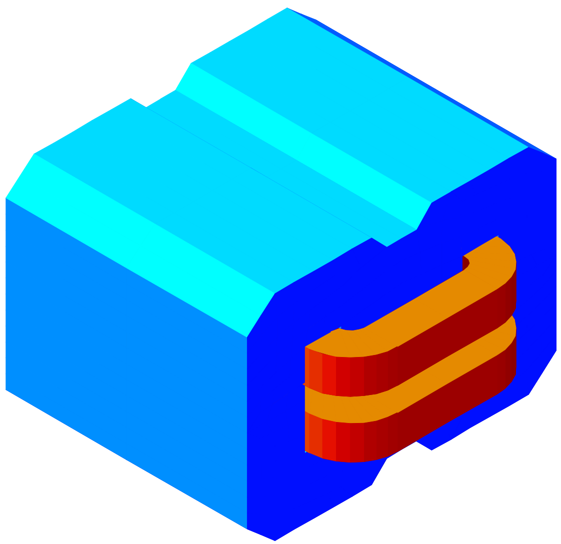


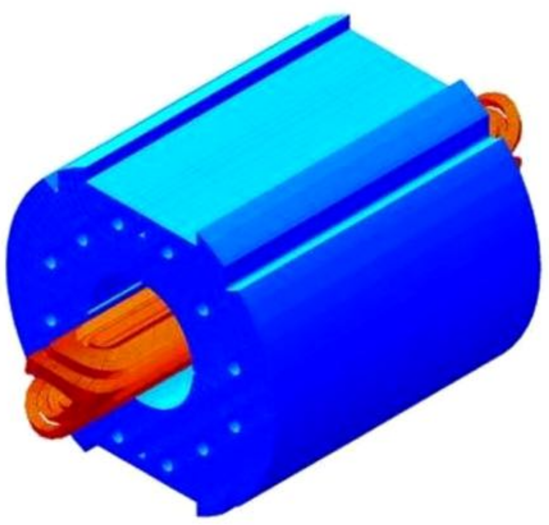


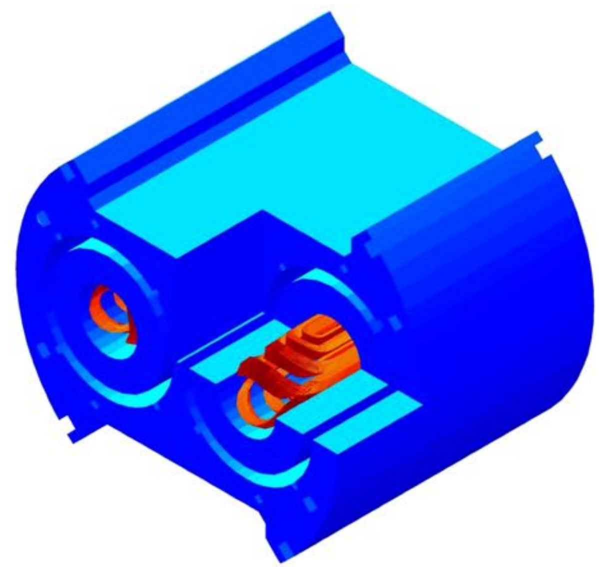
Magnet Extremities











Total Scalar Potential

$$\operatorname{curl} \mathbf{H} = 0$$

$$\mathbf{H} = -\operatorname{grad} \phi_{\mathrm{m}} \quad \text{in}\, \Omega,\ \mathbf{J} = 0$$

$$\operatorname{div} (\mu \operatorname{grad} \phi_{\mathrm{m}}) = 0 \quad \text{in}\, \Omega,\ \mathbf{J} = 0$$

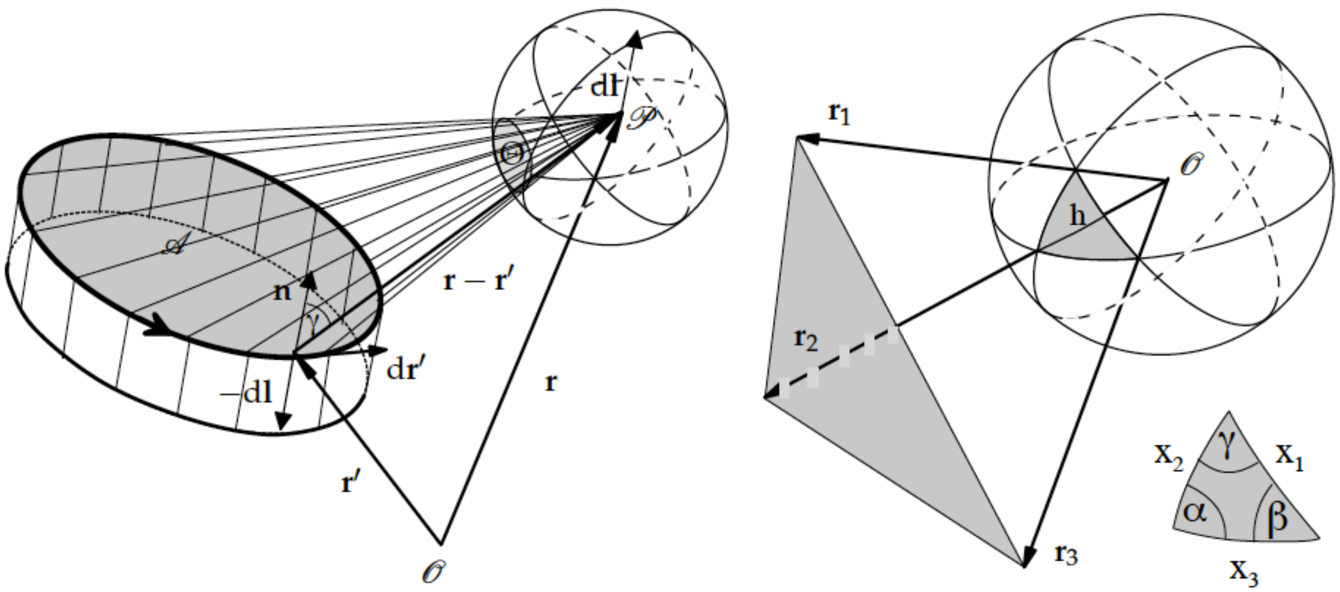
$$\begin{aligned} \mu_0 \operatorname{div} \operatorname{grad} \phi_{\mathrm{m}} &= 0 \quad \text{in}\, \Omega_{\mathrm{a}},\ \mathbf{J} = 0 \\ \nabla^2 \phi_{\mathrm{m}} &= 0 \quad \text{in}\, \Omega_{\mathrm{a}},\ \mathbf{J} = 0 \end{aligned}$$

$$\begin{aligned} \mathbf{H}_{\mathrm{t}} = \mathbf{0} \quad \rightarrow \quad \mathbf{n} \times (\operatorname{grad} \phi_{\mathrm{m}} \times \mathbf{n}) &= \mathbf{0} \quad \text{on}\ \Gamma_H \\ B_{\mathrm{n}} = 0 \quad \rightarrow \quad \mu \mathbf{n} \cdot \operatorname{grad} \phi_{\mathrm{m}} &= 0 \quad \text{on}\ \Gamma_B \end{aligned}$$

$$\begin{aligned} [\mu \mathbf{n} \cdot \operatorname{grad} \phi_{\mathrm{m}}]_{\mathrm{ai}} &= 0 \quad \text{on}\ \Gamma_{\mathrm{ai}} \\ [\operatorname{grad} \phi_{\mathrm{m}} \times \mathbf{n}]_{\mathrm{ai}} &= \mathbf{0} \quad \text{on}\ \Gamma_{\mathrm{ai}} \end{aligned}$$

