

Emergence of Pion and Proton parton distributions



J. Rodríguez-Quintero



In collaboration with: D. Binosi, Z-F. Cui, M. Ding, J.M. Morgado, J. Papavassiliou, K. Raya, C.D. Roberts, S. Schmidt.

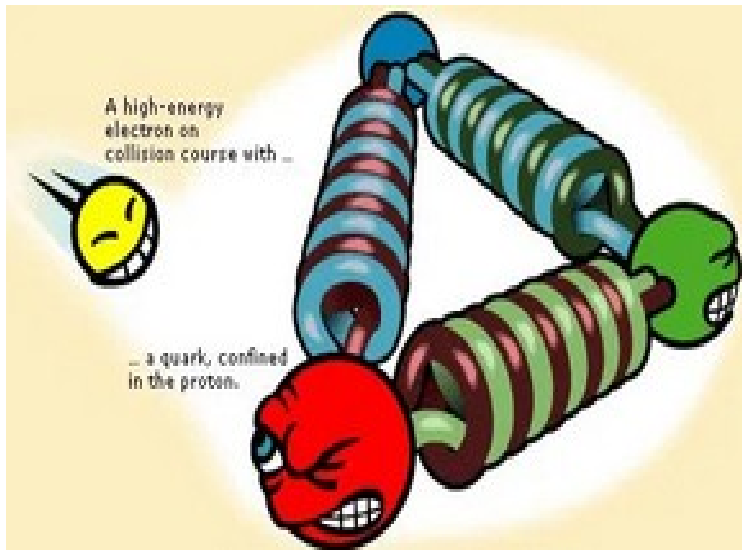


QCD: Basic Facts

- **QCD** is characterized by two **emergent** phenomena:
confinement and dynamical generation of mass (**DGM**).



- ◆ Quarks and gluons not *isolated* in nature.
- ➔ Formation of colorless bound states: “**Hadrons**”
- ➔ **1-fm scale** size of hadrons?



- ◆ Emergence of hadron masses (**EHM**) from QCD **dynamics**

$$\mathcal{L}_{\text{QCD}} = \sum_{j=u,d,s,\dots} \bar{q}_j [\gamma_\mu D_\mu + m_j] q_j + \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a$$

$$D_\mu = \partial_\mu + ig \frac{1}{2} \lambda^a A_\mu^a,$$

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf^{abc} A_\mu^b A_\nu^c,$$

Higgs mechanism

Quarks
Mass = 1.78×10^{-26} g

~ 1% of proton mass

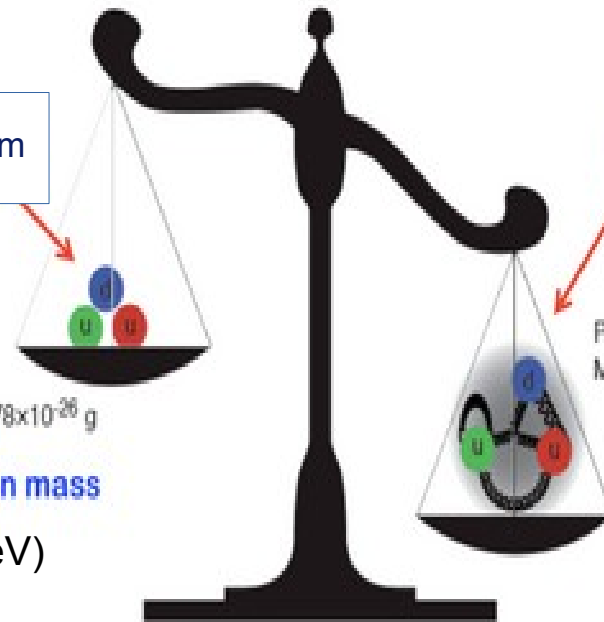
(~ 10 MeV)

QCD dynamics

Proton
Mass = 168×10^{-26} g

~ 99% of proton mass

(~ 928 MeV)



QCD: Basic Facts

➤ **QCD** is characterized by two **emergent** phenomena: **confinement** and dynamical generation of mass (**DGM**).

Can we trace them down to fundamental d.o.f ?

$$\mathcal{L}_{\text{QCD}} = \sum_{j=u,d,s,\dots} \bar{q}_j [\gamma_\mu D_\mu + m_j] q_j + \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a$$

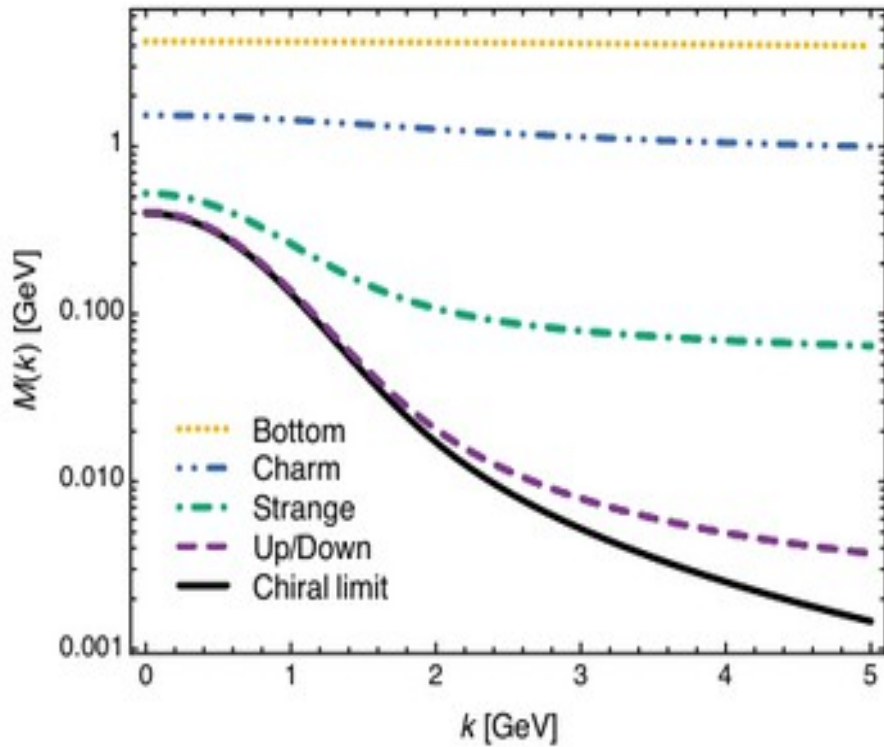
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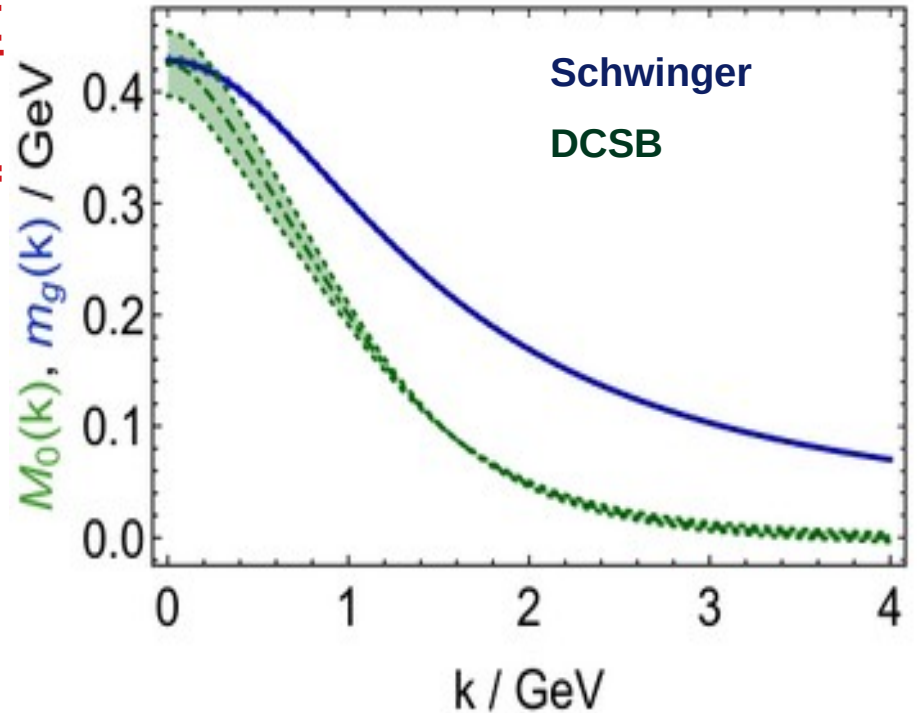
◆ Emergence of hadron masses (**EHM**) from QCD **dynamics**

Dynamical masses
(Dynamical Chiral Symmetry Breaking)



$$S_f^{-1}(p) = Z_f^{-1}(p^2)(i\gamma \cdot p + \mathbf{M}_f(p^2))$$

"Higgs" masses

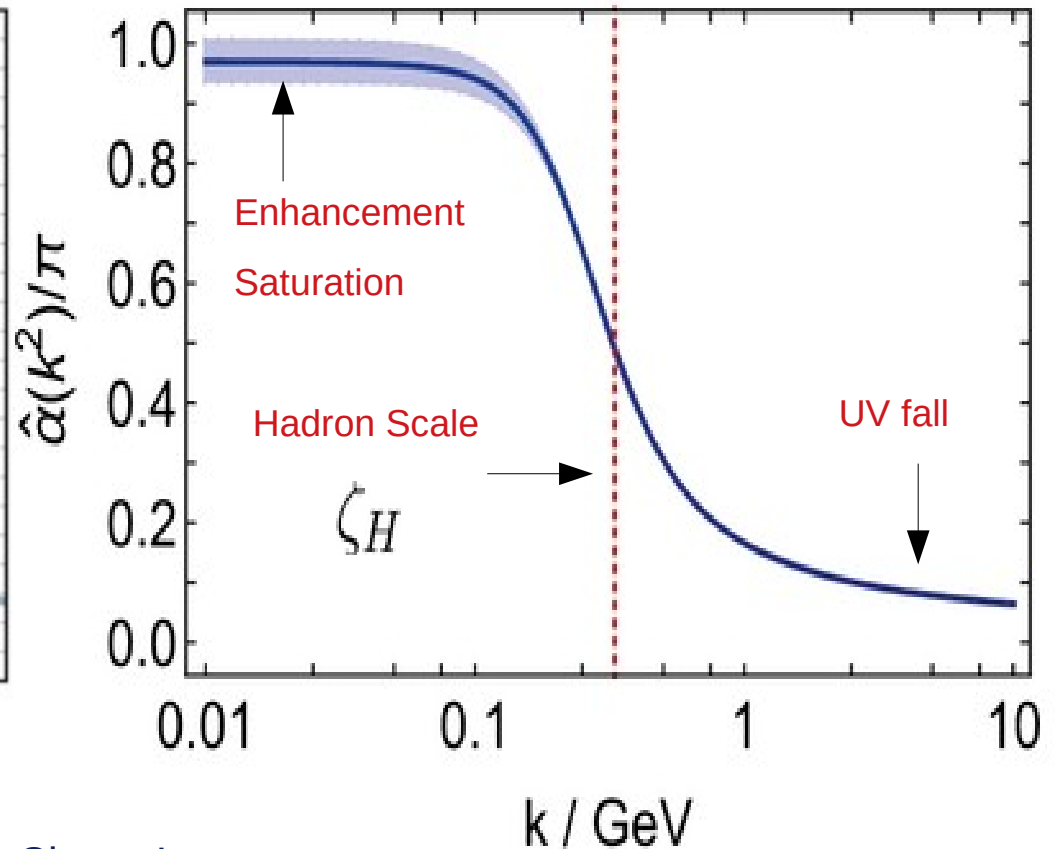
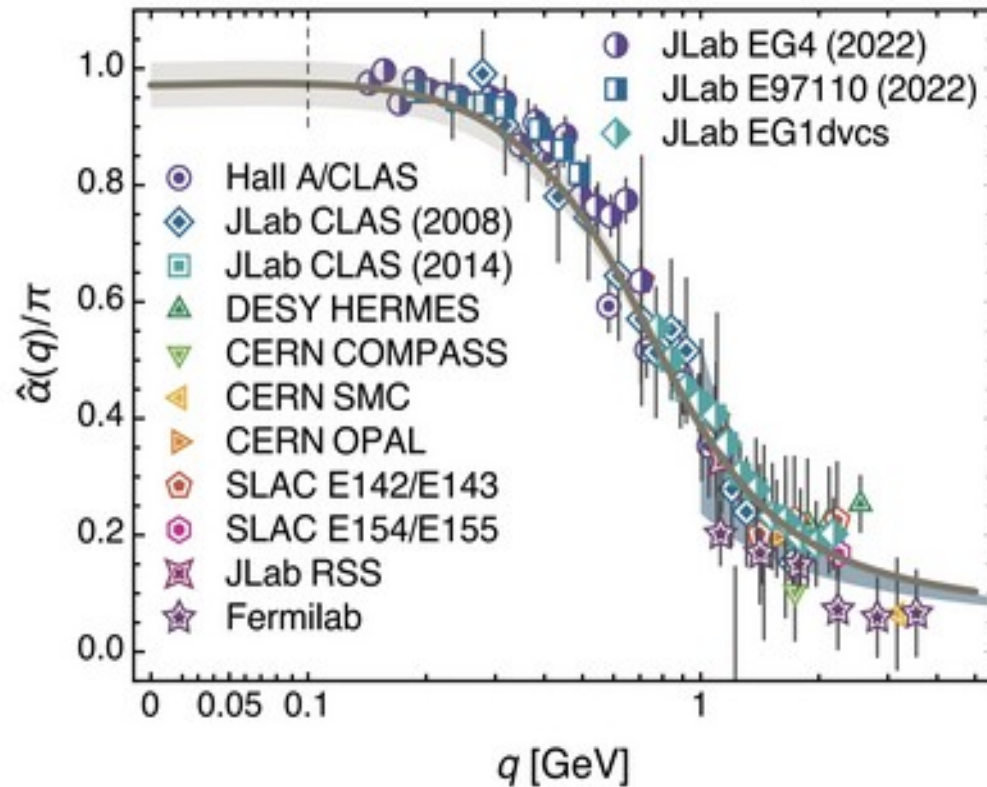


