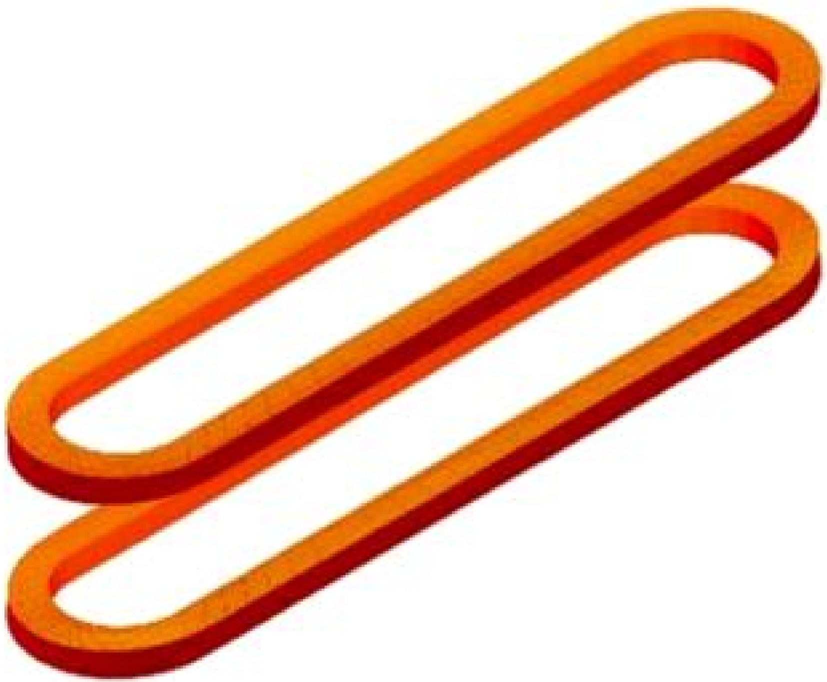
# Electromagnetic Design of Accelerator Magnets and ROXIE User's Course