Solid Angle and Magnetic Scalar Potential

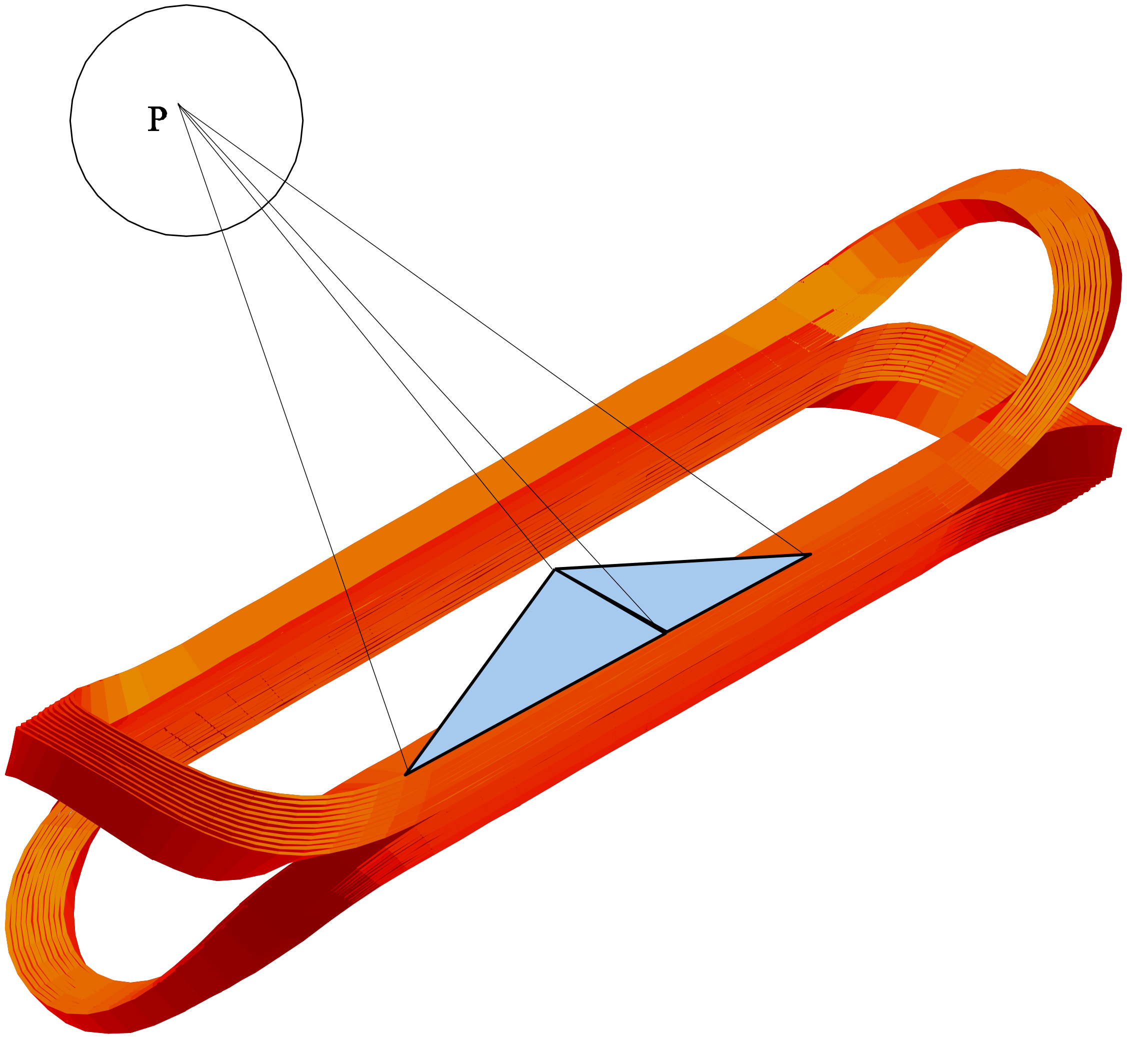
Solid angle (easy to compute) gives the magnetic scalar potential of a current loop



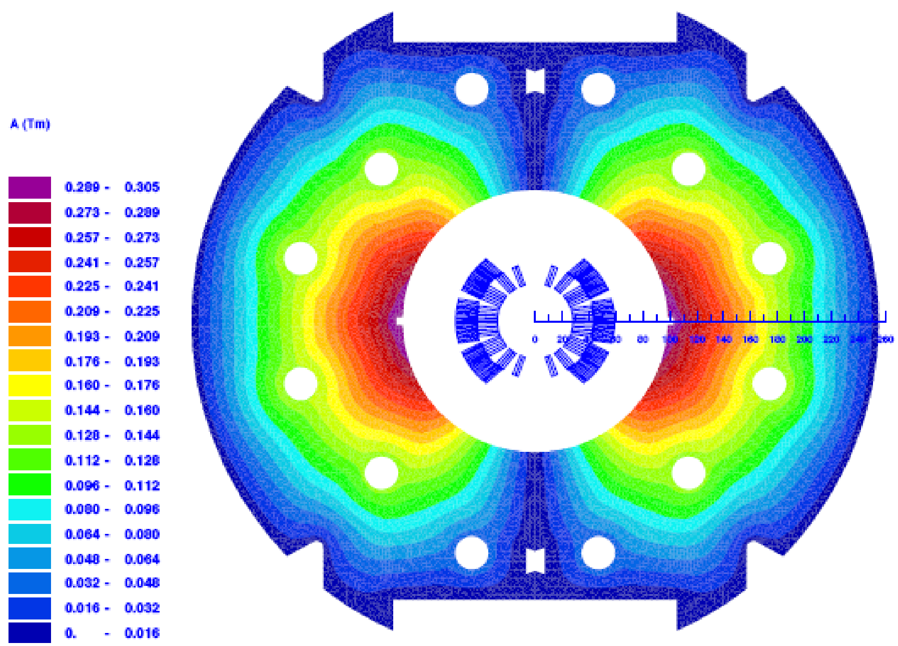
$$\phi_m(\mathbf{r}) = \frac{I}{4\pi}\Theta \,.$$

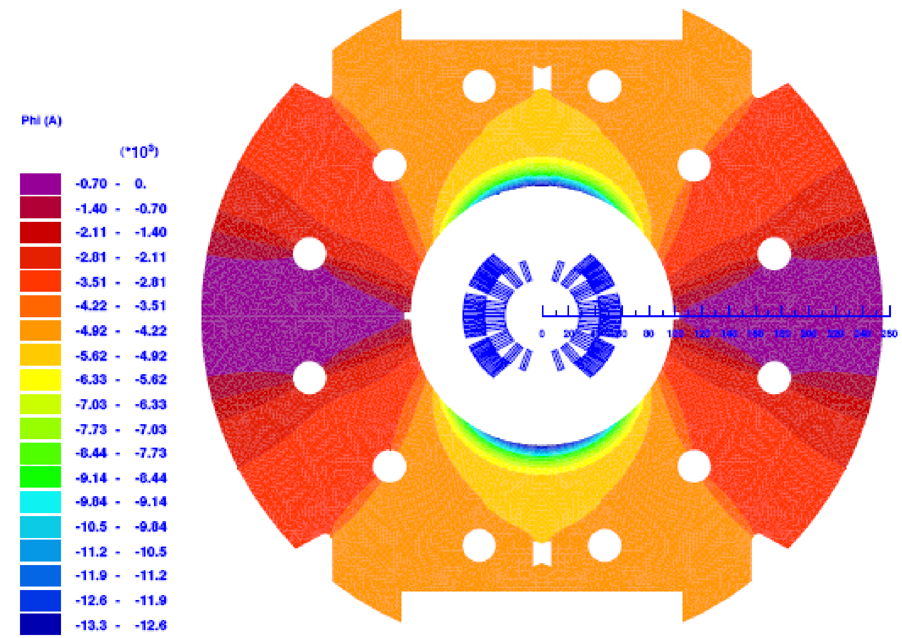
$$\tan\left(\frac{\Theta}{2}\right) = \frac{\mathbf{r}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_3)}{r_1 r_2 r_3 + (\mathbf{r}_1 \cdot \mathbf{r}_2) r_3 + (\mathbf{r}_1 \cdot \mathbf{r}_3) r_2 + (\mathbf{r}_2 \cdot \mathbf{r}_3) r_1} \,.$$