Gluon and quark *running masses*

QCD: Basic Facts

- **Confinement** and the **EHM** are tightly connected with **QCD's running coupling**.

(figure: D. Binosi's courtesy!)



Modern picture of **QCD** coupling. 'Effective Charge'

Combined continuum + QCD lattice analysis

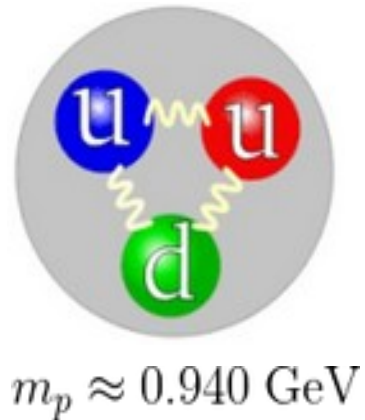
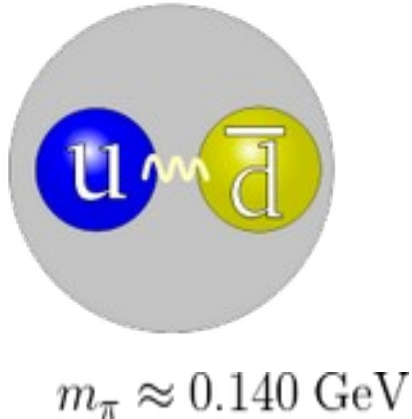
ζ_H : Fully dressed **valence** quarks express all hadron's properties

Why bother about pions?

- **Pions** and **kaons** emerge as (pseudo)-**Goldstone** bosons of **DCSB**.

(besides being 'simple' bound states)

- Their study is **crucial** to understand the **EHM** and the **hadron structure**:



- Dominated by **QCD** dynamics
- Simultaneously explains the mass of the **proton** and the **masslessness** of the **pion**

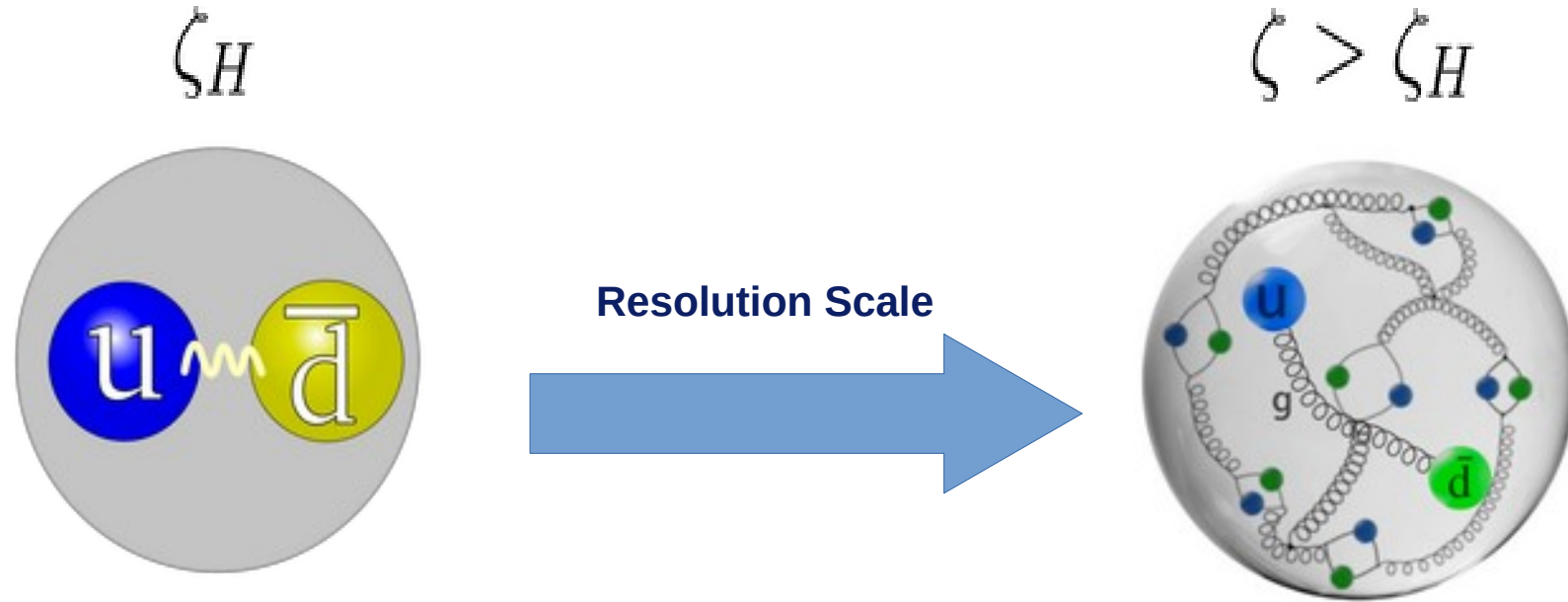
'Higgs' masses

$m_{u/d} \approx 0.004 \text{ GeV}$
 $m_s \approx 0.095 \text{ GeV}$



- Interplay between **Higgs** and **strong** mass generating mechanisms.

Parton distributions: **energy scales**

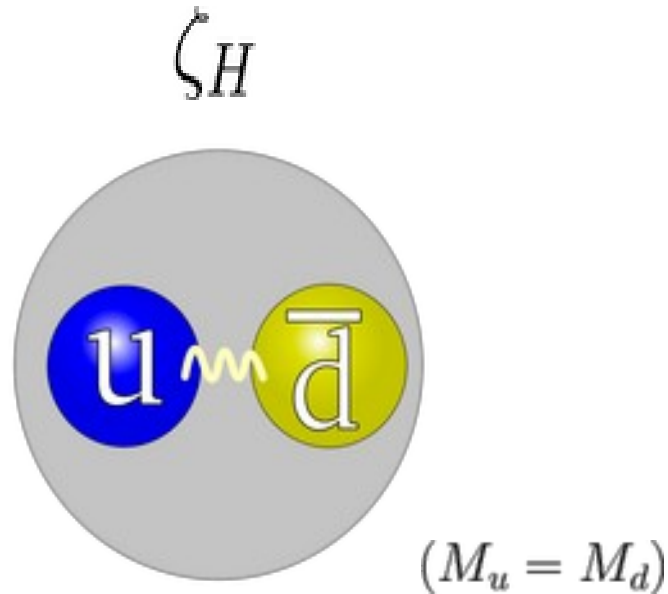


- Fully-dressed **valence quarks**

(quasiparticles)

- Unveiling of **glue and sea d.o.f.**

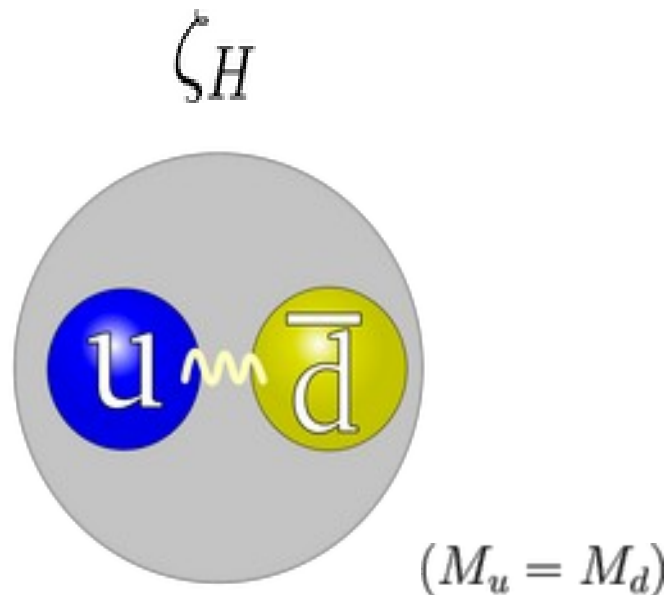
(partons)



- **Fully-dressed valence quarks**

- At this scale, **all properties** of the hadron are contained within their valence quarks.
- **QCD constraints** are defined from here (e.g. large- x behavior of the PDF)