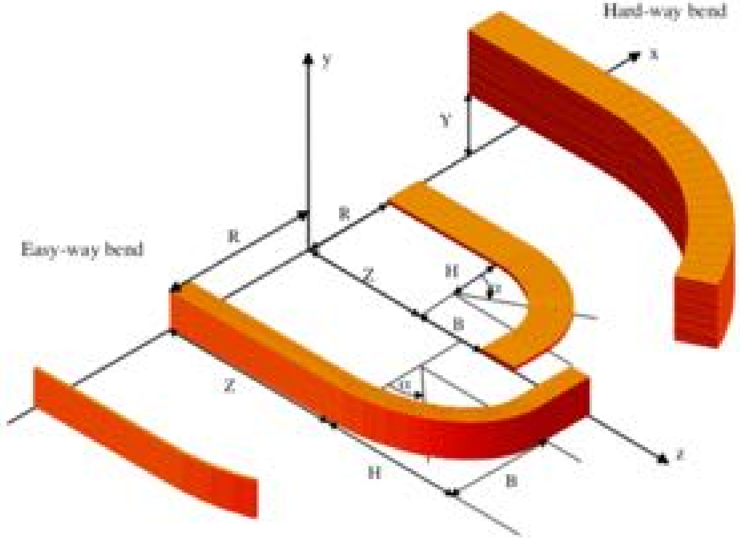
Coil-end design

Stephan Russenschuck, CERN, 2022

# Racetrack and Constant Perimeter Coils

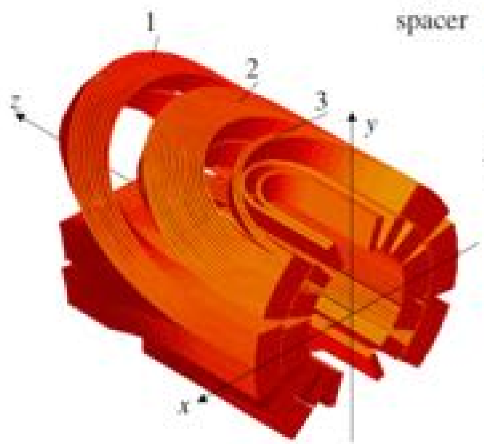


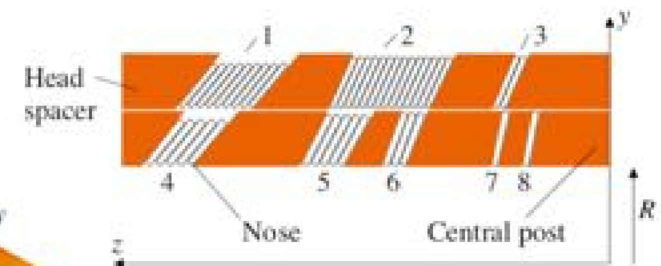
# A Few Conventions

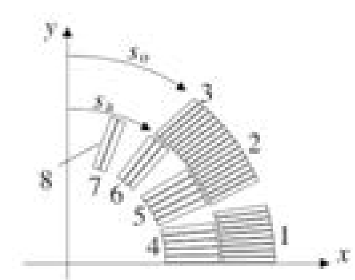



Hard-way bend

Soft-way bend

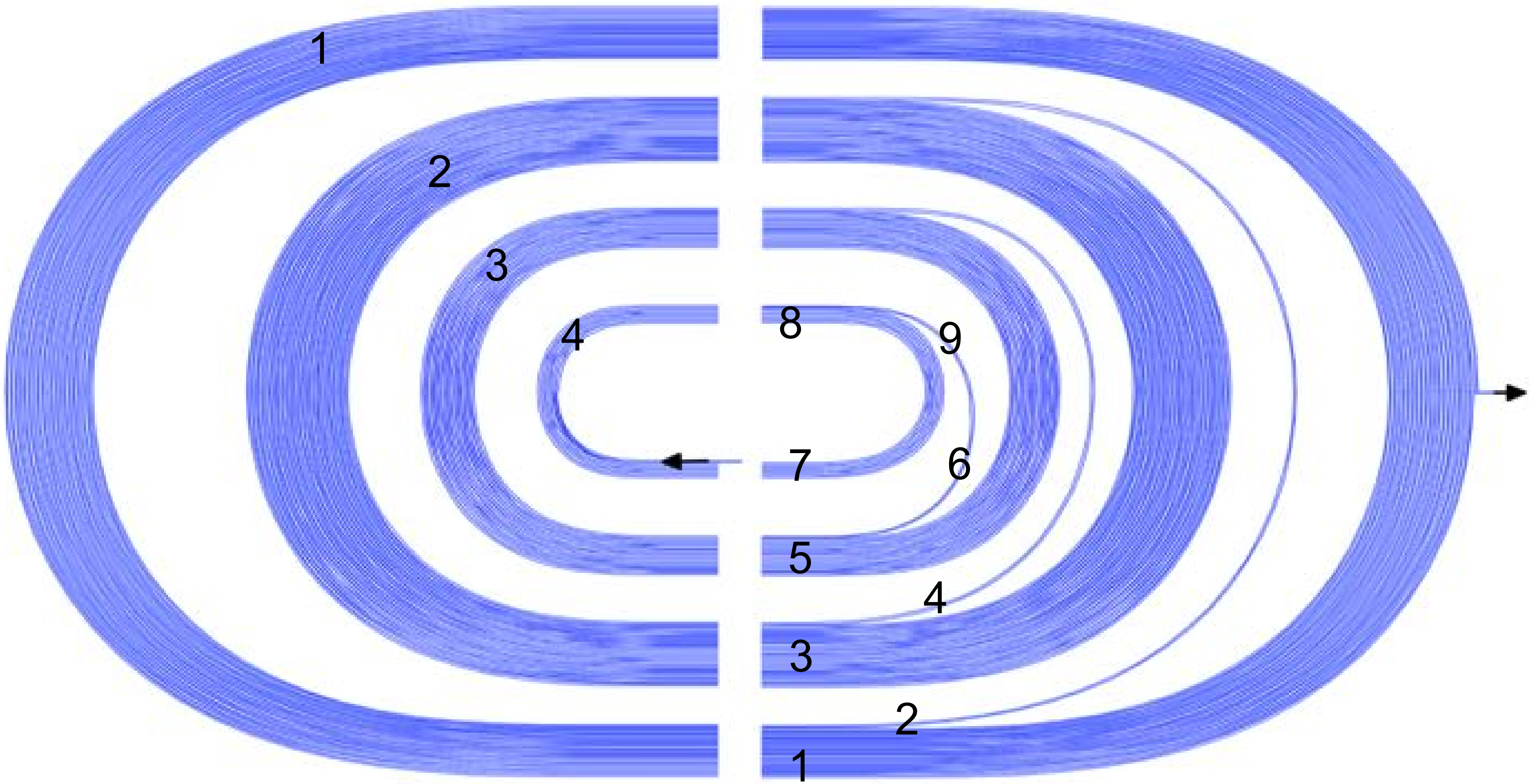


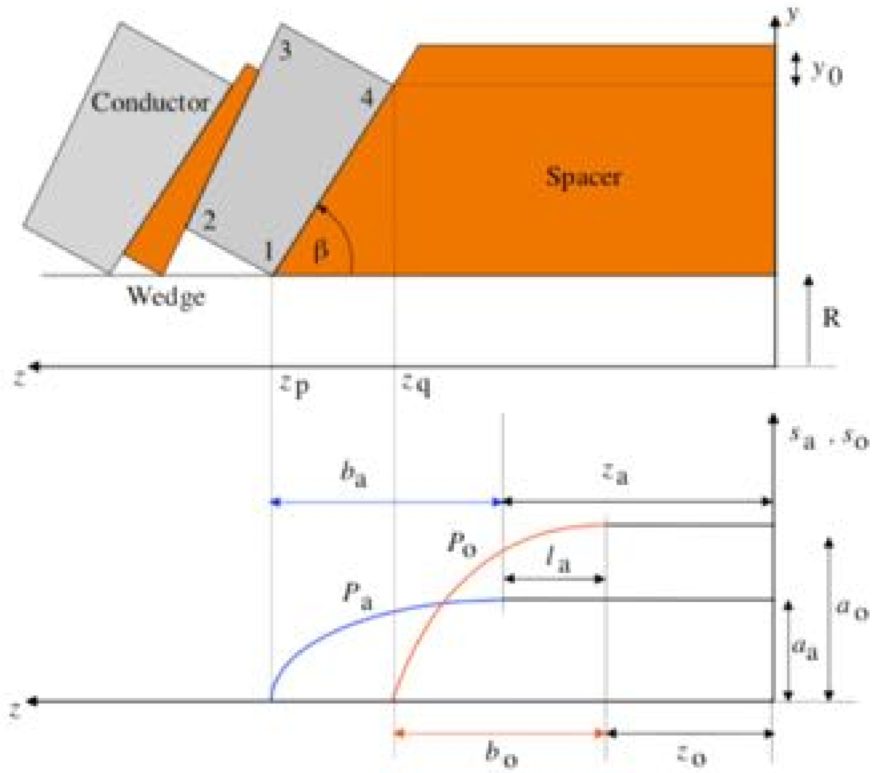



# Coil and End-spacers

# The Winding Scheme



# Constant-Perimeter Ends



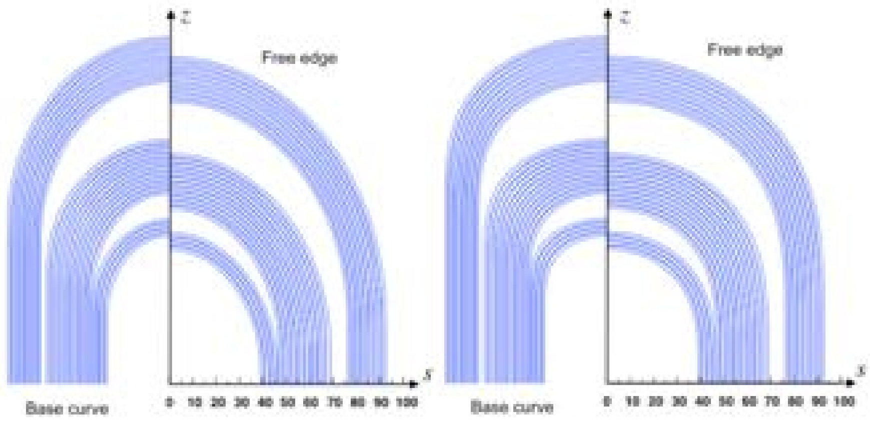

$$P_o \approx \frac{1}{4}\pi(a_o + b_o)\left(1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{64}\right), \qquad \lambda := \frac{a_o - b_o}{a_o + b_o}.$$

$$P_a \approx \frac{1}{4}\pi(a_a + b_a)\left(1 + \frac{\nu^2}{4} + \frac{\nu^4}{64}\right), \qquad \nu := \frac{a_a - b_a}{a_a + b_a}.$$

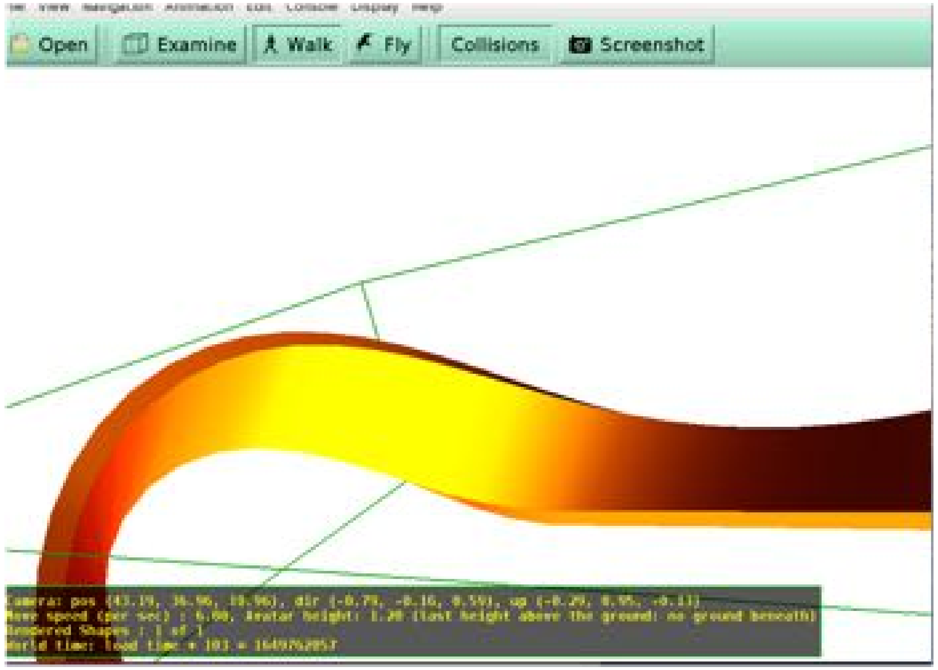
Perimeter of outer (free) edge

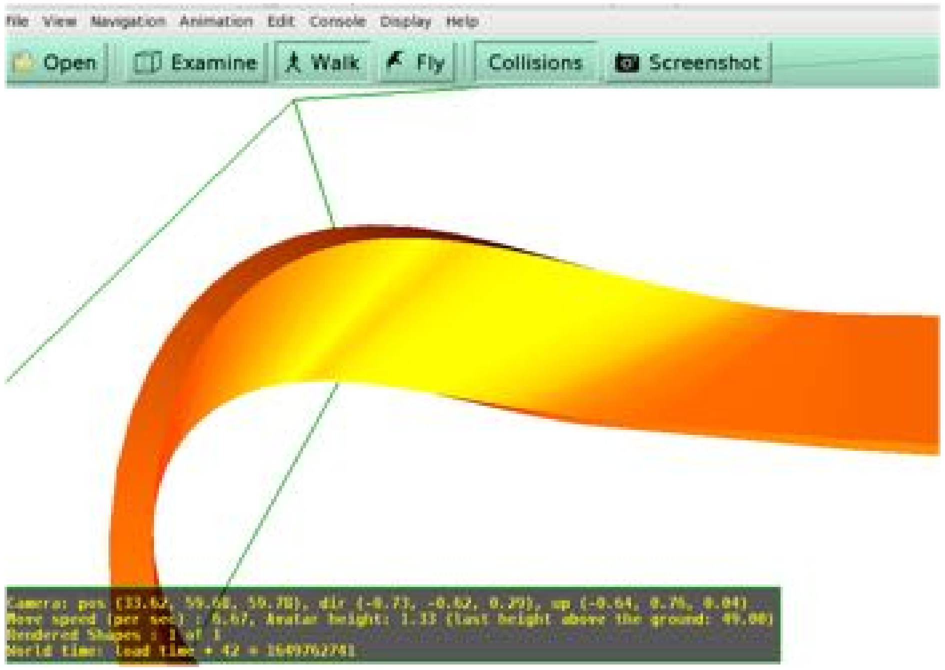
$$P_o = b_o E\left(\frac{\pi}{2}, e\right)$$

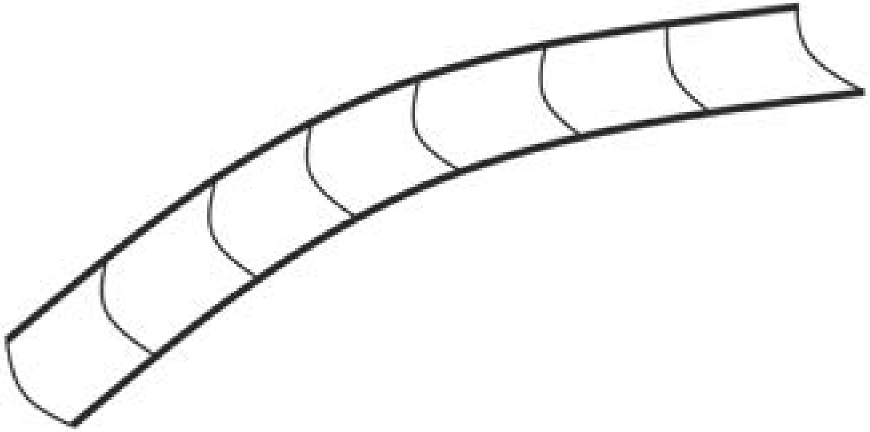
$$e := \sqrt{1 - \frac{a_o^2}{b_o^2}}$$



# Constant Perimeter Ends are not (Necessarily) Developable



# Freney Frame of Space Curves

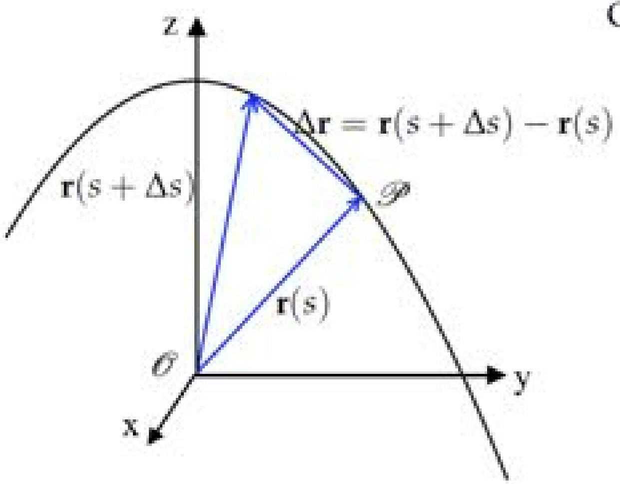
$$\frac{d\mathbf{r}(t)}{dt} = \mathbf{v}(t) = \frac{d}{dt}(r\mathbf{e}_r + z\mathbf{e}_z) = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} + \frac{dz}{dt}\mathbf{e}_z + z\frac{d\mathbf{e}_z}{dt},$$