Total Magnetic Scalar Potential



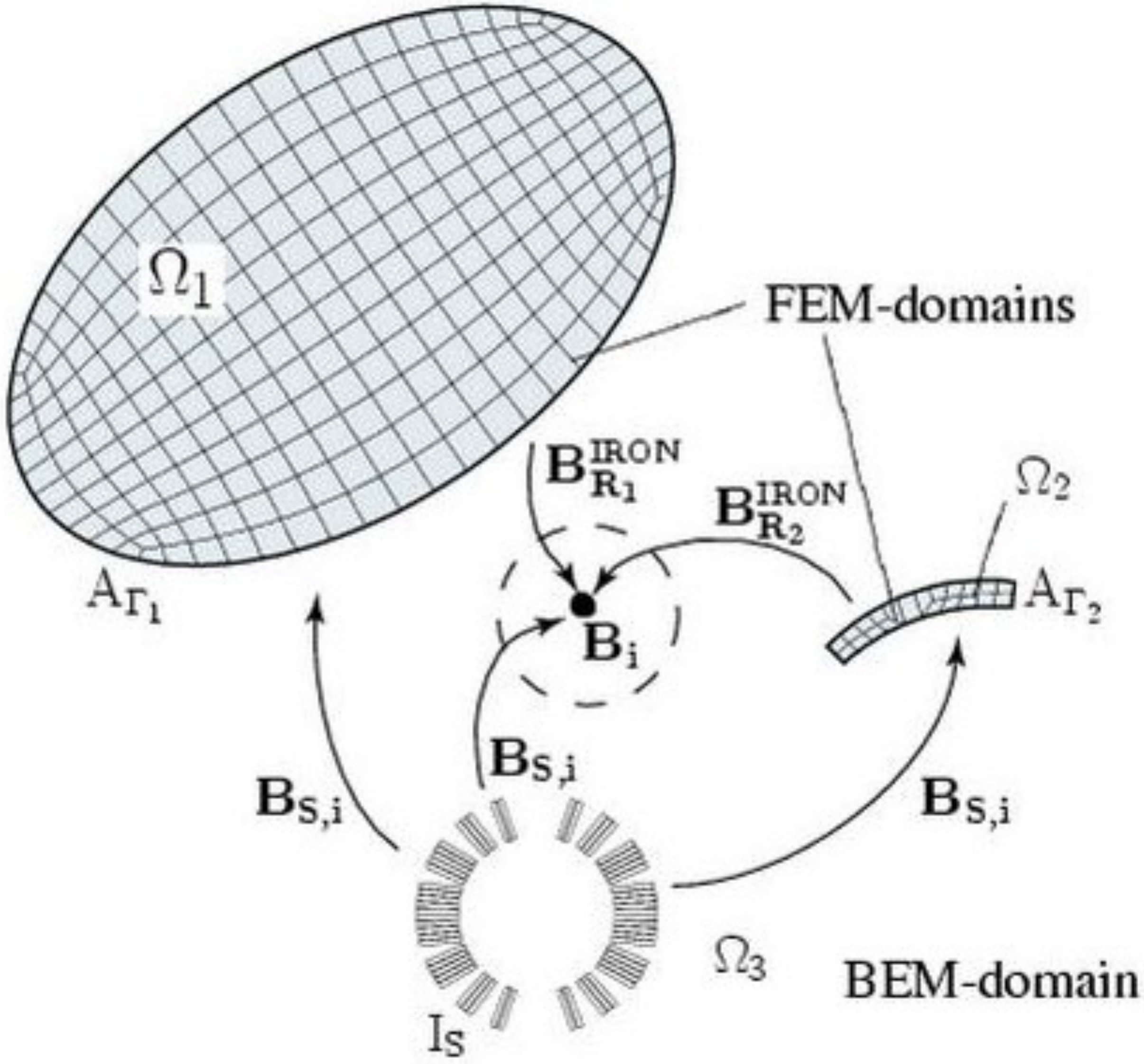
Vector Potential and Total Scalar Potential





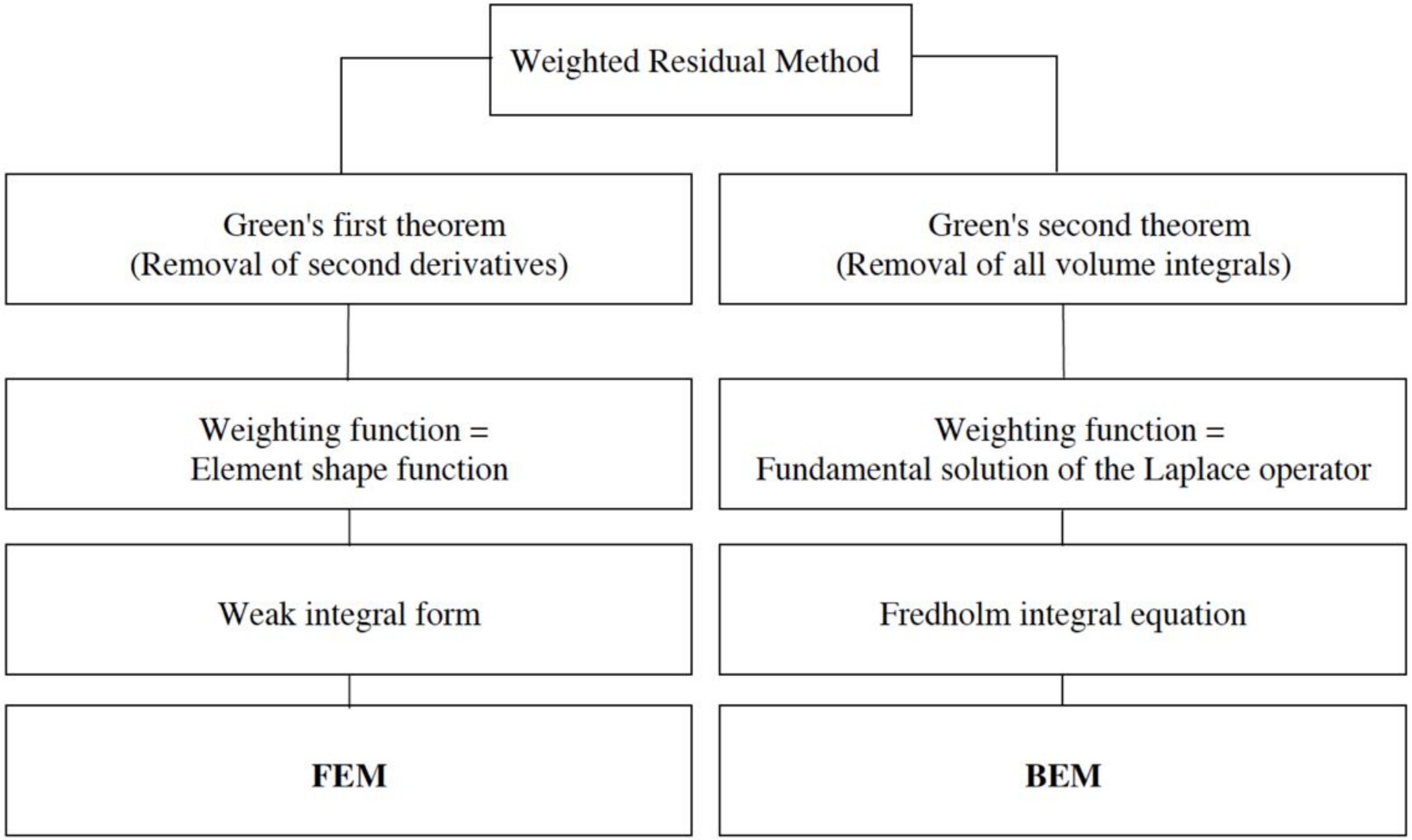
Number of finite elements	60	178	449	787	2799	6233
Total scalar potential	65.8	72.1	13.0	5.0	3.8	15.7
Vector potential	-40.5	-27.4	-7.4	-4.8	-3.8	25.0

BEM-FEM Coupling (Elementary Model Problem)