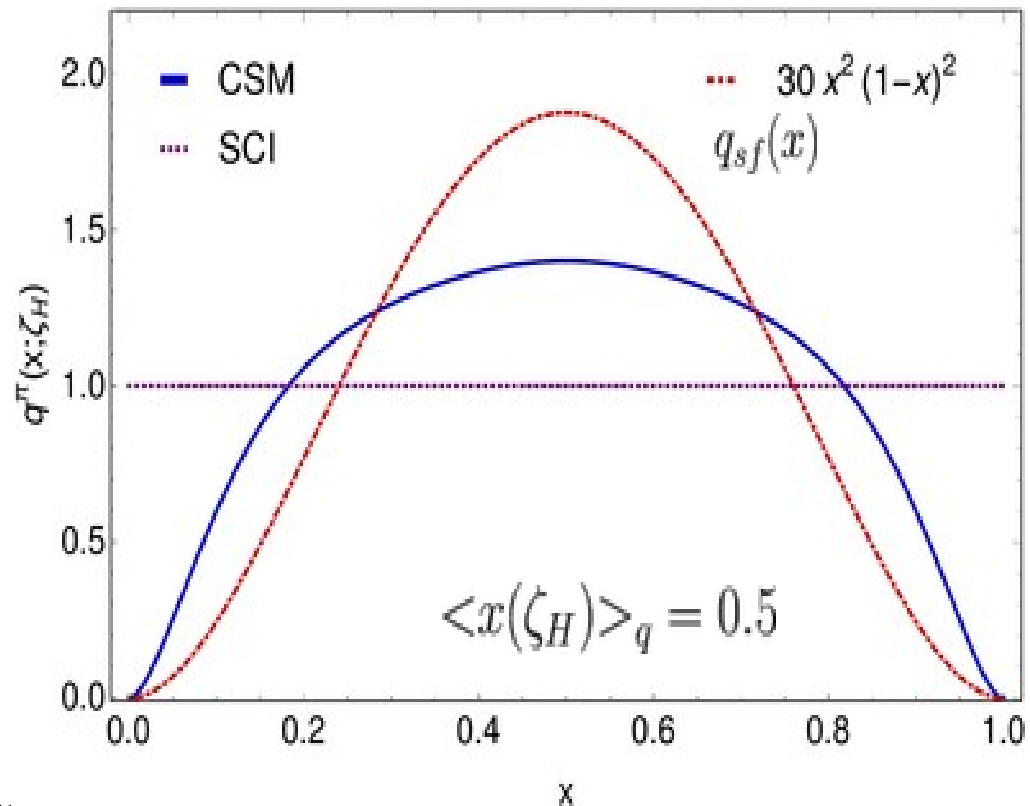
$$u^\pi(x; \zeta) \stackrel{x \approx 1}{\sim} (1-x)^{\beta = 2 + \gamma(\zeta)}$$



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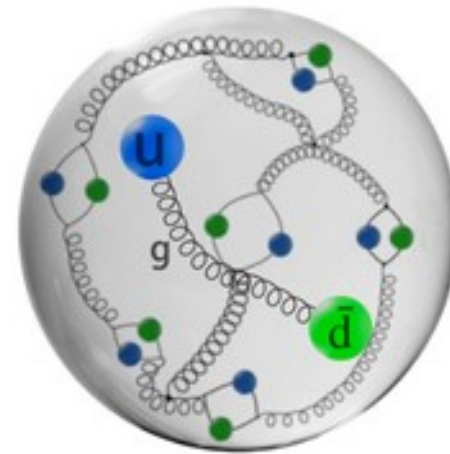
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- **CSM** results produce:

- **EHM-induced** dilated distributions
- Soft end-point behavior

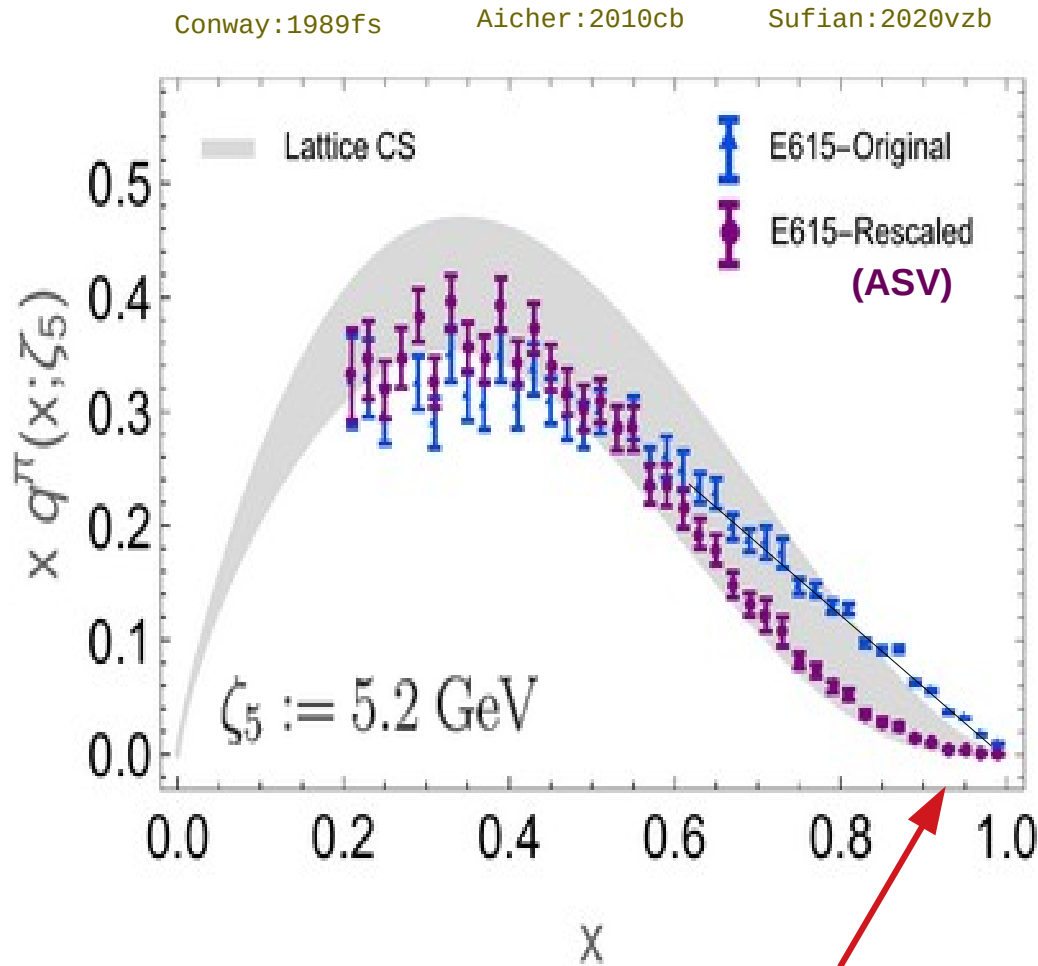
$$\zeta > \zeta_H$$



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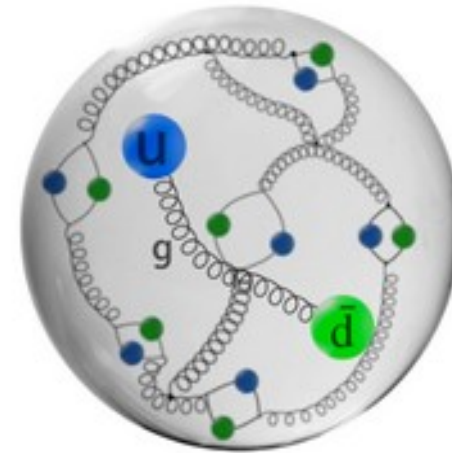
- **Experimental** data is given **here**.
- The interpretation of parton distributions from cross sections demands **special care**.
- In addition, the synergy with **lattice QCD** and phenomenological approaches is welcome.

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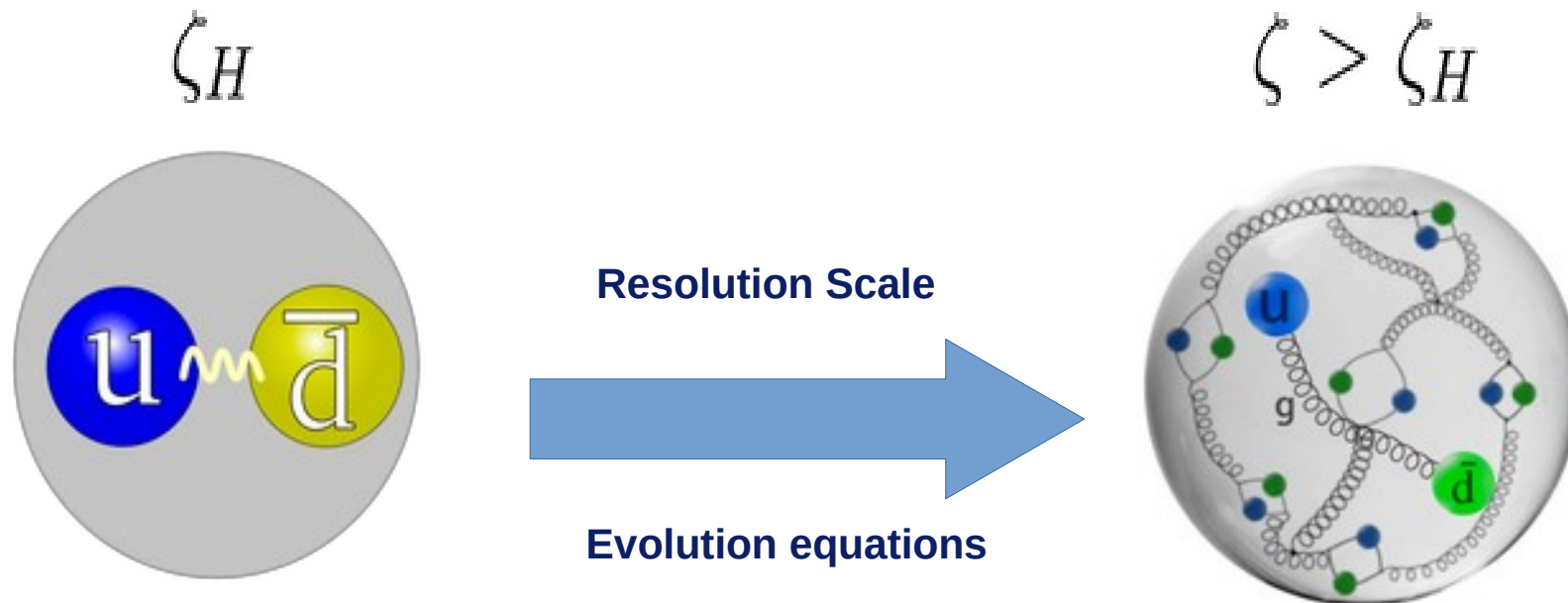
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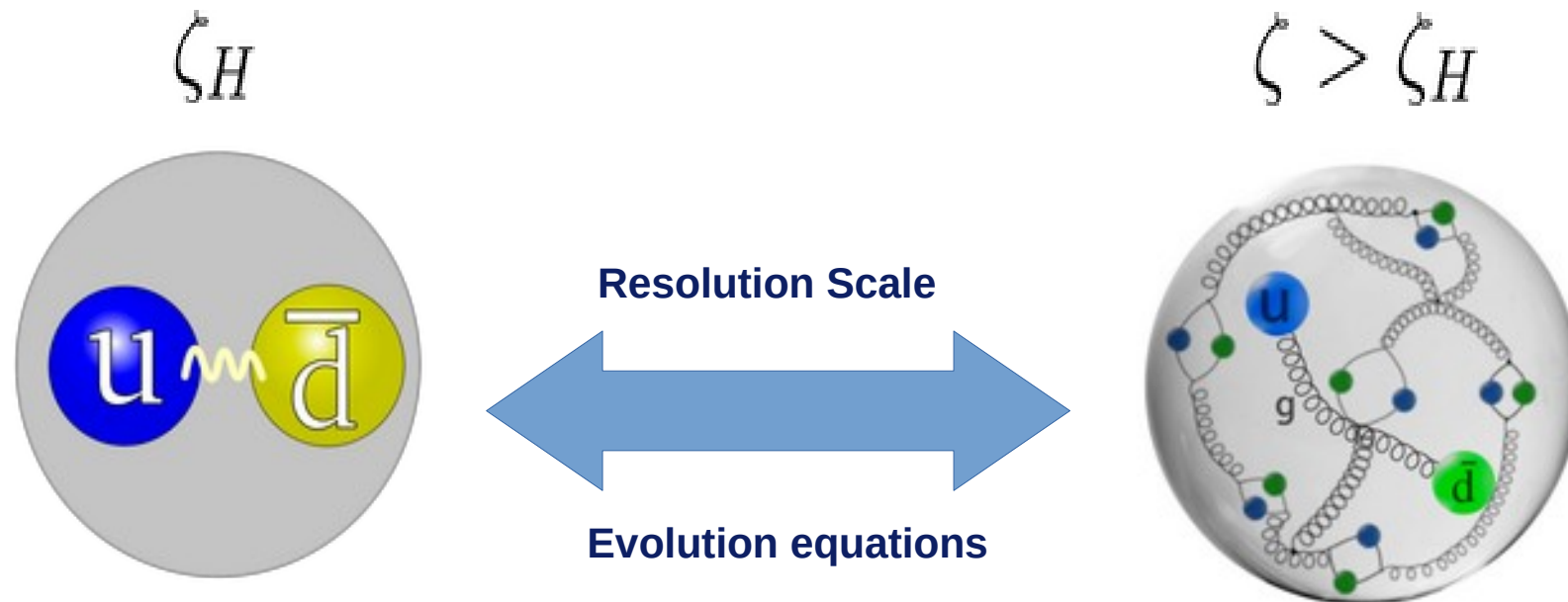
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Have a nice end of the world.