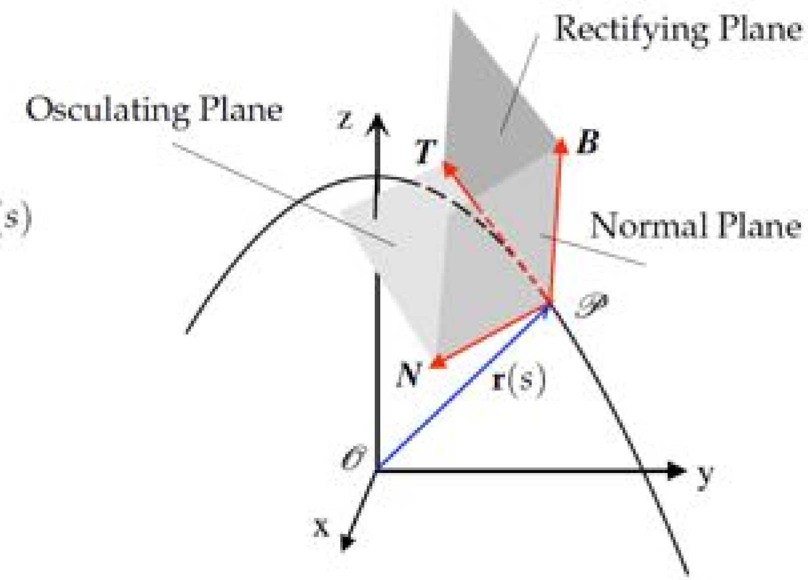
Unit speed

$$\frac{df(s(t))}{dt} = \frac{df}{ds}\frac{ds}{dt} = v\frac{df}{ds}$$

$$\mathbf{T}(t) := \frac{\mathbf{v}(t)}{v(t)} = T_x(t)\mathbf{e}_x + T_y(t)\mathbf{e}_y + T_z(t)\mathbf{e}_z.$$

$$\mathbf{T}(s) = \frac{1}{v}\frac{d\mathbf{r}(t)}{dt} = \frac{d\mathbf{r}(s)}{ds}$$

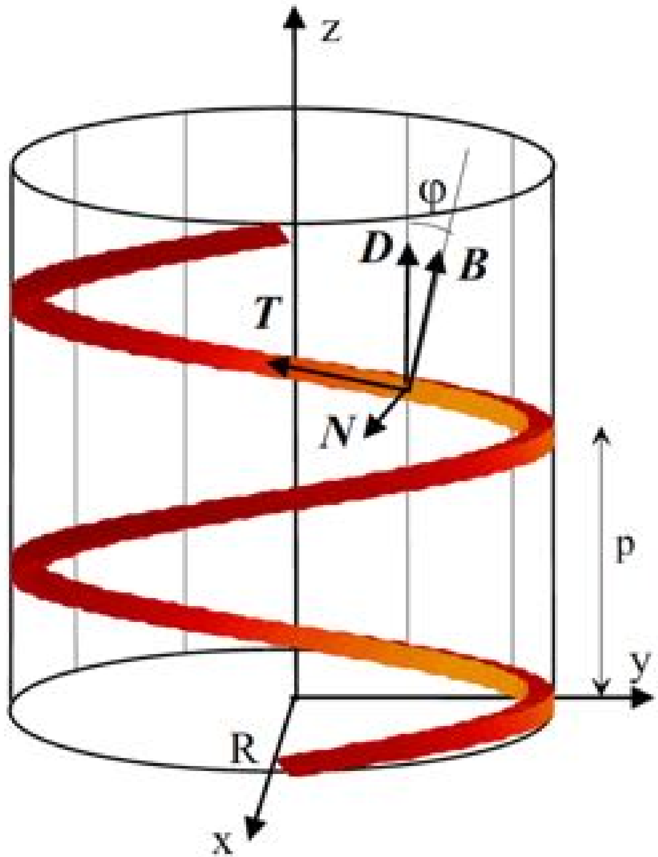



$$\mathbf{T}' = \mathbf{r}'' = |\mathbf{T}'|\, \mathbf{N} =: \kappa\,\mathbf{N},$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

# The Helix and the Darboux Vector




$$\mathbf{D} = \tau\,\mathbf{T} + \kappa\,\mathbf{B} = \frac{1}{\sqrt{R^2+q^2}}\,\mathbf{e}_z\,.$$

Axis of rotation of the Frenet frame

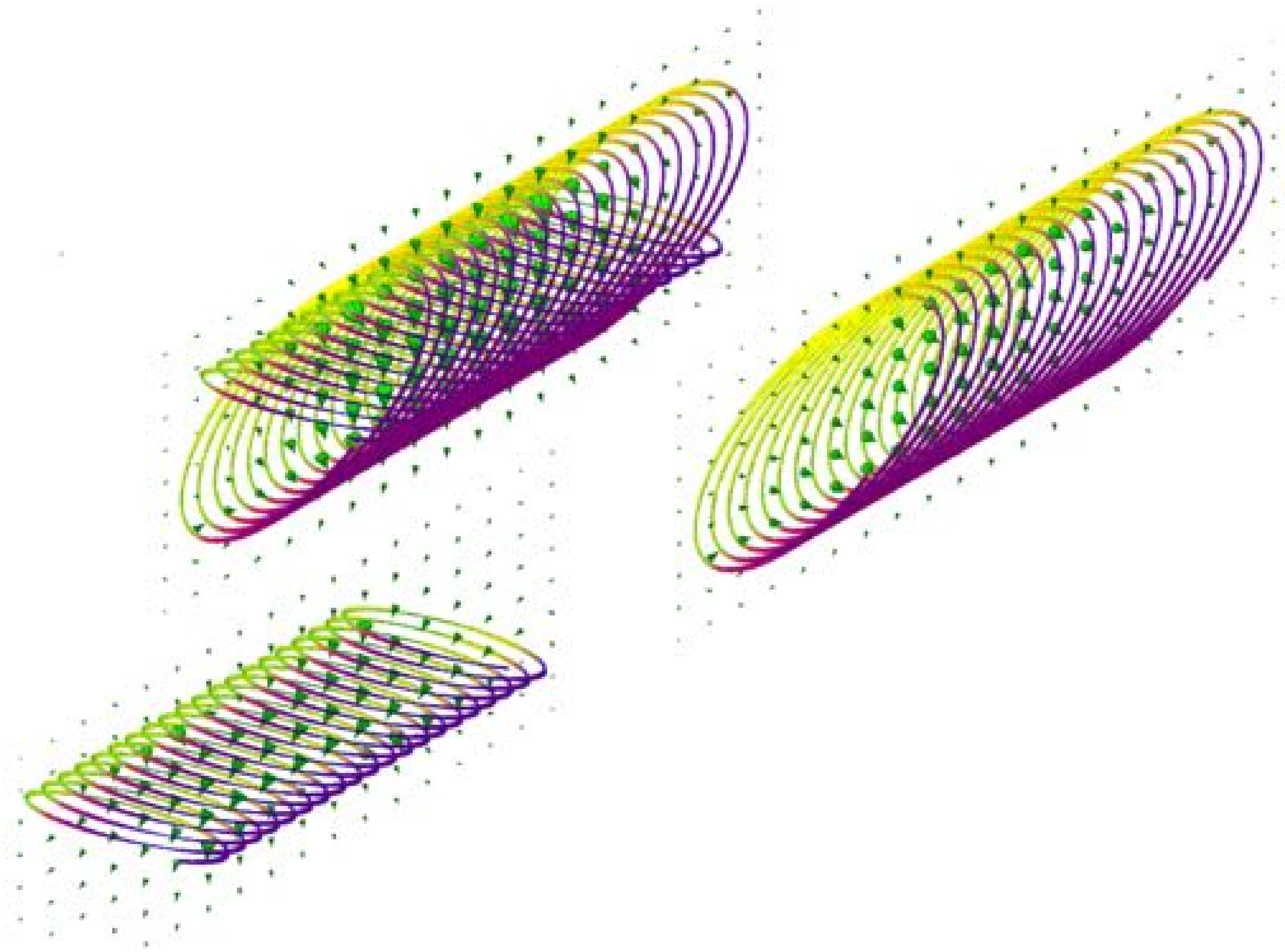
Frenet frame

$$\mathbf{T}' = \kappa\,\mathbf{N}\,, \qquad \mathbf{N}' = \tau\,\mathbf{B} - \kappa\,\mathbf{T}\,, \qquad \mathbf{B}' = -\tau\,\mathbf{N}$$

$$\mathbf{T}' = \mathbf{D}\times\mathbf{T}\,, \qquad \mathbf{N}' = \mathbf{D}\times\mathbf{N}\,, \qquad \mathbf{B}' = \mathbf{D}\times\mathbf{B}$$