Green's First and Second Identities in FEM and BEM

$$\int_\Omega (\,\mathrm{grad}\,\phi \cdot\ \mathrm{grad}\,\psi + \phi\nabla^2\psi)\,\mathrm{d}V = \int_\Gamma \phi\ \mathrm{grad}\,\psi\cdot\mathbf{n}\,\mathrm{d}a\,,$$



$$\int_\Omega \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\,\mathrm{d}V = \int_\Gamma (\phi\partial_{\mathbf{n}}\psi - \psi\partial_{\mathbf{n}}\phi)\,\mathrm{d}a\,,$$

The FEM Part (Vector Laplace Equation)

$$\boxed{-\frac{1}{\mu_0}\nabla^2\mathbf{A} = \mathbf{J} + \operatorname{curl}\mathbf{M}} \qquad \text{in } \Omega_i,$$

$$\mathbf{A}\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma_H,$$

$$\frac{1}{\mu_0}\operatorname{div}\mathbf{A} = 0 \qquad \text{on } \Gamma_B,$$

$$\mathbf{n}\times(\mathbf{A}\times\mathbf{n}) = \mathbf{0} \qquad \text{on } \Gamma_B,$$

$$\frac{1}{\mu}(\operatorname{curl}\mathbf{A})\times\mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma_H,$$

$$\left[\frac{1}{\mu_0}\operatorname{div}\mathbf{A}_\mathrm{a}\right]_\mathrm{ai} = 0 \qquad \text{on } \Gamma_\mathrm{ai},$$

$$\boxed{\frac{1}{\mu_0}(\operatorname{curl}\mathbf{A}_\mathrm{i} - \mu_0\mathbf{M})\times\mathbf{n}_\mathrm{i} + \frac{1}{\mu_0}(\operatorname{curl}\mathbf{A}_\mathrm{a})\times\mathbf{n}_\mathrm{a} = \mathbf{0}} \qquad \text{on } \Gamma_\mathrm{ai},$$

$$[\mathbf{A}]_\mathrm{ai} = \mathbf{0} \qquad \text{on } \Gamma_\mathrm{ai}.$$

FEM Part

$$\frac{1}{\mu_0}\int_{\Omega_\mathrm{i}} \mathrm{grad}\,(\mathbf{A}\cdot\mathbf{e}_a)\cdot\,\mathrm{grad}\,w_a\,\mathrm{d}\Omega_\mathrm{i} - \frac{1}{\mu_0}\oint_{\Gamma_\mathrm{ai}}\left(\frac{\partial\mathbf{A}}{\partial n_\mathrm{i}} - (\mu_0\mathbf{M}\times\mathbf{n}_\mathrm{i})\right)\cdot\mathbf{w}_a\,\mathrm{d}\Gamma_\mathrm{ai} = \int_{\Omega_\mathrm{i}}\mathbf{M}\cdot\,\mathrm{curl}\,\mathbf{w}_a\,\mathrm{d}\Omega_\mathrm{i}$$

$$\frac{\partial\mathbf{A}_\mathrm{i}^\mathrm{FEM}}{\partial n_\mathrm{i}} - (\mu_0\mathbf{M}\times\mathbf{n}_\mathrm{i}) + \frac{\partial\mathbf{A}_\mathrm{a}^\mathrm{BEM}}{\partial n_\mathrm{a}} = \mathbf{0} \quad \text{on } \Gamma_\mathrm{ai}$$

$$\frac{1}{\mu_0}\int_{\Omega_\mathrm{i}} \mathrm{grad}\,(\mathbf{A}\cdot\mathbf{e}_a)\cdot\,\mathrm{grad}\,w_a\,\mathrm{d}\Omega_\mathrm{i} - \frac{1}{\mu_0}\oint_{\Gamma_\mathrm{ai}}\mathbf{Q}_{\Gamma_\mathrm{ai}}\cdot\mathbf{w}_a\,\mathrm{d}\Gamma_\mathrm{ai} = \int_{\Omega_\mathrm{i}}\mathbf{M}\cdot\,\mathrm{curl}\,\mathbf{w}_a\,\mathrm{d}\Omega_\mathrm{i}$$

$$[K]\{A\} - [T]\{Q\} = \{F(\mathbf{M})\}$$

BEM Part

Vector Laplace

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}\,, \qquad \text{in}\,\Omega_\mathrm{a}$$

Weighted Residual (x 3 in Cartesian coordinates)

$$\int_{\Omega_\mathrm{a}} \nabla^2 A w \,\mathrm{d}\Omega_\mathrm{a} = -\int_{\Omega_\mathrm{a}} \mu_0 J w \,\mathrm{d}\Omega_\mathrm{a}$$

Kirchhoff theorem: From Green's second theorem and setting the using the free-space Green function of the Laplace operator as weighting function

$$C\,A(\mathbf{r}) + \int_\Gamma -\partial_{\mathbf{n}_\mathrm{a}} A(\mathbf{r}')\, u^*(\mathbf{r},\mathbf{r}')\,\mathrm{d}a' + \int_\Gamma A(\mathbf{r}')\,\partial_{\mathbf{n}_\mathrm{a}} u^*(\mathbf{r},\mathbf{r}')\;\mathrm{d}a' = \int_{\Omega_\mathrm{a}} \mu_0 J u^*(\mathbf{r},\mathbf{r}')\,\mathrm{d}V'\,,$$

	$u^*(\mathbf{r},\mathbf{r}')$	$q^*(\mathbf{r},\mathbf{r}') := \partial_{\mathbf{n}_\mathrm{a}} u^*(\mathbf{r},\mathbf{r}')$	C
2D	$-\frac{1}{2\pi}\ln\lvert\mathbf{r}-\mathbf{r}'\rvert$	$-\frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{n}_a}{2\pi\lvert\mathbf{r}-\mathbf{r}'\rvert^2}$	$\frac{\beta}{2}$
3D	$\frac{1}{4\pi\lvert\mathbf{r}-\mathbf{r}'\rvert}$	$-\frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{n}_a}{4\pi\lvert\mathbf{r}-\mathbf{r}'\rvert^3}$	$\frac{\Theta}{4\pi}$

$$\nabla^2 w = -\delta(\lvert\mathbf{r}-\mathbf{r}'\rvert)$$

Representation Formula (Fredholm Integral Equation)

$$C\,A(\mathbf{r}) + \int_\Gamma -\partial_{\mathbf{n}_\mathrm{a}} A(\mathbf{r}')\, u^*(\mathbf{r},\mathbf{r}')\,\mathrm{d}a' + \int_\Gamma A(\mathbf{r}')\,\partial_{\mathbf{n}_\mathrm{a}} u^*(\mathbf{r},\mathbf{r}')\;\mathrm{d}a' = \int_{\Omega_\mathrm{a}} \mu_0 J u^*(\mathbf{r},\mathbf{r}')\,\mathrm{d}V',$$

Single-layer potential

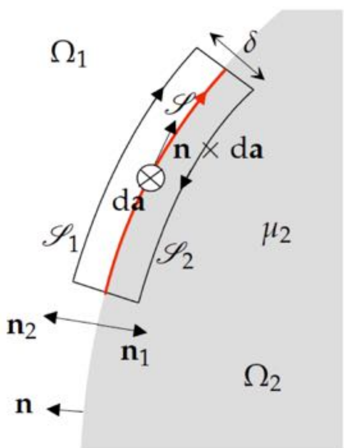
$$\boldsymbol{\alpha}(\mathbf{r}') := -\frac{1}{\mu}\partial_{\mathbf{n}_\mathrm{a}} A(\mathbf{r}')$$

$$[\boldsymbol{\alpha}] = 1\ \mathrm{A\,m^{-1}}$$

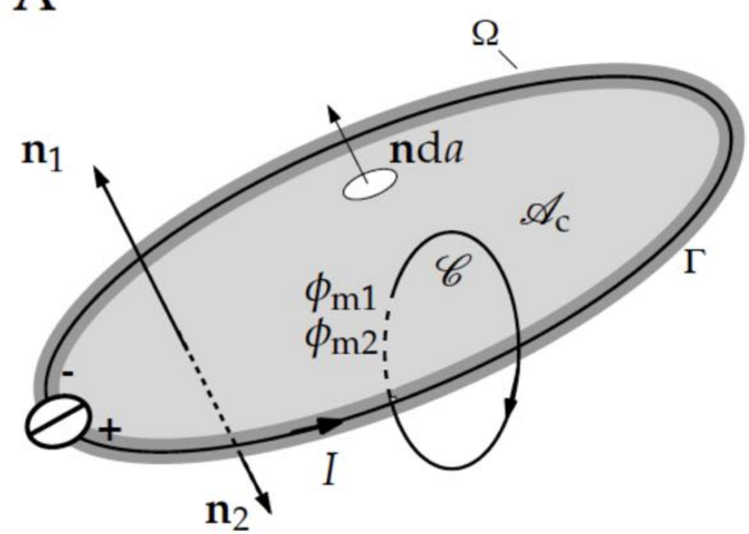
Double-layer potential

$$\boldsymbol{\tau}(\mathbf{r}') := \frac{1}{\mu} A(\mathbf{r}')$$

$$[\boldsymbol{\tau}] = 1\ \mathrm{A}$$



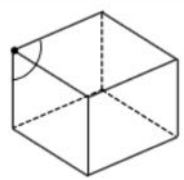
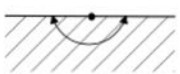
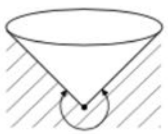
$$\boldsymbol{\alpha} = \mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2)$$