EVOLUTION

SUMMER

WINTER 1 2011

www.countingdown.com

Countdown
to 2012

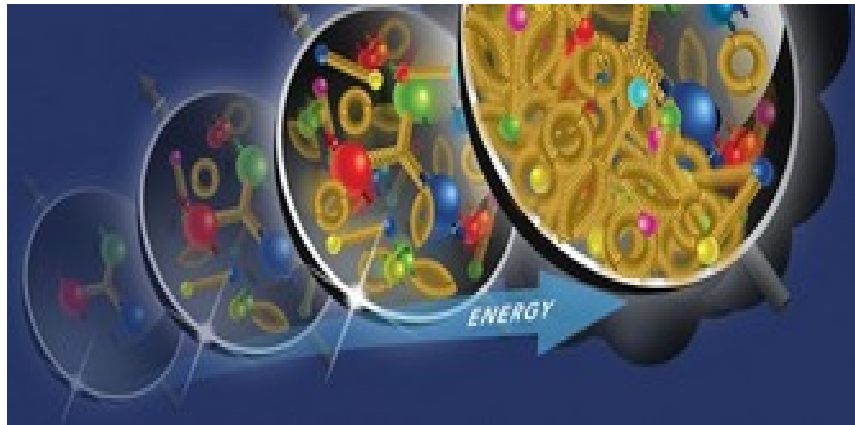
DGLAP: All orders evolution

Raya:2021zrz

Cui:2020tdf

$$\left\{ \zeta^2 \frac{d}{d\zeta^2} \int_0^1 dy \delta(y-x) - \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \begin{pmatrix} P_{qq}^{\text{NS}}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^{\text{S}}\left(\frac{\mathbf{x}}{\mathbf{y}}\right) \end{pmatrix} \right\} \begin{pmatrix} H_{\pi}^{\text{NS},+}(y, t; \zeta) \\ \mathbf{H}_{\pi}^{\text{S}}(y, t; \zeta) \end{pmatrix} = 0$$

DGLAP leading-order evolution equations



DGLAP: All orders evolution

Assumption: define an **effective** charge such that

Raya:2021zrz

Cui:2020tdf

Starting from fully-dressed
quasiparticles, at ζ_H



Sea and **Glun** content unveils,
as prescribed by **QCD**

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DGLAP ~~leading order~~ evolution equations

- **Not** the **LO** QCD coupling but an **effective** one.
- Making this equation **exact**.
- Connecting with the **hadron scale**, at which the **fully-dressed** valence-**quarks** express **all** of the hadron's properties.

(thus carrying all the momentum)



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$$\left\{ \zeta^2 \frac{d}{d\zeta^2} \mathbb{1} + \frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} \gamma_{qq}^{(n)} & 0 & 0 \\ 0 & \gamma_{qq}^{(n)} & 2n_f \gamma_{qg}^{(n)} \\ 0 & \gamma_{gq}^{(n)} & \gamma_{gg}^{(n)} \end{pmatrix} \right\} \begin{pmatrix} \langle x^n \rangle_{NS}(\zeta) \\ \langle x^n \rangle_S(\zeta) \\ \langle x^n \rangle_g(\zeta) \end{pmatrix} = 0$$

DGLAP ~~leading order~~ evolution equations

$$\gamma_{AB}^{(n)} = - \int_0^1 dx x^n P_{AB}^C(x)$$



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DGLAP: All orders evolution

Cui:2020tdf

Implication 1: valence quark PDF

$$\langle x^n(\zeta_f) \rangle_q = \exp \left(-\frac{\gamma_{qq}^{(n)}}{4\pi} I(\zeta_0, \zeta_f) \right) \langle x^n(\zeta_0) \rangle_q$$

$$I(\zeta_0, \zeta_f) = \int_{2 \ln(\zeta_0/\Lambda_{\text{QCD}})}^{2 \ln(\zeta_f/\Lambda_{\text{QCD}})} dt \alpha(t)$$

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Direct connection bridging from hadron to experimental scale: **only one input** is needed to evolve “all” the Mellin moments up and **reconstruct the PDF**.

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$$q(x; \zeta) \underset{x \rightarrow 1}{\sim} (1-x)^{\beta(\zeta)} (1 + \mathcal{O}(1-x))$$

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DGLAP: All orders evolution

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Reconstruction after evolving a CSM PDF

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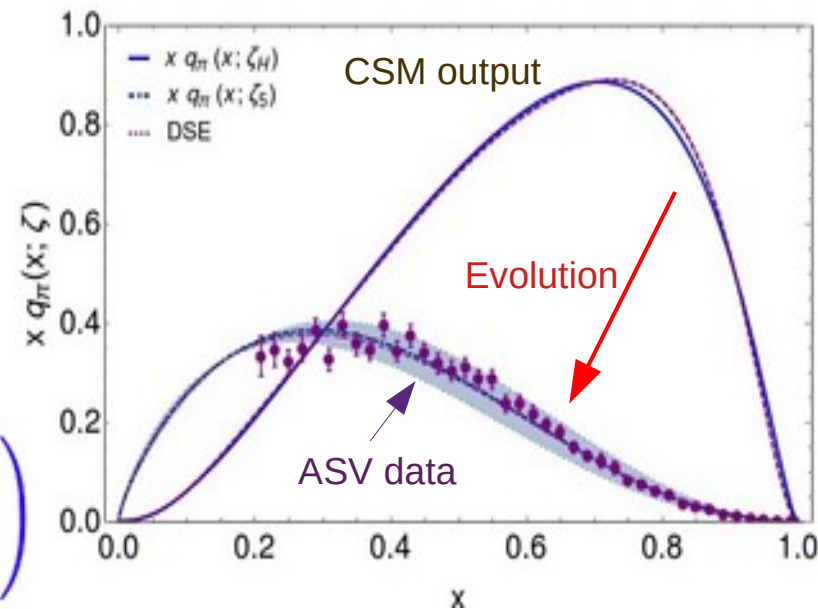
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DGLAP: All orders evolution

Implication 2: glue and sea-quark distributions ($n_f=4$)

$$\langle 2x(\zeta_f) \rangle_q = \exp\left(-\frac{8}{9\pi} I(\zeta_H, \zeta_f)\right), \quad q = u, \bar{d};$$

✦ Obtained from valence-quark inputs

$$\langle x(\zeta_f) \rangle_{\text{sea}} = \langle x(\zeta_f) \rangle_{\sum_q q + \bar{q}} - (\langle x(\zeta_f) \rangle_u + \langle x(\zeta_f) \rangle_{\bar{d}}),$$

$$= \frac{3}{7} + \frac{4}{7} \langle 2x(\zeta_f) \rangle_u^{7/4} - \langle 2x(\zeta_f) \rangle_u$$

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$$\zeta_f / \zeta_H \rightarrow \infty$$

Asymptotic (massless) limit is manifestly in agreement with textbook results: G. Altarelli, Phys. Rep. 81, 1 (1982)

DGLAP: All orders evolution

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$$\begin{aligned} \langle x(\zeta_f) \rangle_{\text{sea}} &= \langle x(\zeta_f) \rangle_{\sum_q q + \bar{q}} - (\langle x(\zeta_f) \rangle_u + \langle x(\zeta_f) \rangle_{\bar{d}}), \\ &= \frac{3}{7} + \frac{4}{7} \langle 2x(\zeta_f) \rangle_u^{7/4} - \langle 2x(\zeta_f) \rangle_u \end{aligned}$$

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R.S. Sufian et al., arXiv:2001.04960

ζ_5	$\langle 2x \rangle_q^\pi$	$\langle x \rangle_q^\pi$	$\langle x \rangle_{\text{sea}}^\pi$
Ref.[55]	0.412(36)	0.449(19)	0.138(17)
Herein	0.40(4)	0.45(2)	0.14(2)

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Implication 2: glue and sea-quark distributions ($n_f=4$)

$$\langle 2x(\zeta_f) \rangle_q = \exp\left(-\frac{8}{9\pi} I(\zeta_H, \zeta_f)\right), \quad q = u, \bar{d};$$

$$\begin{aligned} \langle x(\zeta_f) \rangle_{\text{sea}} &= \langle x(\zeta_f) \rangle_{\sum_q q + \bar{q}} - (\langle x(\zeta_f) \rangle_u + \langle x(\zeta_f) \rangle_{\bar{d}}), \\ &= \frac{3}{7} + \frac{4}{7} \langle 2x(\zeta_f) \rangle_u^{7/4} - \langle 2x(\zeta_f) \rangle_u \end{aligned}$$

$$\langle x(\zeta_f) \rangle_g = \frac{4}{7} \left(1 - \langle 2x(\zeta_f) \rangle_u^{7/4}\right);$$

★ Obtained from valence-quark inputs

Momentum sum rule:

$$\langle 2x(\zeta_f) \rangle_q + \langle x(\zeta_f) \rangle_{\text{sea}} + \langle x(\zeta_f) \rangle_g = 1$$

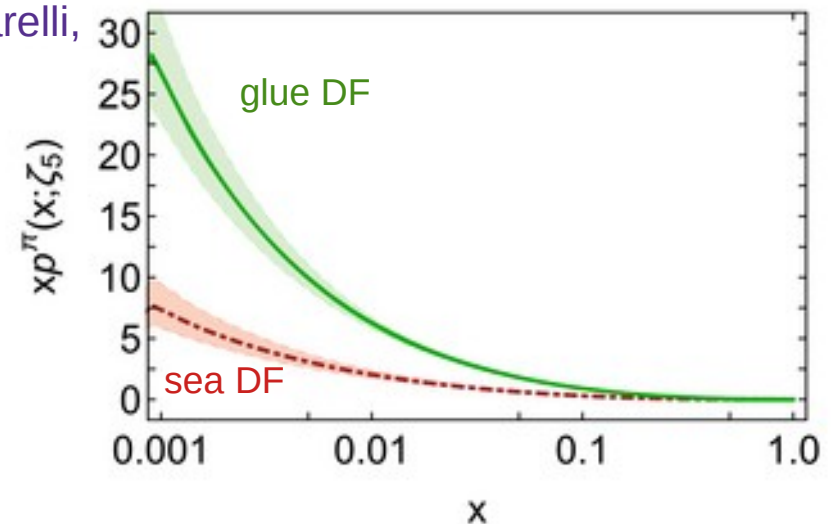
$$\zeta_f / \zeta_H \rightarrow \infty$$

Asymptotic (massless) limit is manifestly in agreement with textbook results: G. Altarelli, Phys. Rep. 81, 1 (1982)

R.S. Sufian et al., arXiv:2001.04960

ζ_5	$\langle 2x \rangle_q^\pi$	$\langle x \rangle_q^\pi$	$\langle x \rangle_{\text{sea}}^\pi$
Ref.[55]	0.412(36)	0.449(19)	0.138(17)
Herein	0.40(4)	0.45(2)	0.14(2)

Compute all the moments and reconstruct:



DGLAP: All orders evolution

Implication 3: recursion of Mellin moments

$$\langle x^{2n+1} \rangle_{u_\pi}^\zeta = \frac{(\langle 2x \rangle_{u_\pi}^\zeta)^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\dots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j \rangle_{u_\pi}^\zeta (\langle 2x \rangle_{u_\pi}^\zeta)^{-\gamma_0^j/\gamma_0^1} .$$

- Since **isospin symmetry** limit implies:
 $q(x; \zeta_H) = q(1 - x; \zeta_H)$
- **Odd** moments can be expressed in terms of previous **even** moments.
- Thus arriving at the recurrence **relation** on the left which is satisfied **if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale** .