# The CCT Magnet Concept

CCT is a better term than titled helix because the quotient of torsion to curvature is not constant



# The Space Curve and the Frenet Frame

$$\mathbf{r}(\varphi) = R\cos(\varphi)\,\mathbf{e}_x + R\sin(\varphi)\,\mathbf{e}_y + s(\varphi)\,\mathbf{e}_z,$$

with

$$s(\varphi) = R\tan(\alpha)\sin(n\varphi) + p\frac{\varphi}{2\pi}.$$

$$\mathbf{r} = R\,\mathbf{e}_r + s(\varphi)\,\mathbf{e}_z = R\,\mathbf{e}_r + \left(R\tan(\alpha)\sin(n\varphi) + \frac{p}{2\pi}\varphi\right)\,\mathbf{e}_z\,,$$

$$\mathbf{v} = R\,\mathbf{e}_\varphi + s'(\varphi)\,\mathbf{e}_z = R\,\mathbf{e}_\varphi + \left(nR\tan(\alpha)\cos(n\varphi) + \frac{p}{2\pi}\right)\,\mathbf{e}_z\,,$$

$$\mathbf{a} = -R\,\mathbf{e}_r + s''(\varphi)\,\mathbf{e}_z = -R\,\mathbf{e}_r - n^2R\tan(\alpha)\sin(n\varphi)\,\mathbf{e}_z\,,$$

$$\mathbf{a}' = -R\,\mathbf{e}_\varphi + s'''(\varphi)\,\mathbf{e}_z = -R\,\mathbf{e}_\varphi - n^3R\tan(\alpha)\cos(n\varphi)\,\mathbf{e}_z\,,$$

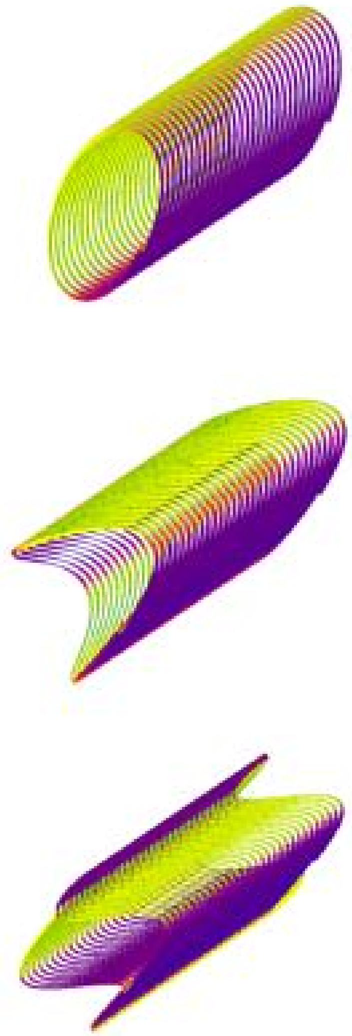
$$\mathbf{v}\times\mathbf{a} = Rs''(\varphi)\,\mathbf{e}_r - Rs'(\varphi)\,\mathbf{e}_\varphi + R^2\,\mathbf{e}_z\,.$$

# The Space Curve and the Frenet Frame

$$\mathbf{r}(\varphi) = R\cos(\varphi)\,\mathbf{e}_x + R\sin(\varphi)\,\mathbf{e}_y + s(\varphi)\,\mathbf{e}_z,$$

with

$$s(\varphi) = R\tan(\alpha)\sin(n\varphi) + p\frac{\varphi}{2\pi}.$$



$$\mathbf{T} = \frac{R\,\mathbf{e}_\varphi + s'(\varphi)\,\mathbf{e}_z}{\sqrt{R^2 + (s'(\varphi))^2}},$$

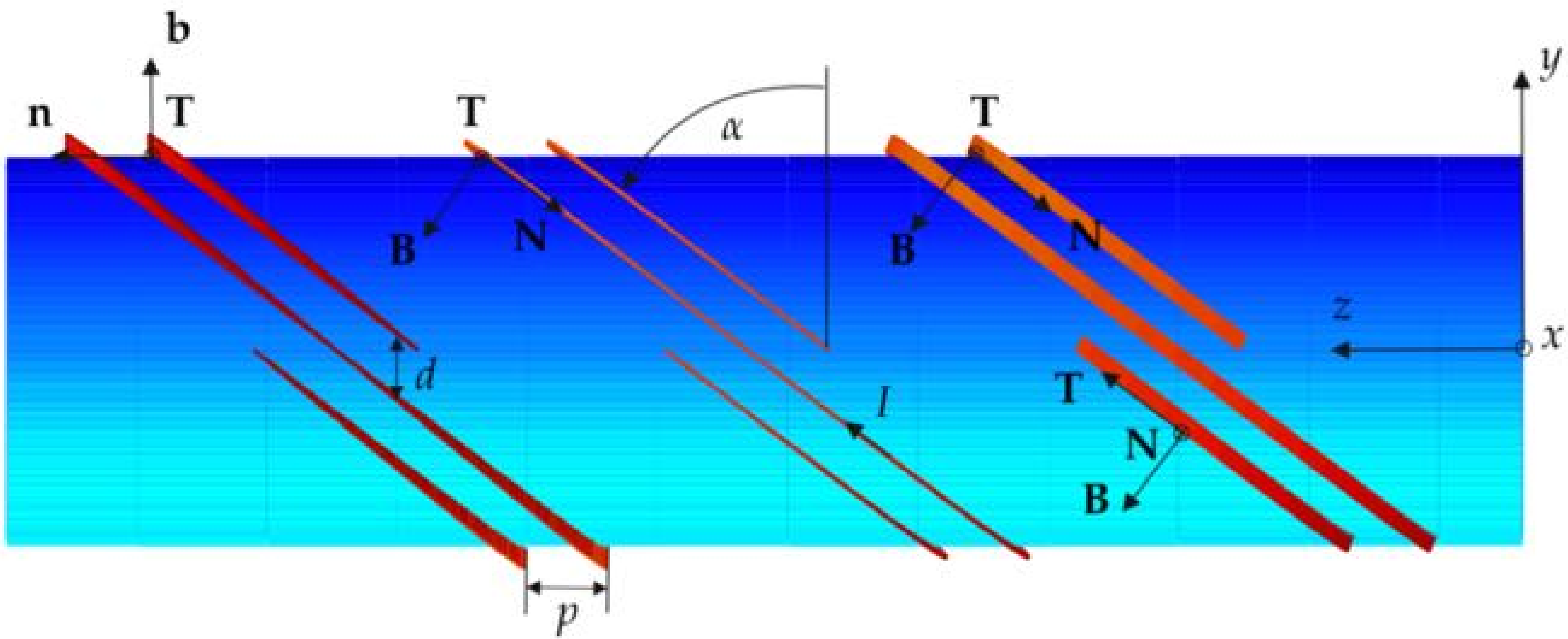
$$\mathbf{B} = \frac{Rs''(\varphi)\,\mathbf{e}_r - Rs'(\varphi)\,\mathbf{e}_\varphi + R^2\,\mathbf{e}_z}{\sqrt{(Rs''(\varphi))^2 + (Rs'(\varphi))^2 + R^4}},$$

$$\mathbf{N} = \frac{-R(s'(\varphi))^2 - R^3\,\mathbf{e}_r - Rs''(\varphi)s'(\varphi)\,\mathbf{e}_\varphi + R^2 s''(\varphi)\,\mathbf{e}_z}{\sqrt{(R^2 + (s'(\varphi))^2)\,(Rs''(\varphi))^2 + (Rs'(\varphi))^2 + R^4)}}.$$

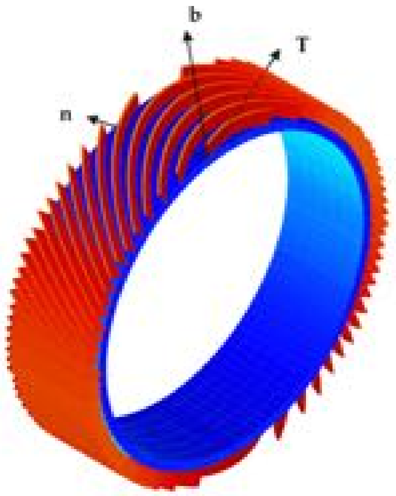
Darboux

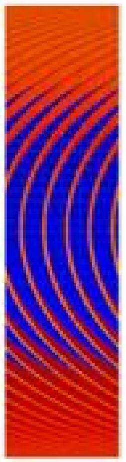
Frenet (hard-way)

Frenet (soft-way)



# The Darboux Frame



Darboux

Frenet

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\vartheta_\mathbf{T}) & -\sin(\vartheta_\mathbf{T}) \\ 0 & \sin(\vartheta_\mathbf{T}) & \cos(\vartheta_\mathbf{T}) \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}$$

$$\tau = \tau + \frac{1}{|\mathbf{v}|} \frac{\mathrm{d}\vartheta_\mathbf{T}}{\mathrm{d}\varphi_\mathrm{c}}, \qquad \kappa_\mathrm{g} = \kappa \cos(\vartheta_\mathbf{T}), \qquad \kappa_\mathrm{n} = \kappa \sin(\vartheta_\mathbf{T})$$

$$\mathbf{T} = \frac{R\,\mathbf{e}_\varphi + s'(\varphi)\,\mathbf{e}_z}{\sqrt{R^2 + (s'(\varphi))^2}},$$

$$\mathbf{b} = \mathbf{e}_r\,,$$

$$\mathbf{n} = -\mathbf{T} \times \mathbf{e}_r = \frac{-s'(\varphi)\,\mathbf{e}_\varphi + R\,\mathbf{e}_z}{\sqrt{R^2 + (s'(\varphi))^2}}\,.$$

The assumption makes the mathematics easy, but also life?