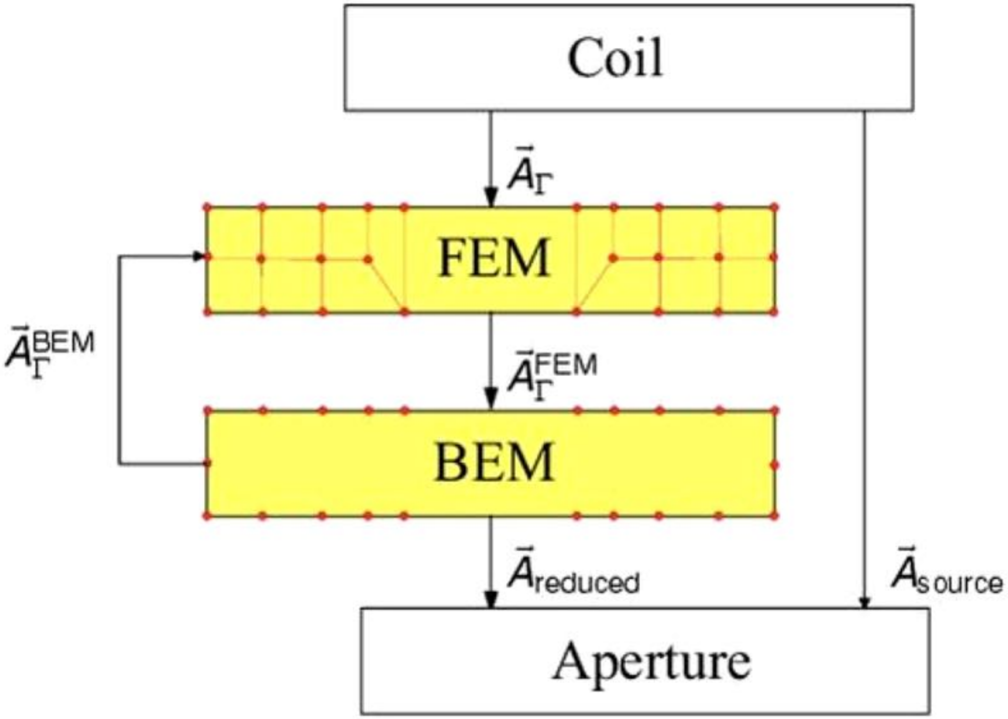
Point-Collocation (Compute One from the Other)

$$Q(\mathbf{r}') := \partial_{\mathbf{n}_a} A(\mathbf{r}') \quad \text{and} \quad q^*(\mathbf{r},\mathbf{r}') := \partial_{\mathbf{n}_a} u^*(\mathbf{r},\mathbf{r}')$$

$$C(\mathbf{r}_p)A(\mathbf{r}_p) + \sum_{e=1}^{E}\int_{\Gamma_e} Q(\mathbf{r})\, u^*(\mathbf{r},\mathbf{r}_p)\mathrm{d}a + \sum_{e=1}^{E}\int_{\Gamma_e} A(\mathbf{r})\, q^*(\mathbf{r},\mathbf{r}_p)\mathrm{d}a = A_\mathrm{s}(\mathbf{r}_p)\,.$$

Ω_a	 90°Corner	 90° Cone inner	 Half-space	 90° Cone outer
Θ	$\frac{1}{2}\pi$	$(2-\sqrt{2})\pi$	2π	$(2+\sqrt{2})\,\pi$
$\frac{\Theta}{4\pi}$	$\frac{1}{8}$	$\frac{2-\sqrt{2}}{4}$	$\frac{1}{2}$	$\frac{2+\sqrt{2}}{4}$

BEM-FEM Coupling



BEM

$$\{Q\} = -[G]^{-1}[H]\{A\} + [G]^{-1}\{A_s\}$$

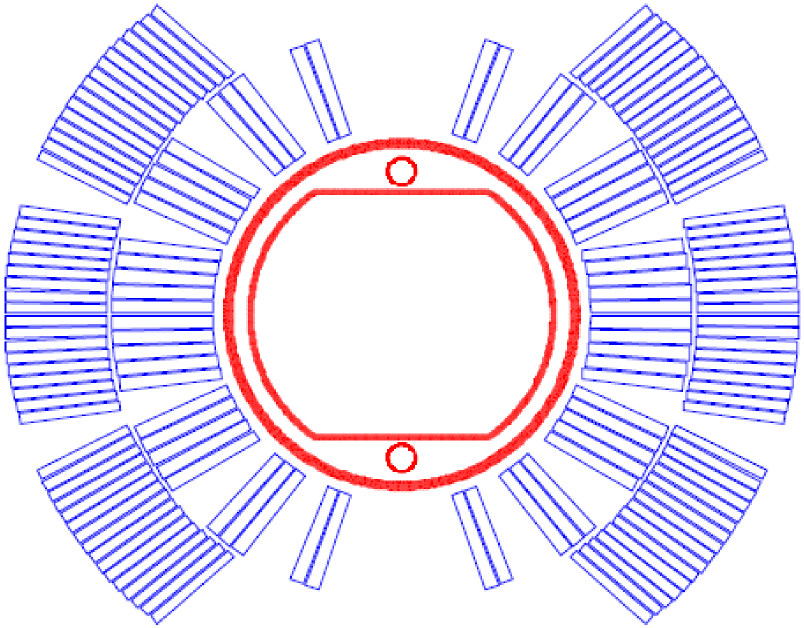
FEM

$$[K]\{A\} - [T]\{Q\} = \{F(\mathbf{M})\}$$

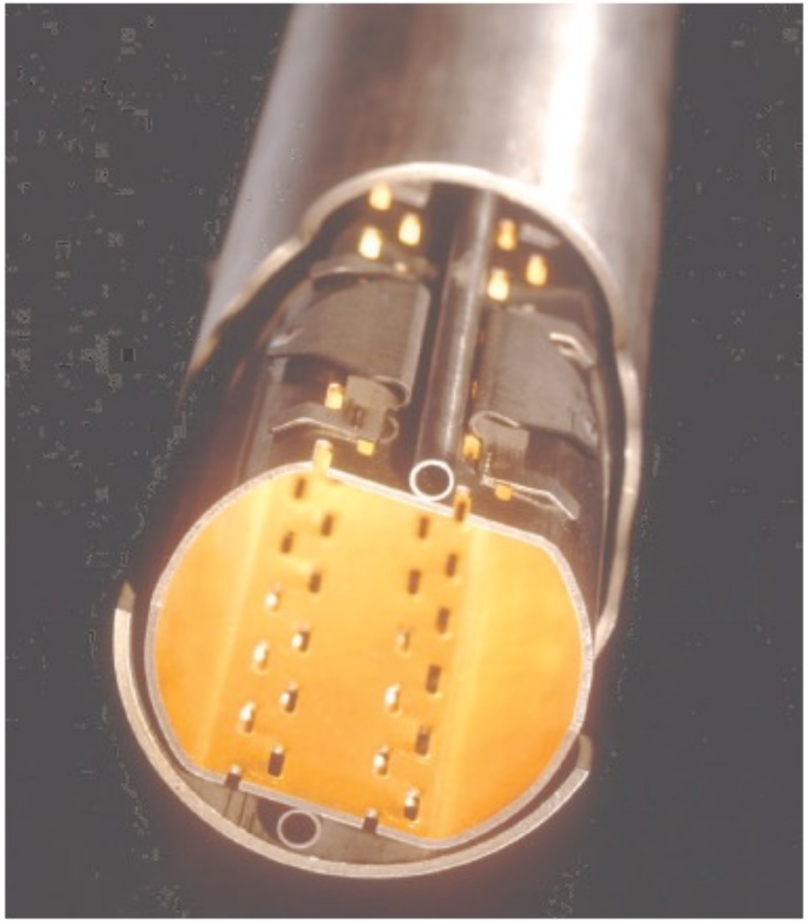
$$\left([K] + [T][G]^{-1}[H]\right)\{A\} = \{F(\mathbf{M})\} + [T][G]^{-1}\{A_s\}$$