DGLAP: All orders evolution

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Reported **lattice moments** agree very well with the **recursion formula**

n	$\langle x^n \rangle_{u_\pi}^{\zeta_5}$	
	Ref. [99]	Eq. (17)
1	0.230(3)(7)	<u>0.230</u>
2	0.087(5)(8)	<u>0.087</u>
3	0.041(5)(9)	<u>0.041</u>
4	0.023(5)(6)	<u>0.023</u>
5	0.014(4)(5)	<u>0.015</u>
6	0.009(3)(3)	<u>0.009</u>
7		0.0078

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Reported **lattice moments** agree very well with the **recursion formula** and so also does and estimate for the 7-th moment from **lattice reconstruction**.

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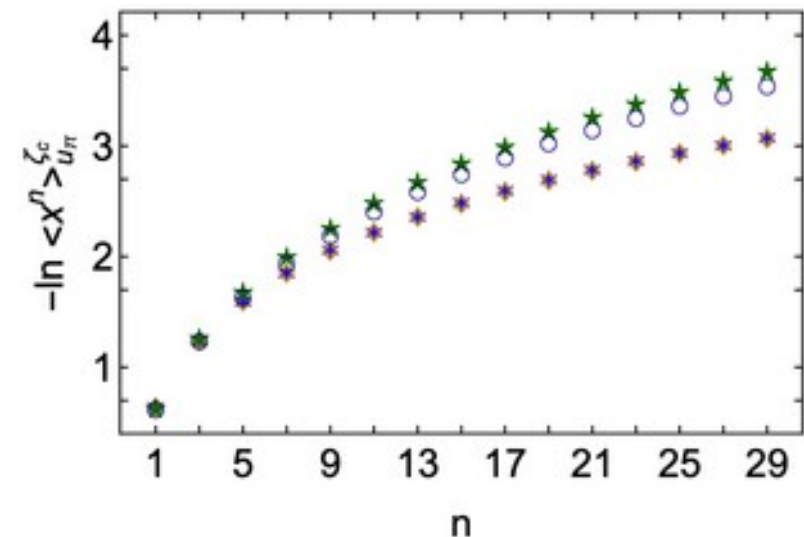
Moments from **global fits** can be also compared to the estimated from recursion !

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Moments computed from: P. Barry et al., PRL127(2021)232001



DGLAP: All orders evolution

Implication 4: physical bounds

$$\langle x^n \rangle_{u_\pi}^\zeta (\langle 2x \rangle_{u_\pi}^\zeta)^{-\gamma_0^n / \gamma_0^1}$$

- Keeping **isospin symmetry**, implying:

$$q(x; \zeta_H) = q(1 - x; \zeta_H)$$

DGLAP: All orders evolution

Implication 4: physical bounds

$$\frac{1}{2^n} \leq \langle x^n \rangle_{u_\pi}^\zeta (\langle 2x \rangle_{u_\pi}^\zeta)^{-\gamma_0^n / \gamma_0^1}$$



$$q(x; \zeta_H) = \delta(x - 1/2)$$

- Keeping **isospin symmetry**, implying:

$$q(x; \zeta_H) = q(1 - x; \zeta_H)$$

- **Lower bound** is imposed by considering the limit of a strongly interacting system of two (then maximally correlated) partons: **both carry half of the momentum.**

DGLAP: All orders evolution

Implication 4: physical bounds

$$\frac{1}{2^n} \leq \langle x^n \rangle_{u_\pi}^\zeta (\langle 2x \rangle_{u_\pi}^\zeta)^{-\gamma_0^n / \gamma_0^1} \leq \frac{1}{1+n}$$

$$q(x; \zeta_H) = \delta(x - 1/2)$$

$$q(x; \zeta_H) = 1$$

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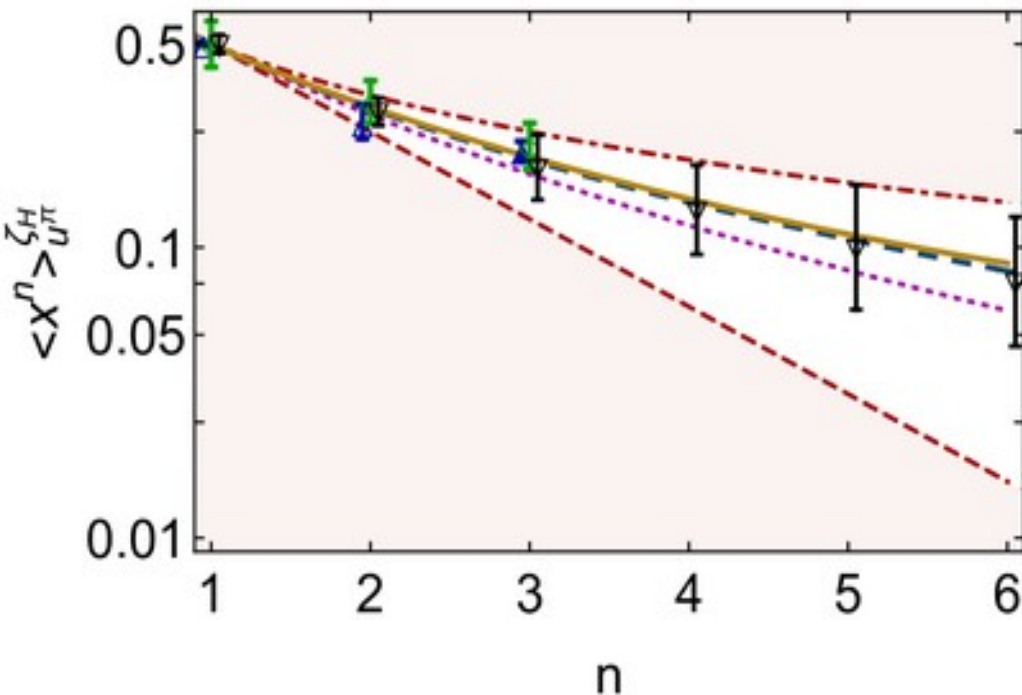
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DGLAP: All orders evolution

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Joo:2019bZR Sufian:2019bol Alexandrou:2021mmi

n	[61]	[62]	[63]
1	0.254(03)	0.18(3)	0.23(3)(7)
2	0.094(12)	0.064(10)	0.087(05)(08)
3	0.057(04)	0.030(05)	0.041(05)(09)
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Lattice moments verifying the **recurrence relation** too.

(guidance)

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PRODUCED BY
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THEY'RE ALL GOING TO
FIGHT IN THE MORNING.

WILLIAM ANDREW WILLIAMS

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Pion PDF: from CSM (DSEs) to the experiment

Symmetry-preserving DSE computation of the valence-quark PDF:

[L. Chang et al., Phys.Lett.B737(2014)23]

[M. Ding et al., Phys.Rev.D101(2020)054014]

$$q^\pi(x; \zeta) = N_c \text{tr} \int_{dk} \delta_n^x(k_\eta) \Gamma_\pi^P(k_{\eta\bar{\eta}}; \zeta) S(k_{\eta\bar{\eta}}; \zeta) \\ \times \left\{ n \cdot \frac{\partial}{\partial k_\eta} [\Gamma_\pi^{-P}(k_{\eta\bar{\eta}}; \zeta) S(k_\eta; \zeta)] \right\}.$$

$$q_{\text{O}}^\pi(x; \zeta_H) = 213.32 x^2 (1-x)^2$$

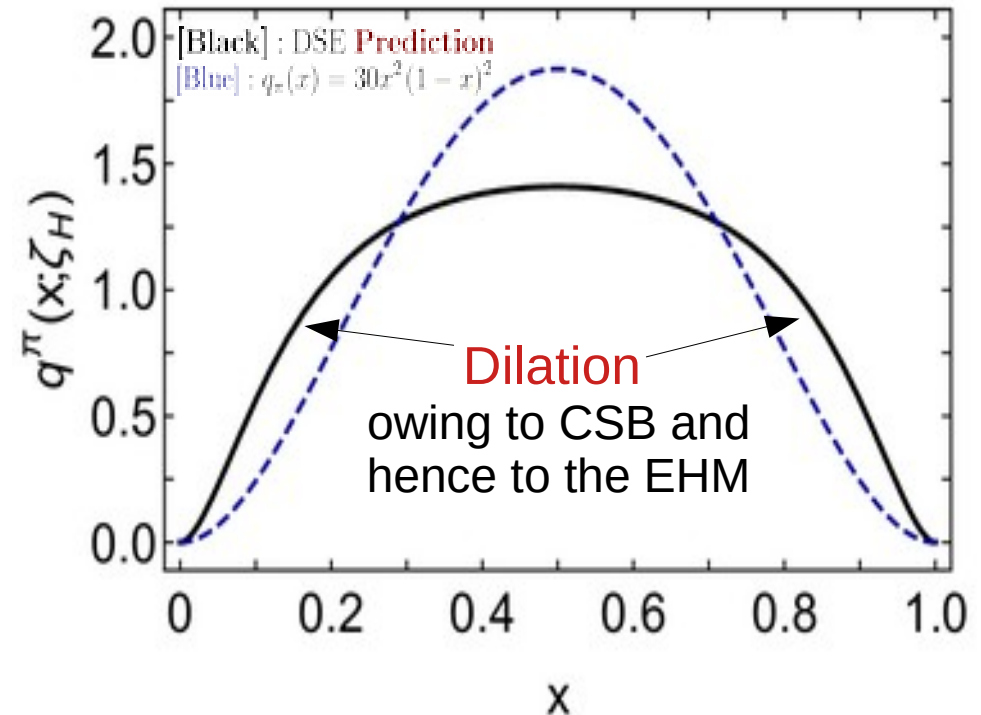
$$\times [1 - 2.9342 \sqrt{x(1-x)} + 2.2911 x(1-x)]$$

$$q(x; \zeta) \underset{x \rightarrow 1}{\sim} (1-x)^{\beta(\zeta)} (1 + \mathcal{O}(1-x)) \\ \beta(\zeta_H) = 2$$