Some hand-waving arguments

$$d = |p\,\mathbf{e}_z \cdot \mathbf{n}| = \frac{Rp}{\sqrt{R^2 + (s'(\varphi_c))^2}} = \frac{Rp}{|\mathbf{v}|}$$

$$B_1 = \frac{\mu_0 I_0}{2 r_c p} \tan(\alpha)$$

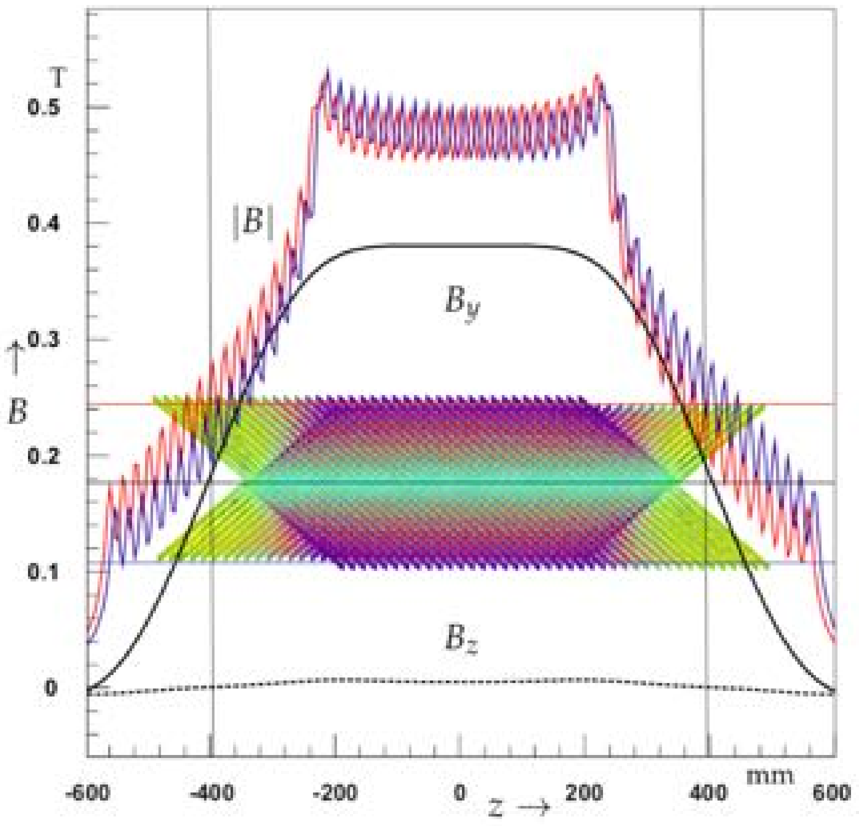
Pitch-averaged surface current

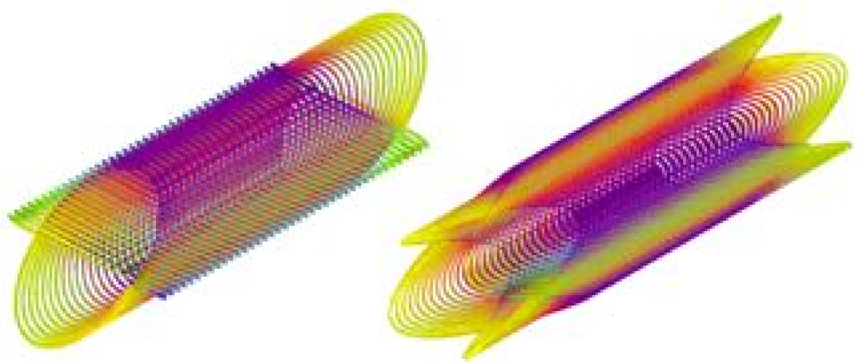
$$B_z(0,0) = \frac{\mu_0 I_0}{p}$$

$$\boldsymbol{\alpha}(\varphi_c) = \frac{I_0\,\mathbf{T}}{d} = \frac{I_0}{Rp}\,\mathbf{v}$$

$$\mathbf{v} = R\,\mathbf{e}_\varphi + s'(\varphi_c)\,\mathbf{e}_z = R\,\mathbf{e}_\varphi + \left(nR\tan(\alpha)\,\cos(n\varphi_c) + \frac{p}{2\pi}\right)\mathbf{e}_z\,,$$

# Local Field Enhancements



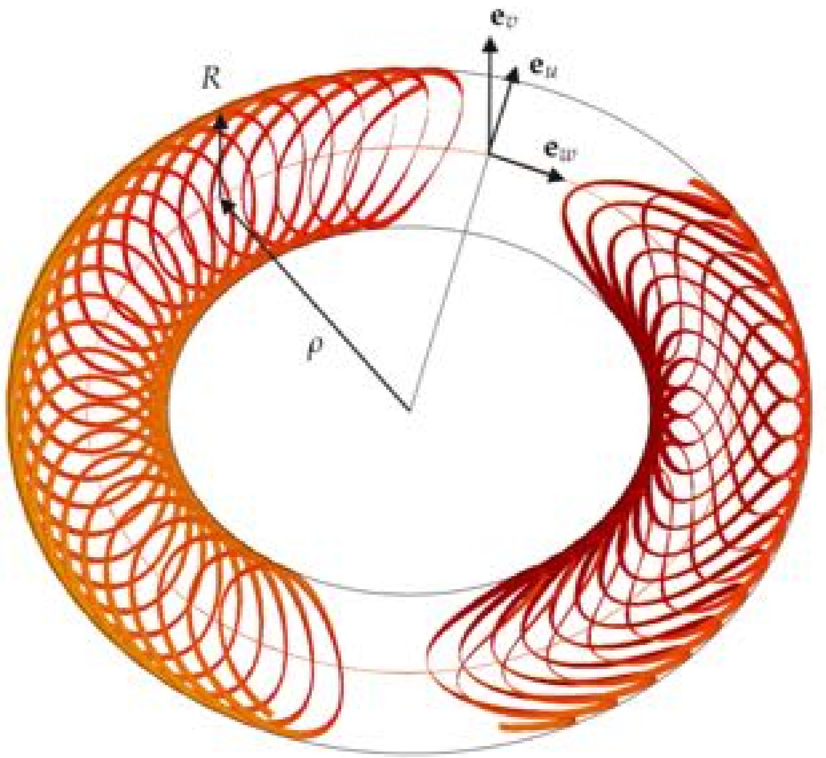



# Curved CCT

$$\mathbf{o}(s) = \rho\cos\left(\frac{s}{\rho}\right)\mathbf{e}_x + \rho\sin\left(\frac{s}{\rho}\right)\mathbf{e}_z$$

$$s(\varphi) = R\tan(\alpha)\sin(n\varphi) + p\frac{\varphi}{2\pi}.$$

$$\begin{aligned}\mathbf{e}_u &= \cos\left(\frac{s}{\rho}\right)\mathbf{e}_x + \sin\left(\frac{s}{\rho}\right)\mathbf{e}_z\,,\\ \mathbf{e}_v &= \mathbf{e}_y\,,\\ \mathbf{e}_w &= -\sin\left(\frac{s}{\rho}\right)\mathbf{e}_x + \cos\left(\frac{s}{\rho}\right)\mathbf{e}_z\,.\end{aligned}$$




$$\begin{aligned}\mathbf{r}(\varphi) &= \mathbf{o}(s(\varphi)) + R\cos(\varphi)\,\mathbf{e}_u(s(\varphi)) + R\sin(\varphi)\,\mathbf{e}_v(s(\varphi))\\ &= \cos\left(\frac{s(\varphi)}{\rho}\right)(\rho + R\cos(\varphi))\,\mathbf{e}_x + R\sin(\varphi)\,\mathbf{e}_y\\ &\quad + \sin\left(\frac{s(\varphi)}{\rho}\right)(\rho + R\cos(\varphi))\,\mathbf{e}_z\,.\end{aligned}$$

# The Darboux Frame of Curved CCTs

$$\mathbf{T} = \frac{v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z}{\sqrt{v_x^2 + v_y^2 + v_z^2}}$$