$$[\overline{K}]\{A\} = \{\overline{F}(A_s, \mathbf{M})\}$$

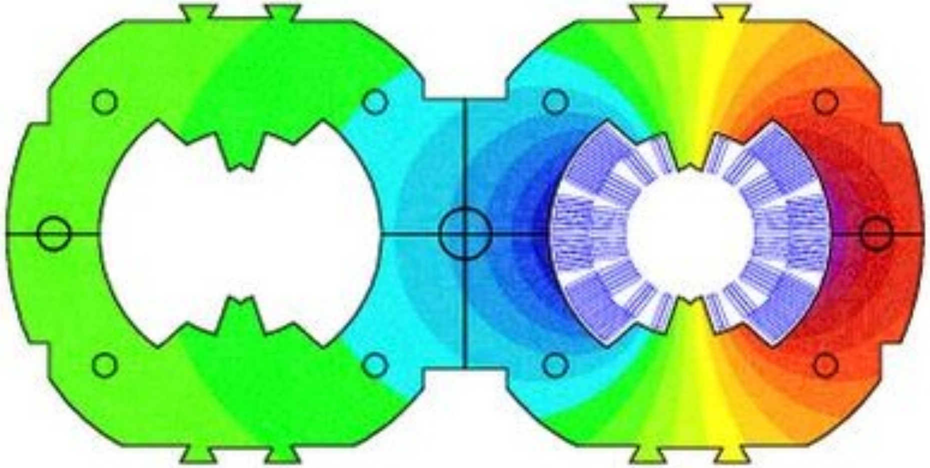
Open Boundary Problems (1)



LHC Beam Screen



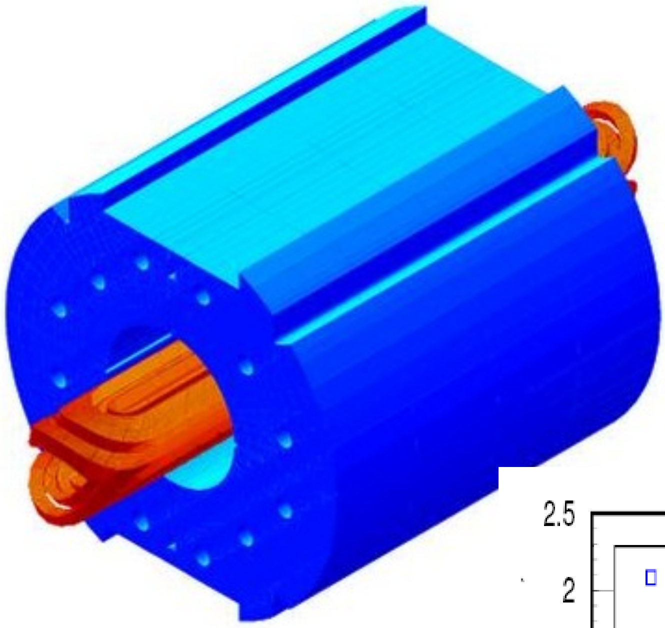
Open Boundary Problem (2)