Farrar, Jackson, Phys.Rev.Lett 35(1975)1416

Berger, Brodsky, Phys.Rev.Lett 42(1979)940

- The EHM-triggered broadening shortens the extent of the domain of convexity lying on the neighborhood of the endpoints, induced too by the QCD dynamics
- It cannot however spoil the asymptotic QCD behaviour at large- x (and, owing to isospin symmetry, at low- x)



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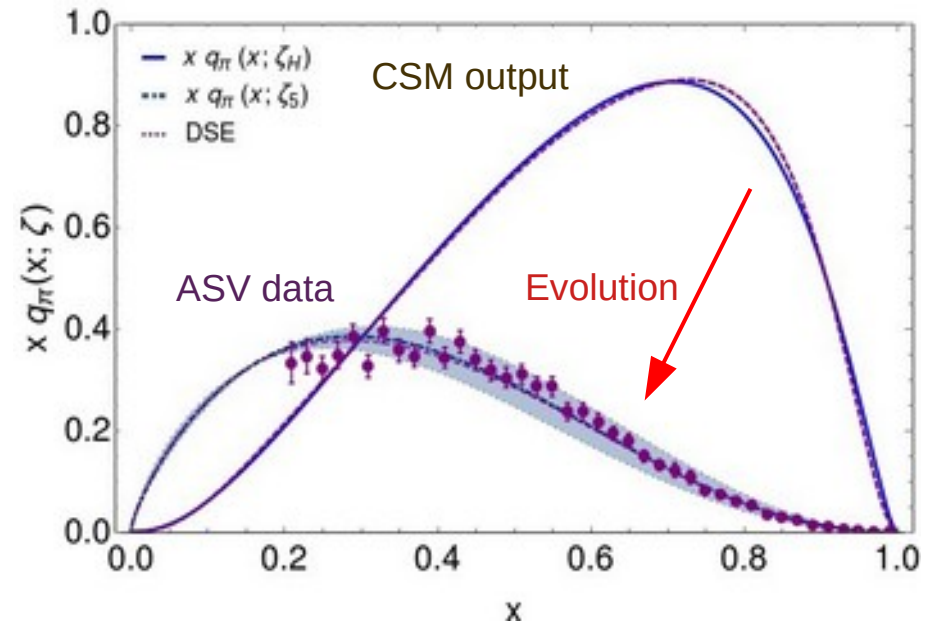
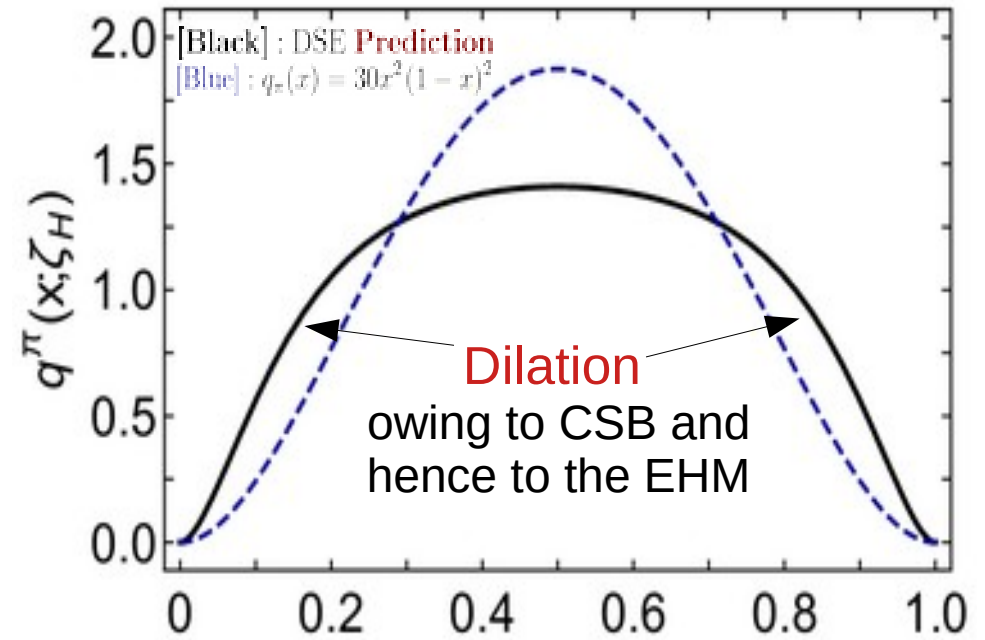
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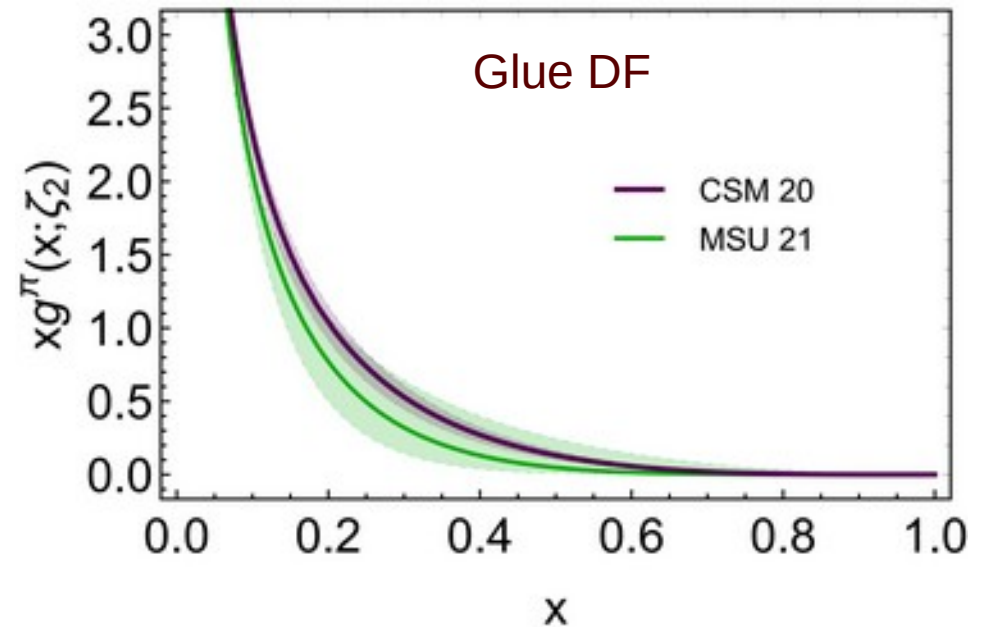
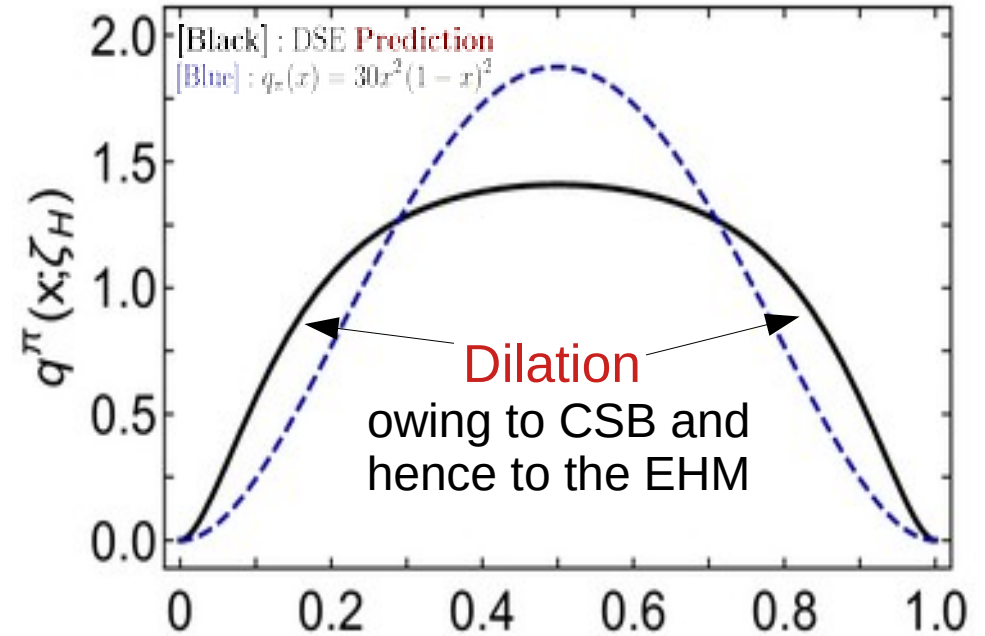
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Proton PDF: from CSM (DSEs) to the experiment

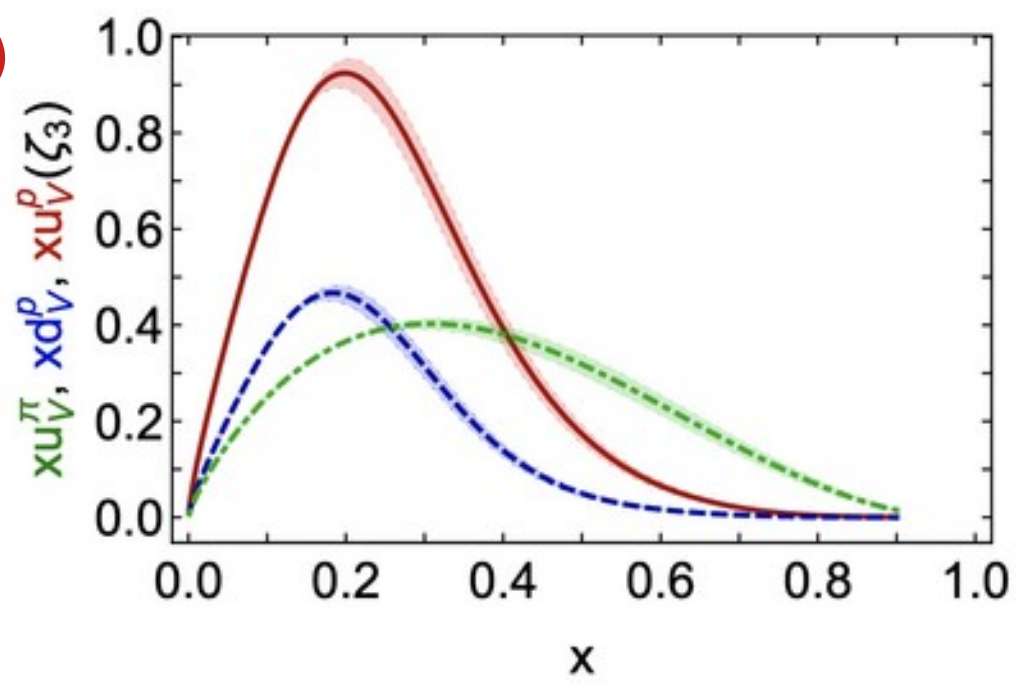
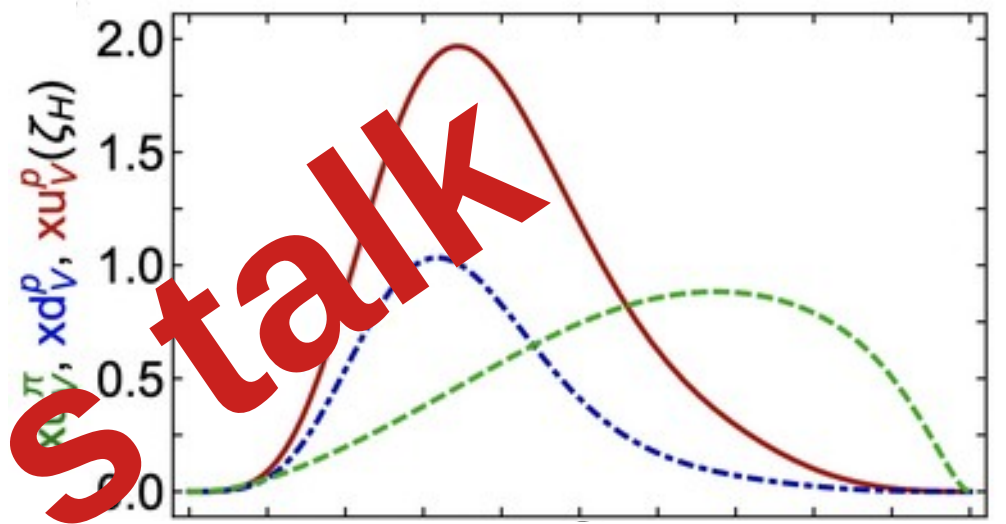
An analogous symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach: [L. Chang et al., Phys.Lett.B, arXiv:2201.07870]