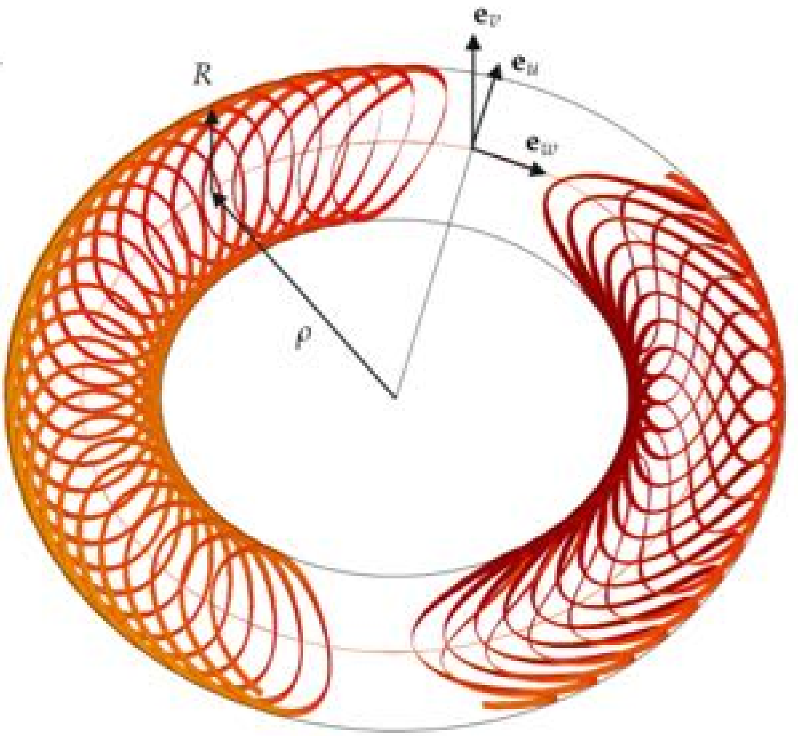
$$v_x(\varphi) = \frac{1}{\rho}\left(-R\rho \sin(\varphi)\cos\left(\frac{s(\varphi)}{\rho}\right) - (R\cos(\varphi) + \rho)\sin\left(\frac{s(\varphi)}{\rho}\right)s'(\varphi)\right),$$

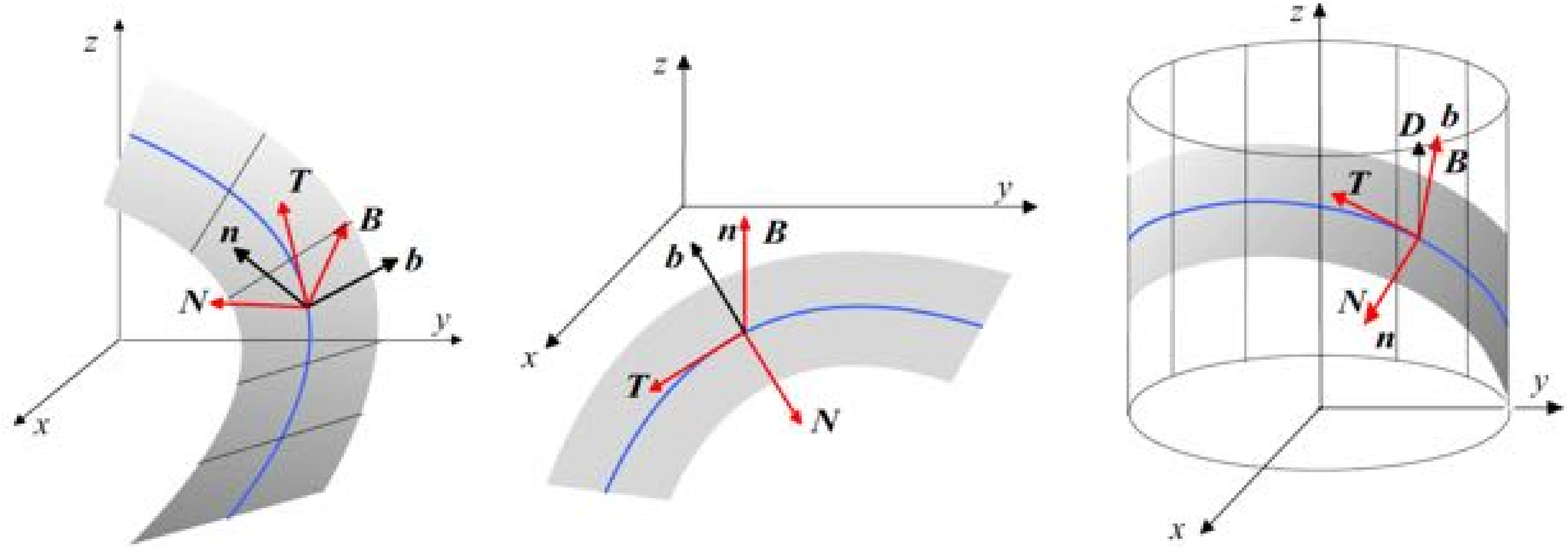
$$v_y(\varphi) = R\cos(\varphi),$$

$$v_z(\varphi) = \frac{1}{\rho}\left(-R\rho \sin(\varphi)\sin\left(\frac{s(\varphi)}{\rho}\right) + (R\cos(\varphi) + \rho)\cos\left(\frac{s(\varphi)}{\rho}\right)s'(\varphi)\right).$$

$$\begin{aligned}\mathbf{b}(\varphi) &= \cos(\varphi)\mathbf{e}_u(s(\varphi)) + \sin(\varphi)\mathbf{e}_v(s(\varphi)) \\ &= \cos(\varphi)\cos\left(\frac{s(\varphi)}{\rho}\right)\mathbf{e}_x + \sin(\varphi)\mathbf{e}_y + \cos(\varphi)\sin\left(\frac{s(\varphi)}{\rho}\right)\mathbf{e}_z,\end{aligned}$$

$$\mathbf{n} = -\mathbf{T} \times \mathbf{b}.$$






$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{n}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_n & -\kappa_g \\ -\kappa_n & 0 & \tau \\ \kappa_g & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}$$

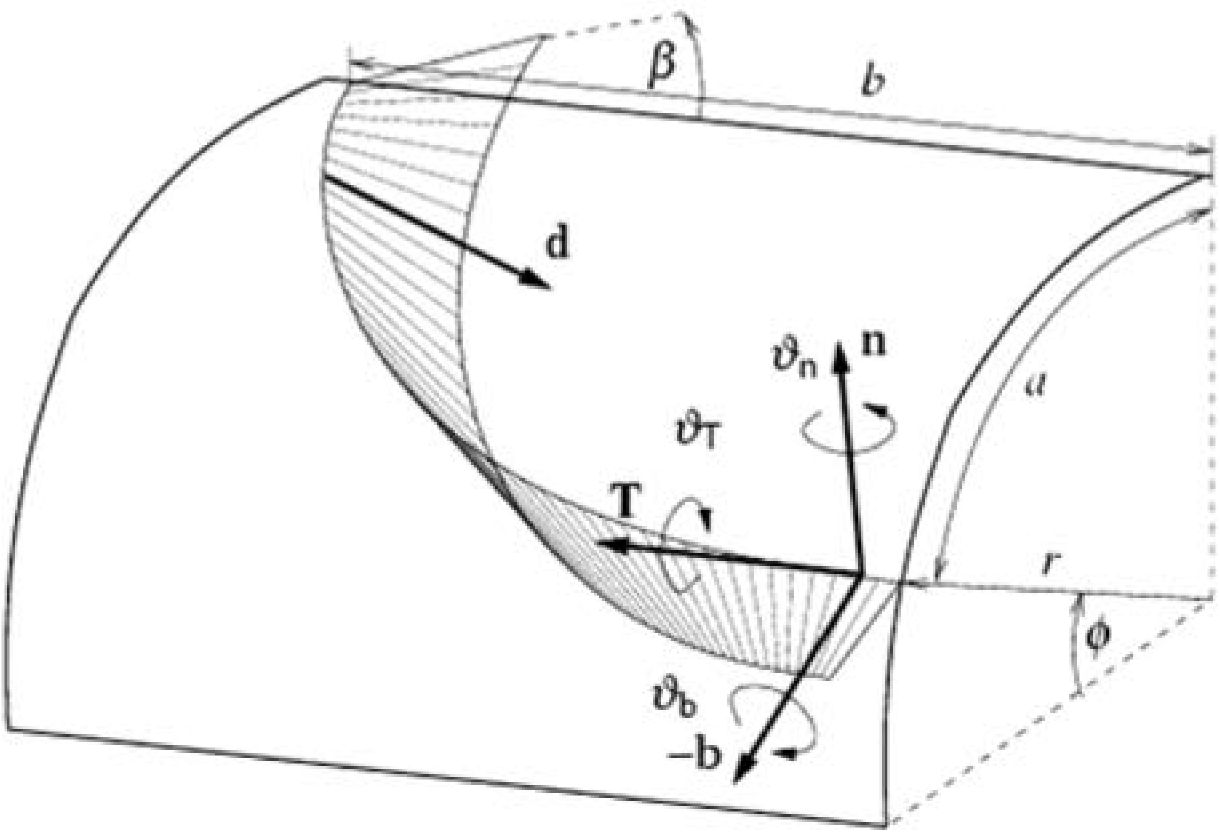
# Limits of the Guiding Strip Theory

- ➔ Rutherford cable is not a strip of zero thickness
- ➔ Transverse cross-section is keystoned
- ➔ The cables must be wound onto each other in groups of up to 30 cables
- ➔ Inclination in the yz cross-section deviates from the ideal angle
- ➔ At the onset, cables lie along a straight line with zero curvature
- ➔ Edge of regression may lay inside the strip surface

- ➔ Minimization of the local hard way deformation through optimal distribution of the twist and optimal order of the hyper-ellipse and the ellipticity ratio

# Twist Angles

$$\tau = \vec{b} \cdot \vec{n}' = \vartheta_{\vec{T}}' \qquad \kappa_g = \vec{T} \cdot \vec{b}' = \vartheta_{\vec{n}}' \qquad \kappa_n = \vec{n} \cdot \vec{T}' = \vartheta_{\vec{b}}'$$