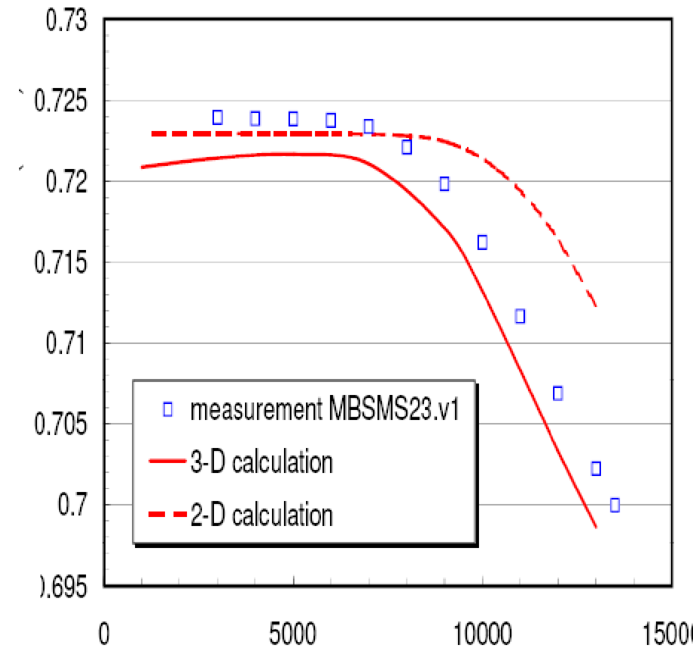


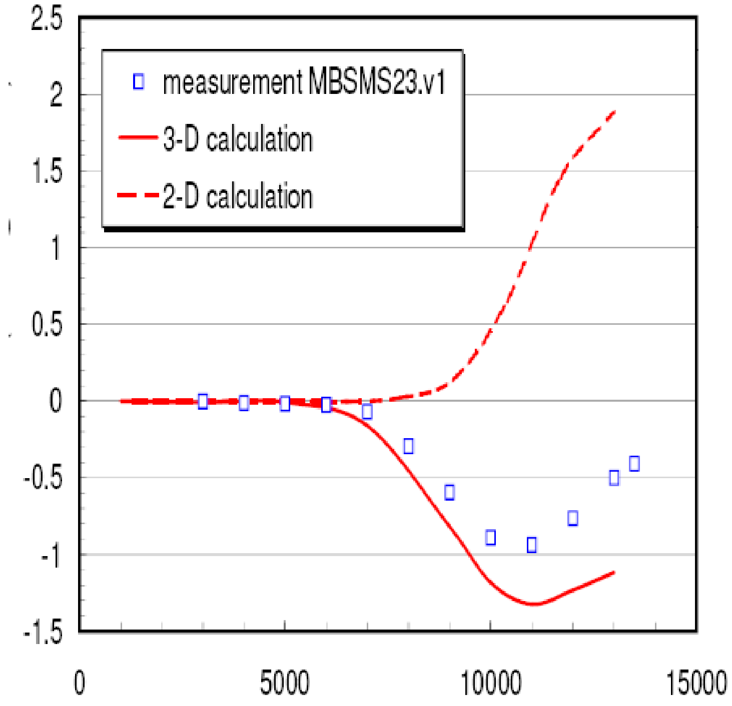
Collared Coil
Field Problem

Collared Coil
Measurements in
Industry

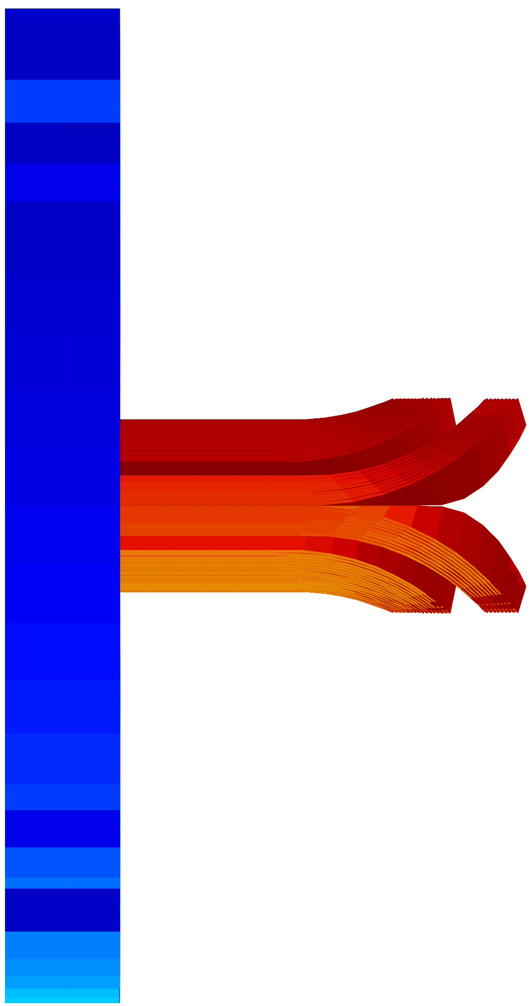
Transfer Function of CTF

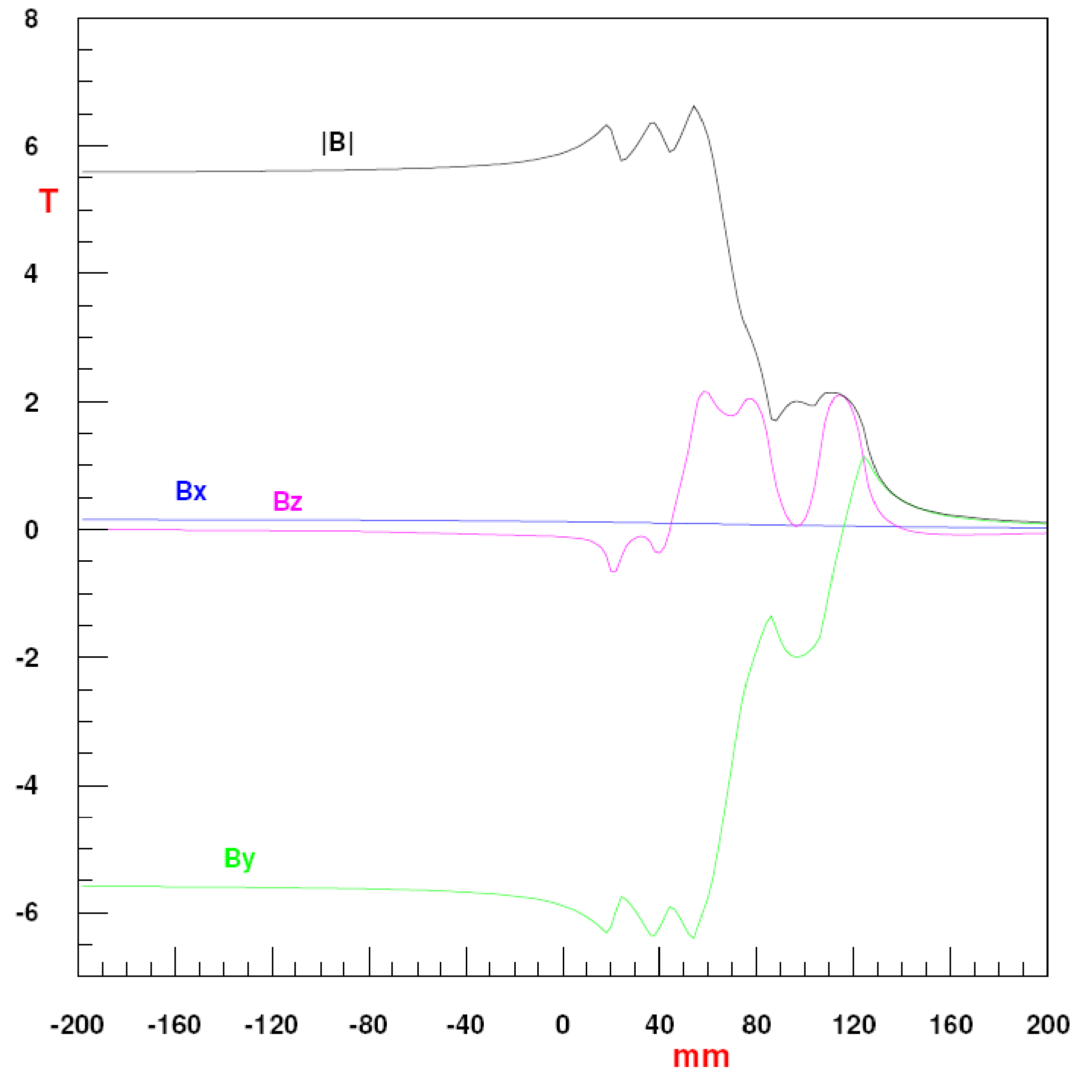




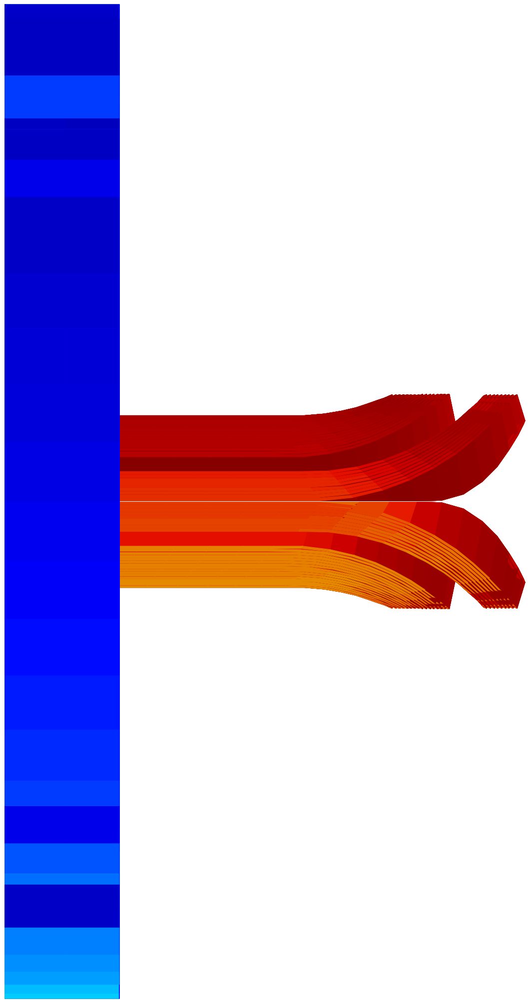


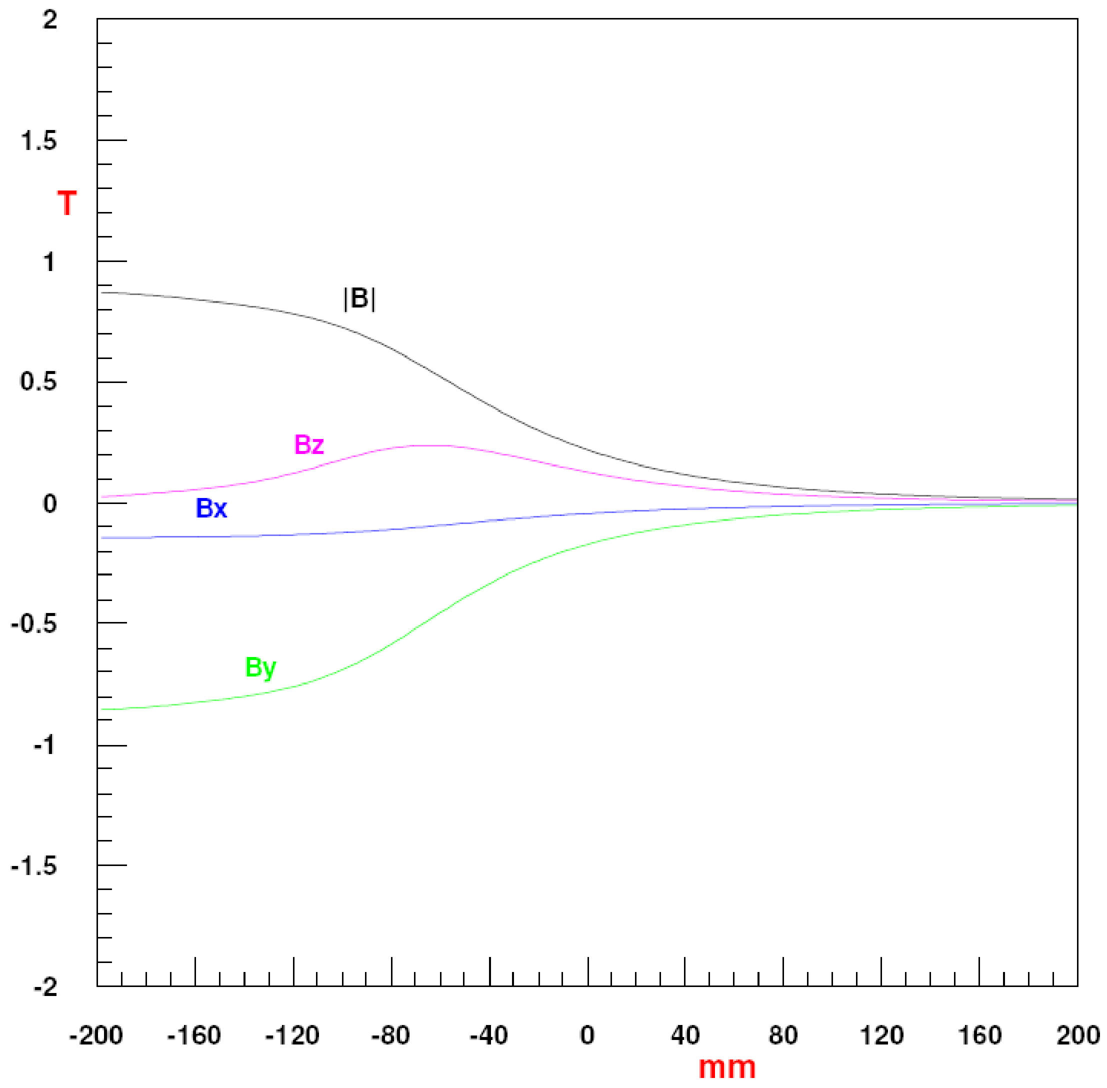