And analogous evolution approach:

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{qH}^\zeta = -\frac{\alpha_{1\ell}(\zeta^2)}{4\pi} \gamma_{qq}^n \langle x^n \rangle_{qH}^\zeta$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{\Sigma_H^q}^\zeta = -\frac{\alpha_{1\ell}(\zeta^2)}{4\pi} \left[\gamma_{qq}^n \langle x^n \rangle_{\Sigma_H^q}^\zeta + 2\mathcal{P}_{qg}^n \langle x^n \rangle_{gH}^\zeta \right]$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{gH}^\zeta = -\frac{\alpha_{1\ell}(\zeta^2)}{4\pi} \left[\sum_q \gamma_{gq}^n \langle x^n \rangle_{\Sigma_H^q}^\zeta + \gamma_{gg}^n \langle x^n \rangle_{gH}^\zeta \right]$$



Craig's talk

Proton PDF: from CSM (DSEs) to the experiment ¹⁰

Let us focus on the evolution equations

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Hadron PDF: **hard-threshold model**

Let us focus on the evolution equations and consider a simpler but insightful model: a system of **massless partons with hard thresholds** for each flavor activation, that can be analytically solved!

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$\theta(\zeta - M_q)$

$\gamma_{qq}^n = \gamma_{uu}^n, \gamma_{gq}^n = \gamma_{gu}^n, \gamma_{qg}^n = \gamma_{ug}^n$

$q = u, d, s, c$

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Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$

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$$\langle x^n \rangle_{\Sigma_{u+d}}^\zeta = \sum_{q=u,d} \langle x^n \rangle_{\Sigma_H^q}^\zeta$$

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle_{\Sigma_{u+d}}^\zeta \\ \langle x^n \rangle_{gH}^\zeta \end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} \gamma_{uu}^n & 4\gamma_{ug}^n \\ \gamma_{gu}^n & \gamma_{gg}^n \end{pmatrix} \begin{pmatrix} \langle x^n \rangle_{\Sigma_{u+d}}^\zeta \\ \langle x^n \rangle_{gH}^\zeta \end{pmatrix} - \frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} 0 \\ \gamma_{gu}^n \langle x^n \rangle_{\Sigma_H^s}^\zeta \end{pmatrix}$$

Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$, such that the singlet combinations can be rearranged and the strange decoupled from the light flavors.

Hadron PDF: **hard-threshold model**

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In pion's (proton's) case

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Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$, such that the singlet combinations can be rearranged and the strange decoupled from the light flavors. Specializing for the averaged momentum fraction.

In pion's (proton's) case

$$\langle x \rangle_{s\pi}^{\zeta_H} = 0$$

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$$\begin{pmatrix} \langle x \rangle_{\Sigma_\pi^{u+d}}^\zeta \\ \langle x \rangle_{g\pi}^\zeta \end{pmatrix} = \begin{pmatrix} \frac{3}{11} + \frac{8}{11} [S(\zeta_H, \zeta)]^{11/8} \\ \frac{8}{11} (1 - [S(\zeta_H, \zeta)]^{11/8}) \end{pmatrix}$$

$$S(\zeta_H, \zeta) = \exp \left(-\frac{\gamma_{uu}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

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Same for all hadrons!

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$$\langle x \rangle_{qH}^\zeta = \langle x \rangle_{qH}^{\zeta_H} S(\zeta_H, \zeta) \quad \langle x \rangle_{gH}^\zeta = \frac{4}{7} - \tau(M_s, M_c) [\langle 2x \rangle_{u\pi}^\zeta]^{7/4}$$

$$\tau(M_s, M_c) = -\frac{12}{175} [\langle 2x \rangle_{u\pi}^{M_c}]^{-7/4} - \frac{24}{275} [\langle 2x \rangle_{u\pi}^{M_c}]^{-3/16} [\langle 2x \rangle_{u\pi}^{M_s}]^{-25/16} + \frac{8}{11} [\langle 2x \rangle_{u\pi}^{M_c} \langle 2x \rangle_{u\pi}^{M_s}]^{-3/16}$$

Capitalizing on the universality of the effective charge, **all hadrons'** momentum fraction averages can be expressed in terms of **pion's** ones.

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3 (starting) active flavors

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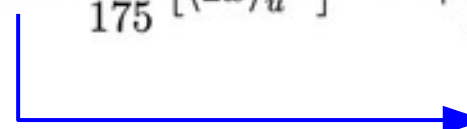
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 $\tau(\zeta_H, \zeta_H) = \frac{4}{7}$

3 (starting) active flavors

4 (always) active flavors

Previous result then recovered!

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Any flavor **sea-quark** momentum fraction can be evaluated and seen to depend explicitly on the **mass threshold**, The same for **all hadrons** in this approximated scheme!

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Momentum conservation!

Summary

I just need
the main ideas



Summary

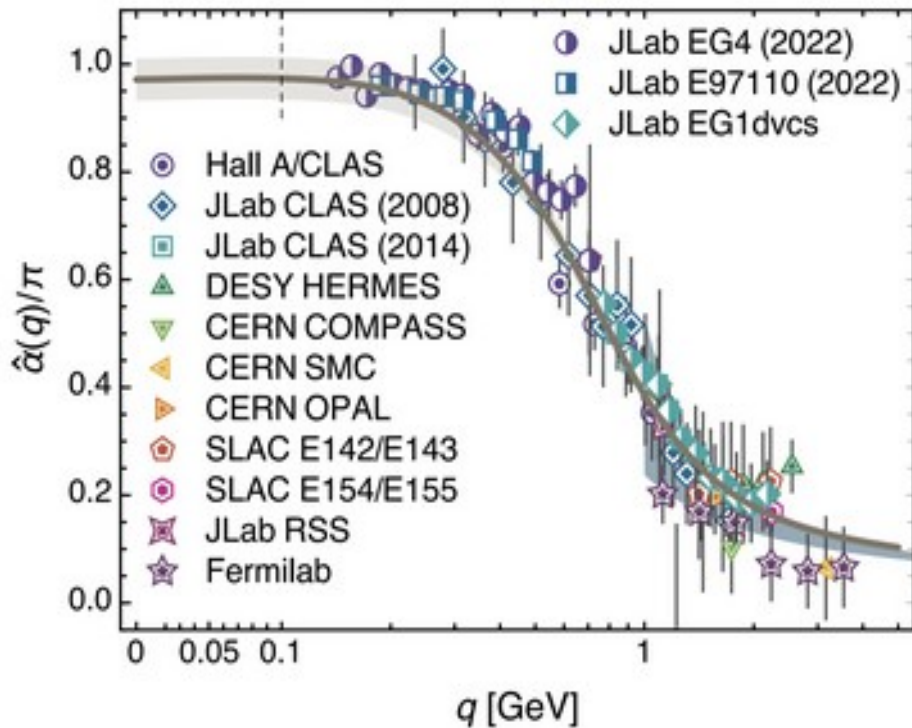
- The **EHM** is argued to be intimately connected to a **PI effective** charge which enters a conformal regime, below a given momentum scale, **where gluons acquiring a dynamical mass decouple from interaction**.
- Capitalizing on the latter, two main ideas emerge: (i) the identification of that decoupling with a **hadronic scale** at which the structure of hadrons can be expressed only in terms of valence dressed partons; and (ii) the reliability of an **all-orders** evolution scheme to describe the splitting of valence into more partons, generating thus the glue and sea, when the resolution scale decreases.
- Key implications stemming from both ideas have been derived and tested for the pion PDFs. Grounding on them, **Lattice QCD** and data from **ASV** or **JAM MF** analyses have been shown to confirm **CSM** results.
- The robustness of the approach based on **all-orders** evolution from **hadronic** to experimental scale has been proved with its application to the proton case (**Craig's talk!**). A model featuring **massless evolution for quark flavors activated after a hard momentum threshold** has been solved analytically, and seen to expose some of the main results implied by the approach.

To be continued...



Backslides

QCD effective charge



Then, we define:

$$\alpha(k^2) = \frac{\gamma_m \pi}{\ln \left[\frac{M^2(k^2)}{\Lambda_{\text{QCD}}^2} \right]}; \quad \alpha(0) = 0.97(4)$$

where

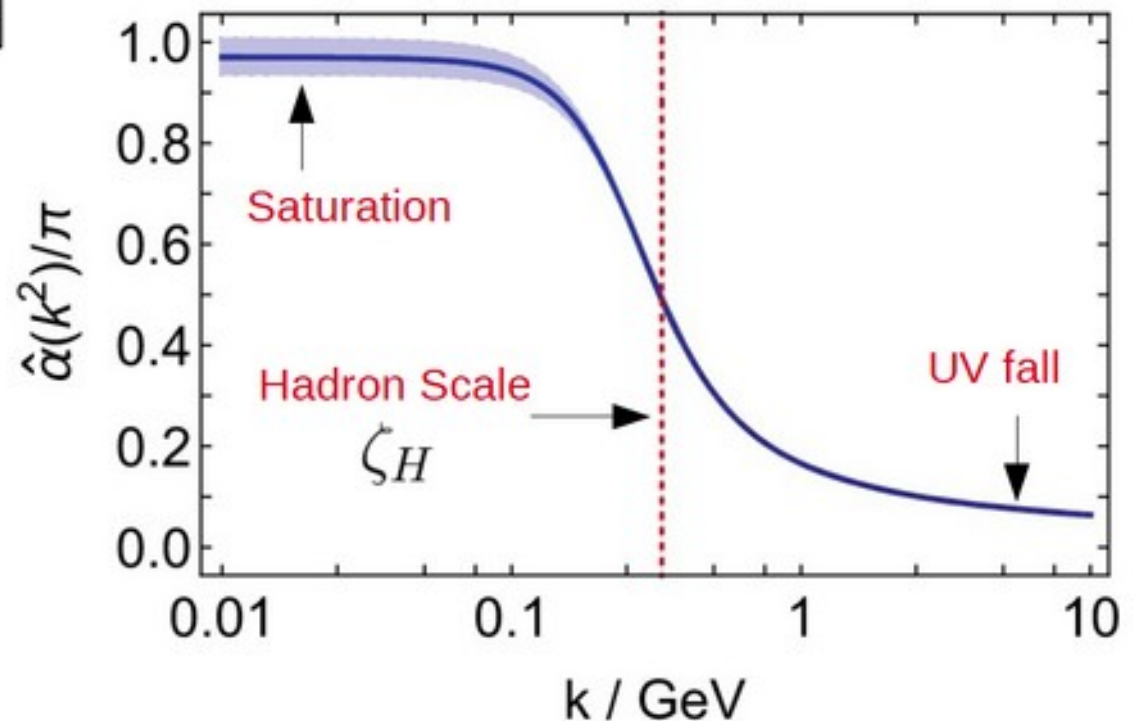
$$\mathcal{M}(k^2 = \Lambda_{\text{QCD}}^2) := m_G = 0.331(2) \text{ GeV}$$

defines the screening mass and an associated wavelength, such that larger gluon modes decouple.