# Design Variables

$$a_{\mathrm{a}} = R(\tfrac{\pi}{2} - \phi)$$

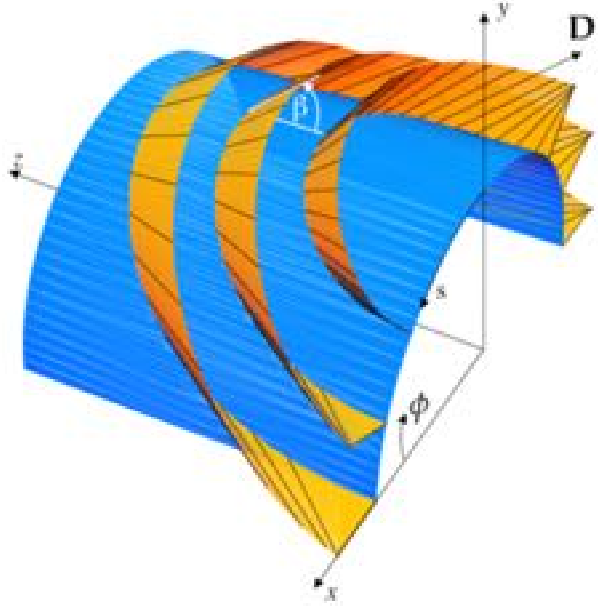
$$\mathbf{r}(t) = R \sin\left(\frac{a_{\mathrm{a}}}{R} \cos^{\frac{2}{n}} t\right) \mathbf{e}_x + R \cos\left(\frac{a_{\mathrm{a}}}{R} \cos^{\frac{2}{n}} t\right) \mathbf{e}_y + b_{\mathrm{a}} \sin^{\frac{2}{n}} t \, \mathbf{e}_z$$

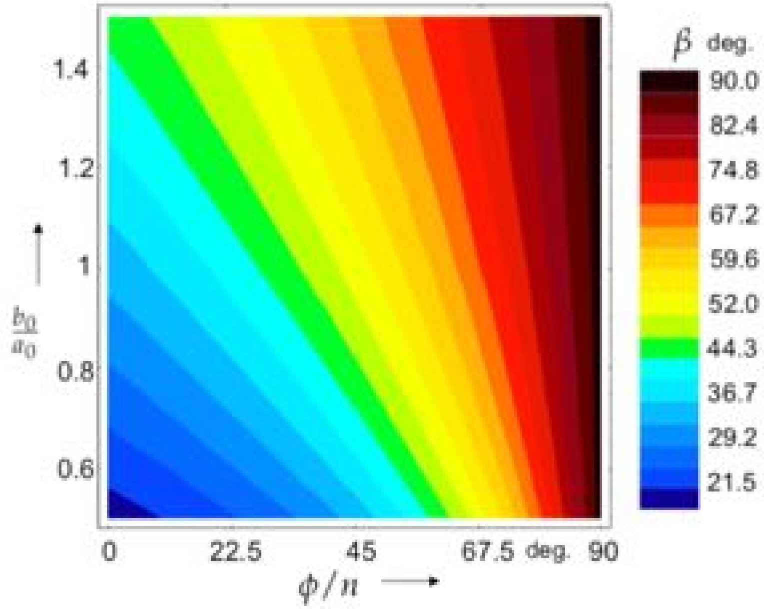
$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}, \qquad \mathbf{B}(t) = \frac{\mathbf{v}(t) \times \mathbf{a}(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|}, \qquad \mathbf{N}(t) = \frac{(\mathbf{v}(t) \times \mathbf{a}(t)) \times \mathbf{v}(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)| \; |\mathbf{v}(t)|},$$

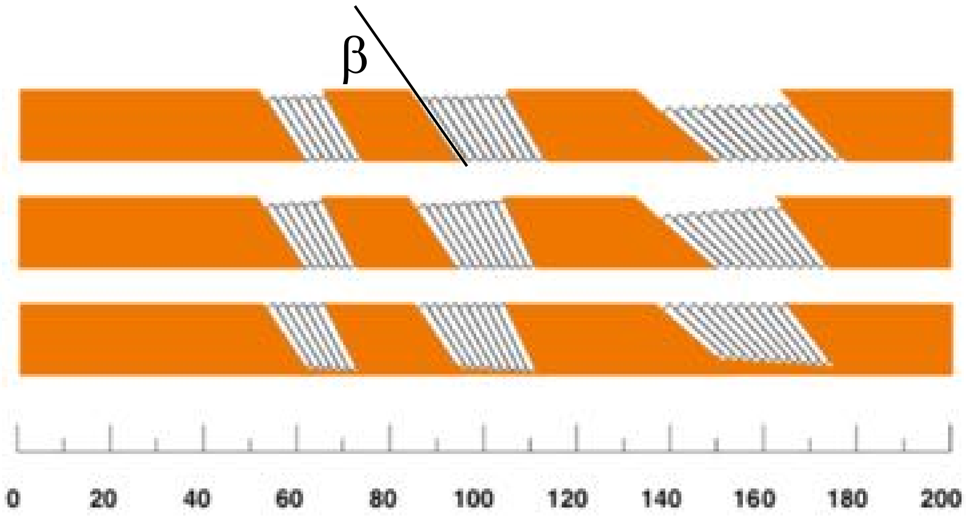
$$\kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3}, \qquad \tau(t) = \frac{(\mathbf{v}(t) \times \mathbf{a}(t)) \cdot \dot{\mathbf{a}}(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|^2}.$$

- The ellipticity of the base curve $\lambda_{\mathrm{e}}$.
- The order $n$ of the base curve.
- The inclination angle $\beta$ between the innermost turn of each coil block.
- Four knots of a cubic spline function allowing for the local adjustment of the cable torsion between the onset of the coil end and the nose, ac-

# The “Natural” Angle

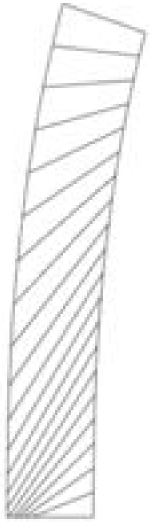


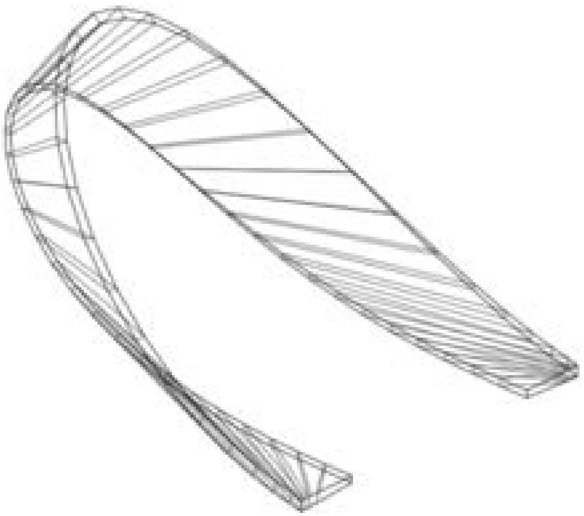



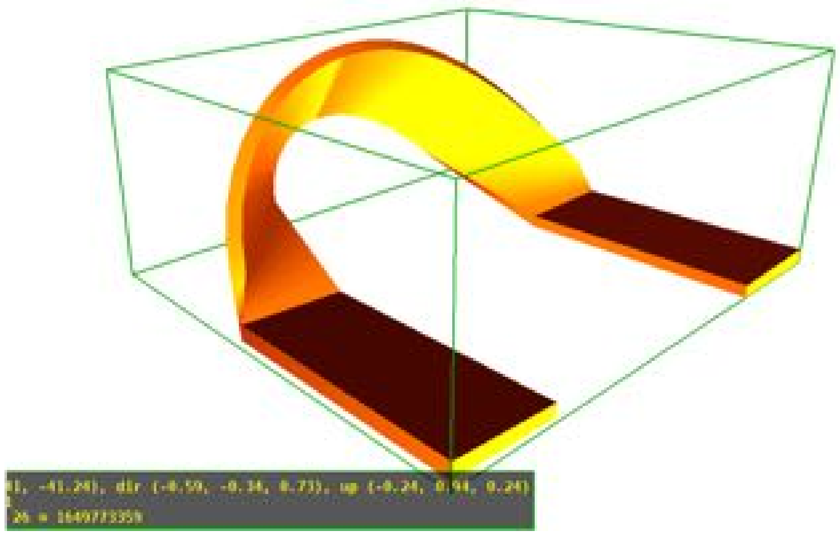



# Objectives

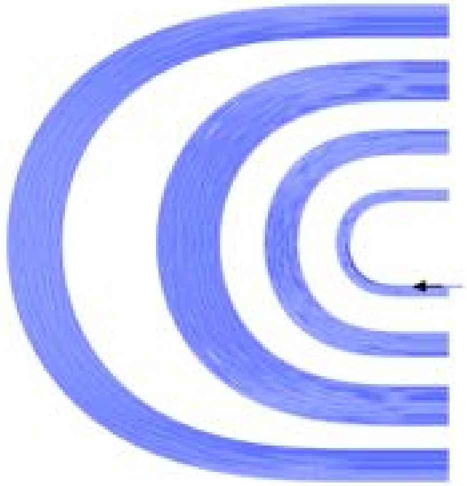
- Integrated squared geodesic curvature of each coil block.
- Maximum curvature parameters in each coil block.
- A parameter indicating an edge of regression violation within the strip surface.