Source Field



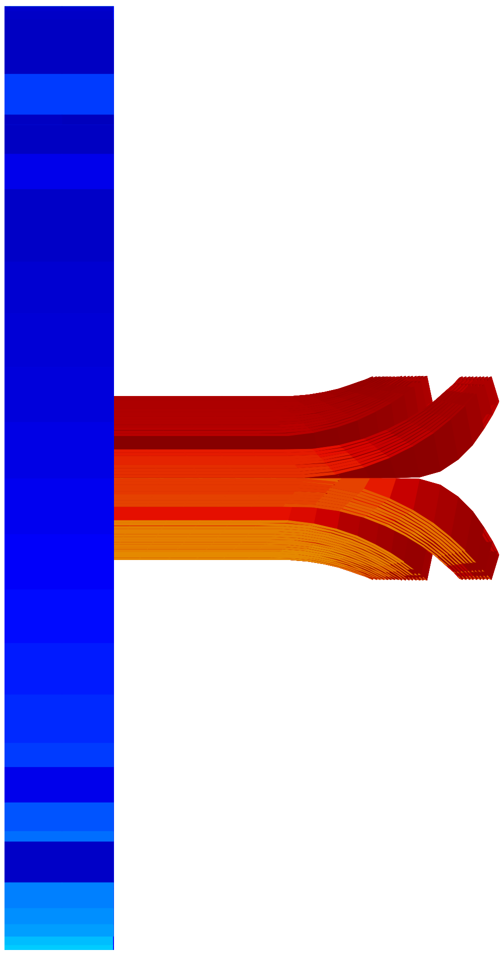


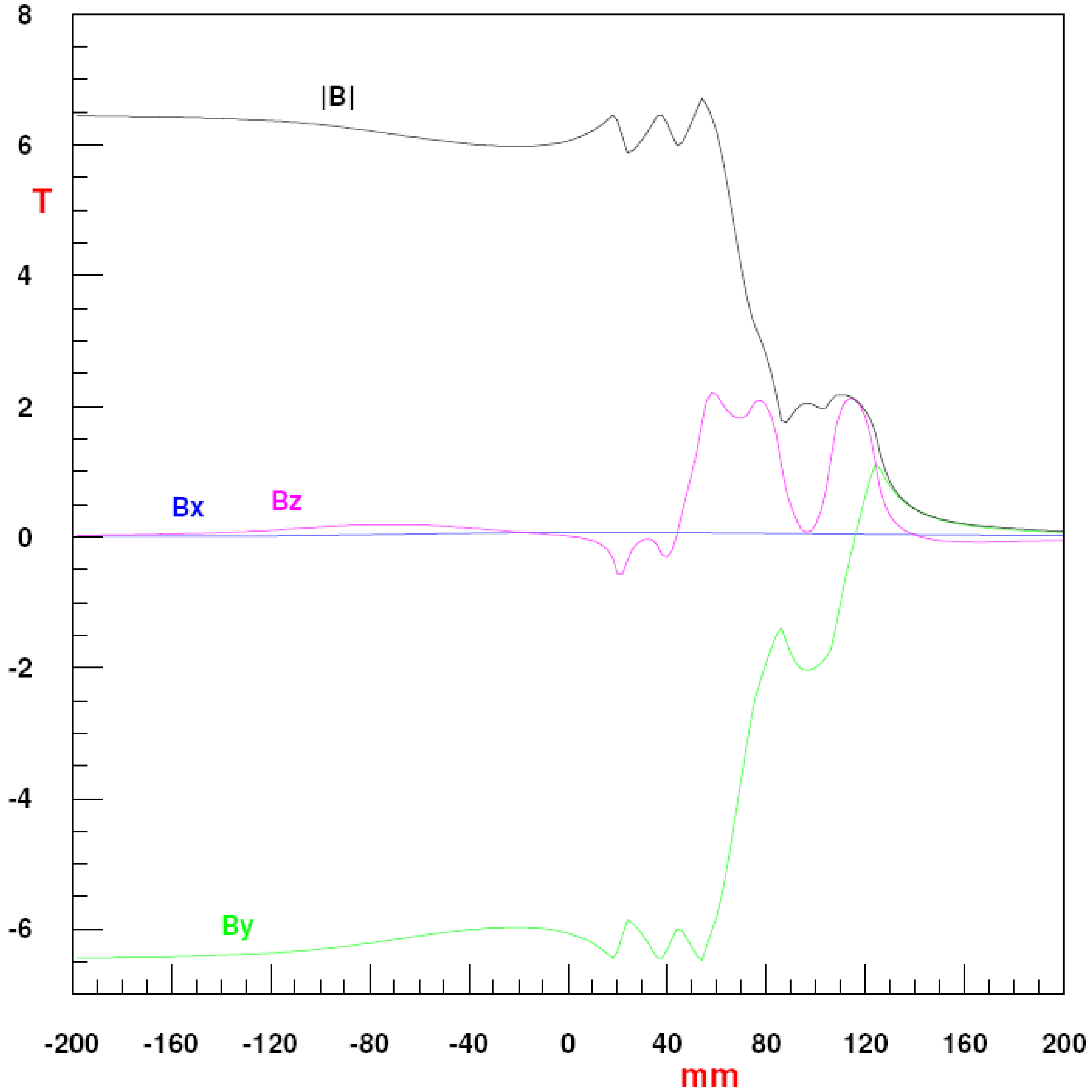
Reduced Field





Total Field





Forces (N) in the Connection Ends of the LHC Main Dipole

I	Fx	Fy	Fz
1	-39.7	-44.0	-45.4
2	-6.5	3.7	-41.7
3	-6.1	88.3	-38.2
4	1.25	3.9	-28.5
5	48.1	-46.7	-48.5
Sum	-2.95	5.2	-202.3