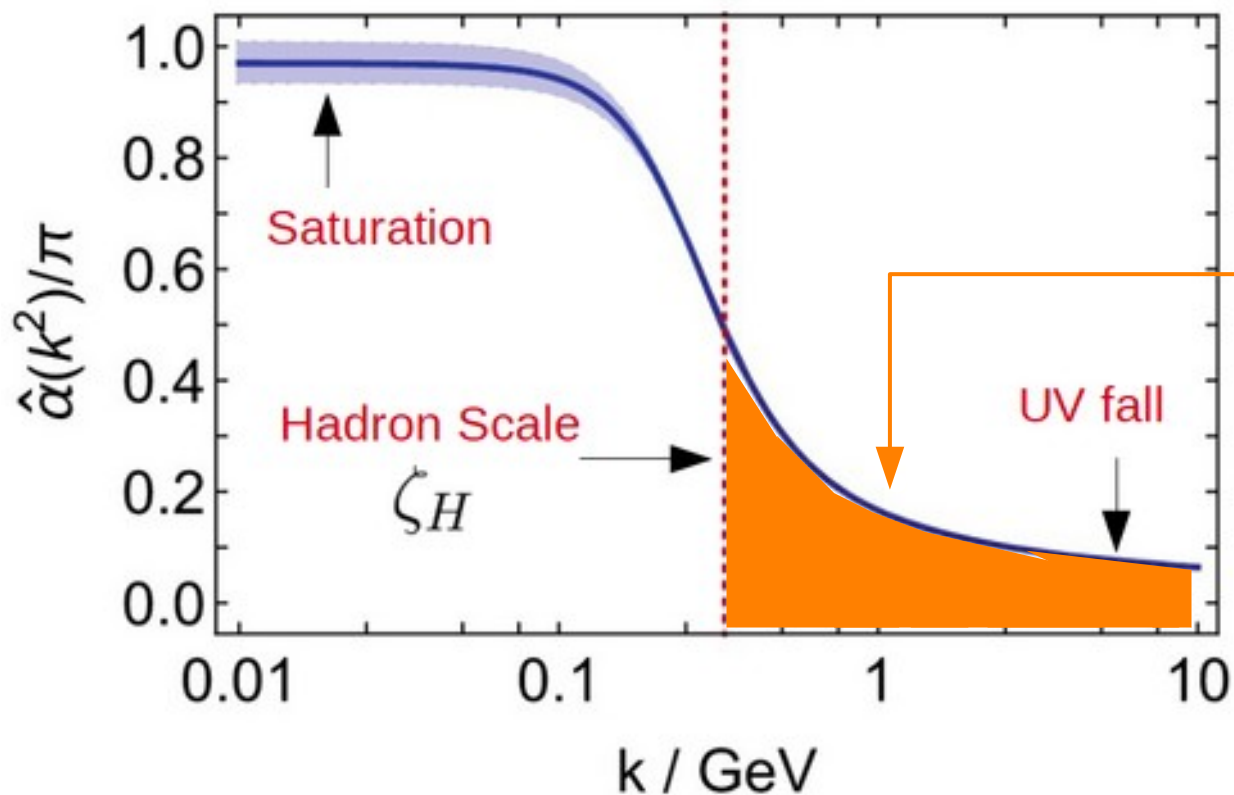
Then, we identify: $\zeta_H := m_G(1 \pm 0.1)$

Modern continuum & lattice QCD analysis in the gauge sector delivers an analogue “Gell-Mann-Low” running charge, from which one obtains a **process-independent, parameter-free prediction** for the **low-momentum saturation**

- No Landau pole
- Below a given mass scale, the interaction becomes scale-independent and QCD practically conformal again (as in the Lagrangian).



QCD effective charge

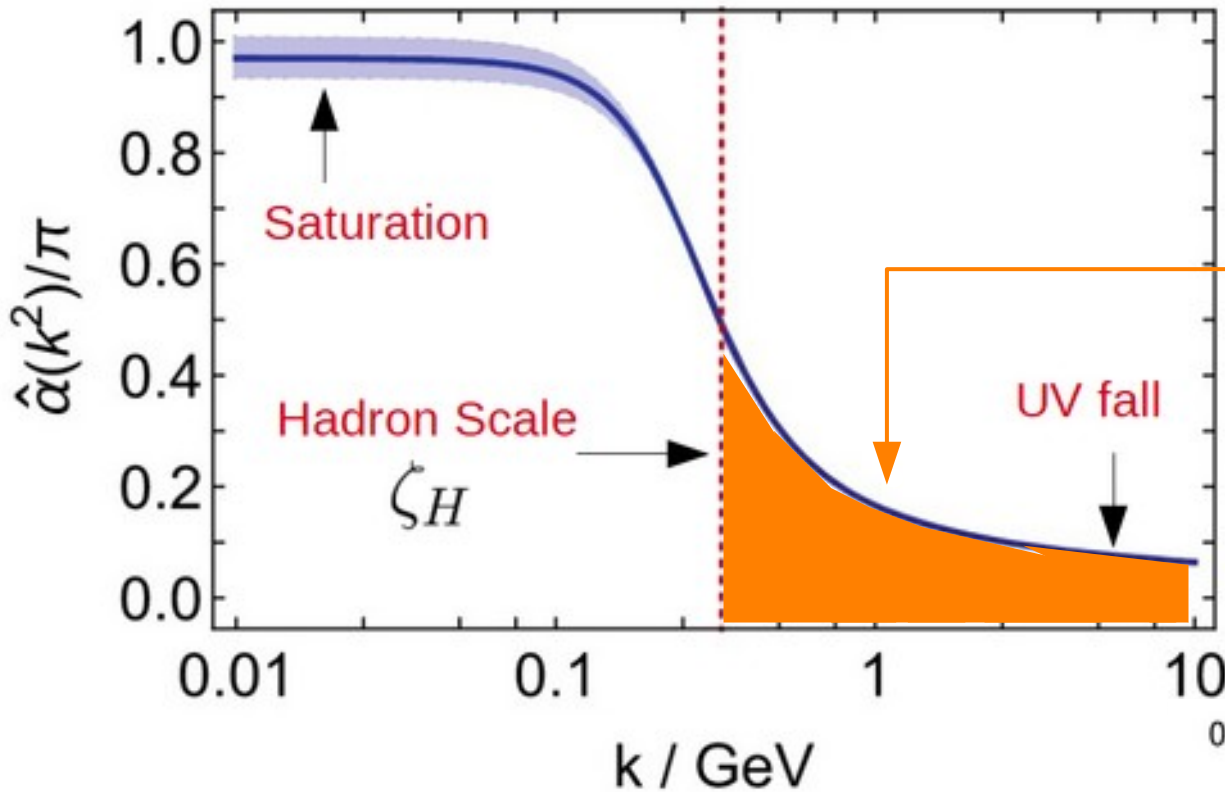


The strength of the charge defines de input for the evolution

$$I(\zeta_H, \zeta_f) = \int_{2\ln(\zeta_H/\Lambda_{\text{QCD}})}^{2\ln(\zeta_f/\Lambda_{\text{QCD}})} dt \hat{\alpha}(t)$$

$$\langle x(\zeta_5) \rangle_q^\pi = \frac{1}{2} \exp\left(-\frac{8}{9\pi} I(\zeta_H, \zeta_5)\right) = 0.20(2)$$

QCD effective charge



The strength of the charge defines de input for the evolution

$$S(\zeta_H, \zeta_f) = \int_{2 \ln(\zeta_H/\Lambda_{\text{QCD}})}^{2 \ln(\zeta_f/\Lambda_{\text{QCD}})} dt \hat{\alpha}(t)$$

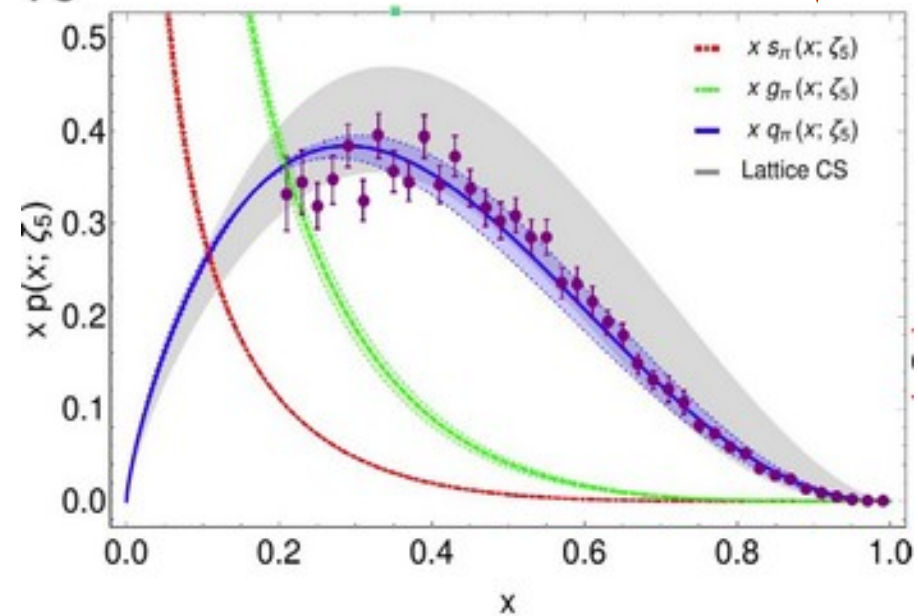
$$\langle x(\zeta_5) \rangle_q^\pi = \frac{1}{2} \exp\left(-\frac{8}{9\pi} S(\zeta_H, \zeta_5)\right) = 0.20(2)$$

[Z-F. Cui et al, EPJC80(2020)11,1064]

[Z-F. Cui et al, EPJA57(2021)1,5]

Then, the glue, valence- and sea-quark DFs can be predicted, with no tuned parameter, on the ground of the effective charge definition, from the LFWF (or, equivalently, from a symmetry-preserving DSE/BSE computation of the valence-quarks Mellin moments

[M. Ding et al, CPC44(2020)3,031002]



Proton PDF: from CSM (DSEs) to the experiment 10

An analogous symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach:
 [L. Chang et al., Phys.Lett.B, arXiv:2201.07870]

And analogous evolution approach:

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{qH}^\zeta = -\frac{\alpha_{1\ell}(\zeta^2)}{4\pi} \gamma_{qq}^n \langle x^n \rangle_{qH}^\zeta$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{\Sigma_H^q}^\zeta = -\frac{\alpha_{1\ell}(\zeta^2)}{4\pi} \left[\gamma_{qq}^n \langle x^n \rangle_{\Sigma_H^q}^\zeta + 2\mathcal{P}_{qg}^\zeta \gamma_{qg}^n \langle x^n \rangle_{gH}^\zeta \right]$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{gH}^\zeta = -\frac{\alpha_{1\ell}(\zeta^2)}{4\pi} \left[\sum_q \gamma_{gq}^n \langle x^n \rangle_{\Sigma_H^q}^\zeta + \gamma_{gg}^n \langle x^n \rangle_{gH}^\zeta \right]$$

Producing an isovector distribution in fair agreement with lattice results
 [H-W. Lin et al., arXiv:2011.14791]