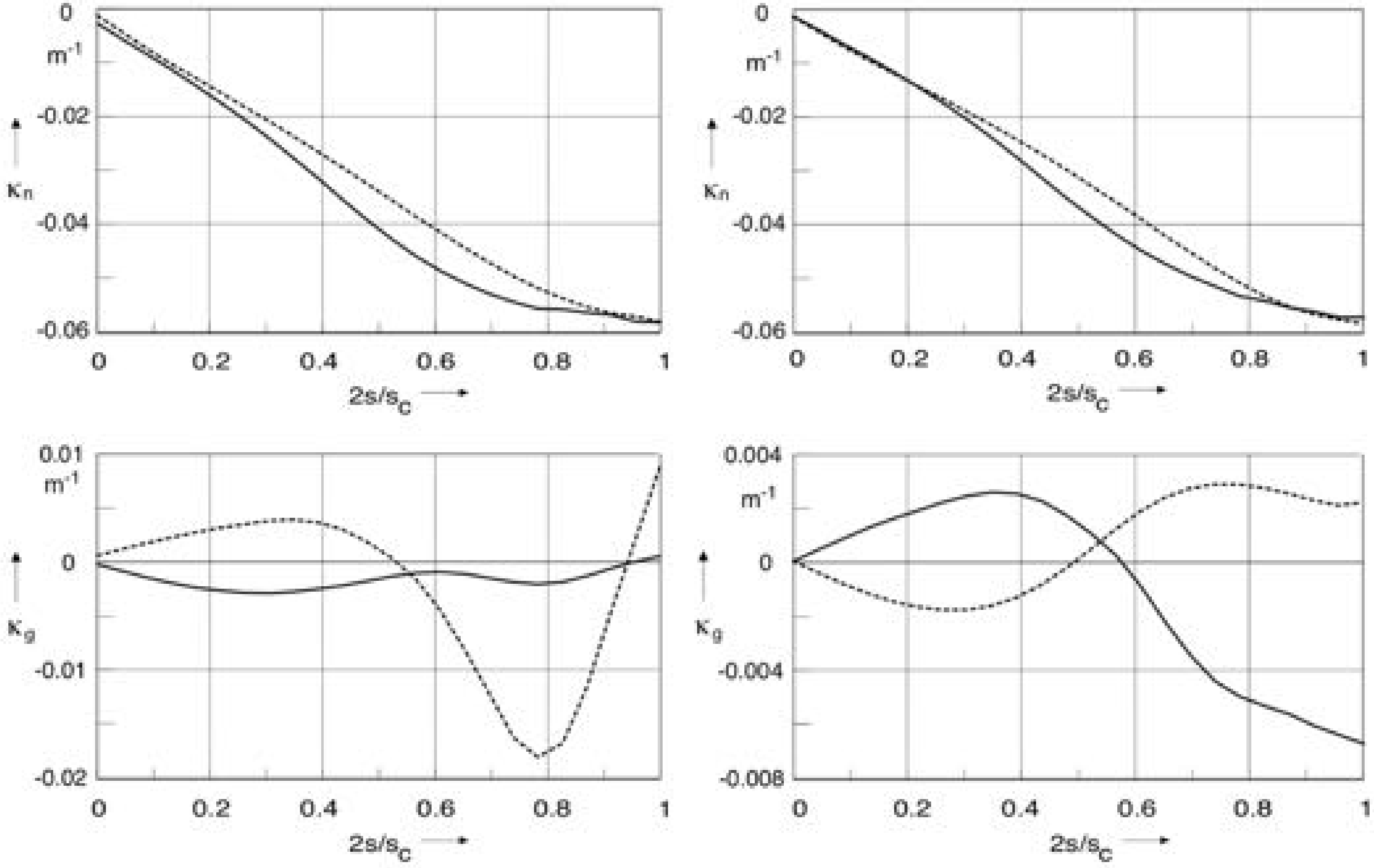




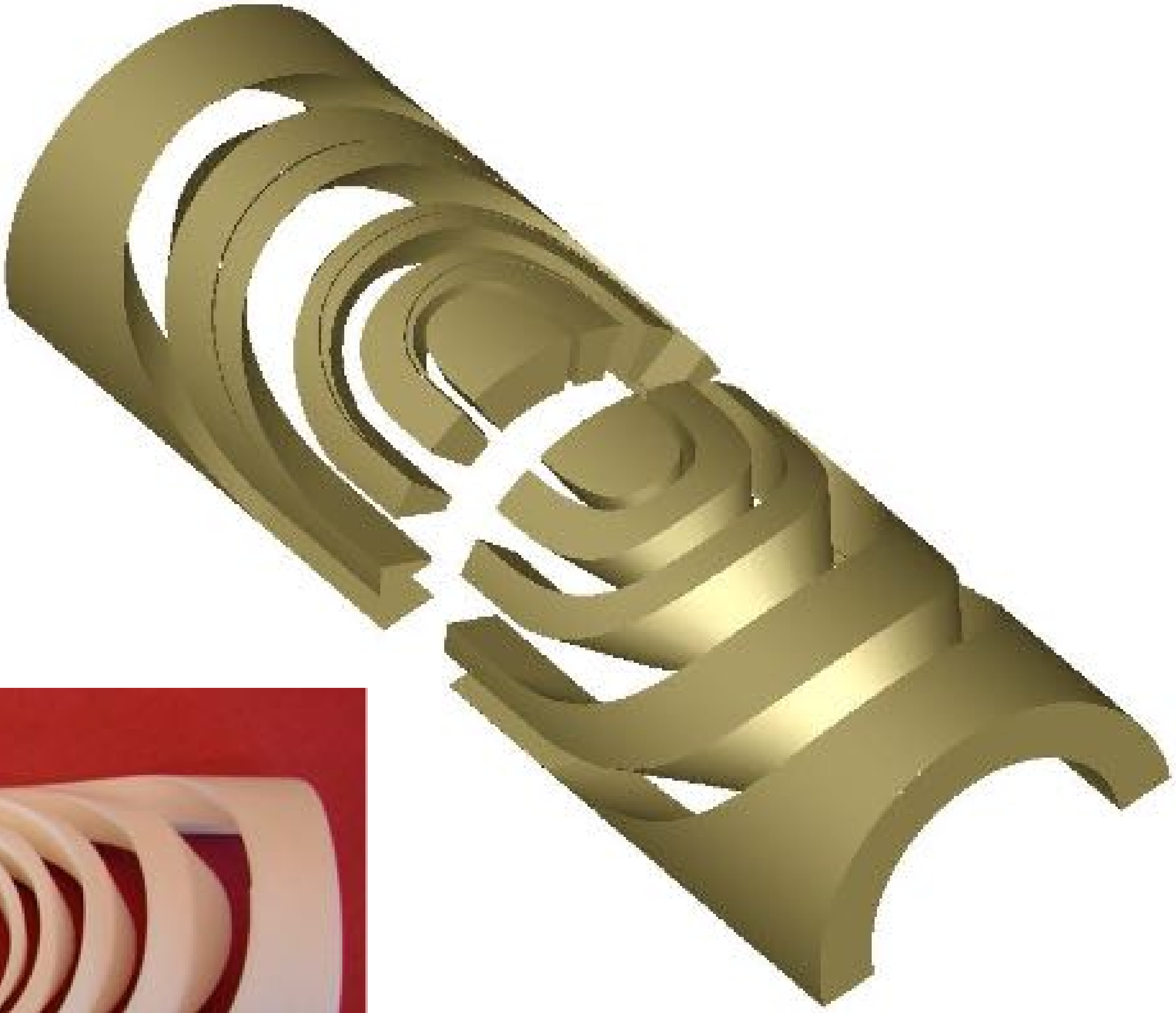


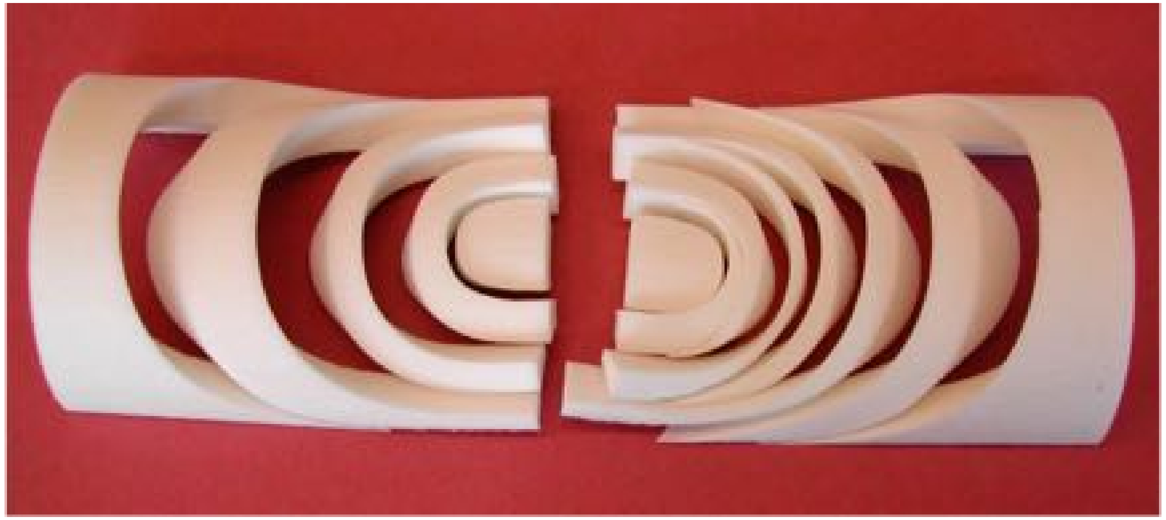
# Normal and Geodesic Curvature





# End-spacer Prototyping





# End-Spacer Prototyping

Sintering (Titanium)

Machining (Epoxy-Glass, G11)

# Endspacer Machining

# Winding Tests