To the light front with contour deformations

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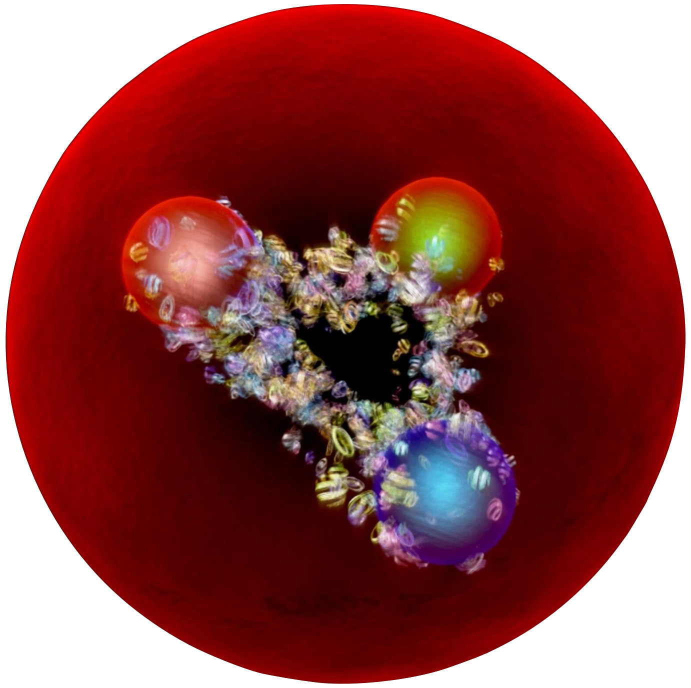
Perceiving the Emergence of Hadron Mass through AMBER@CERN - VII

May 12, 2022

Motivation: why go to the light front?

Strong experimental activities to determine hadron structure

COMPASS/AMBER, JLab, EIC, PANDA, JPARC, …

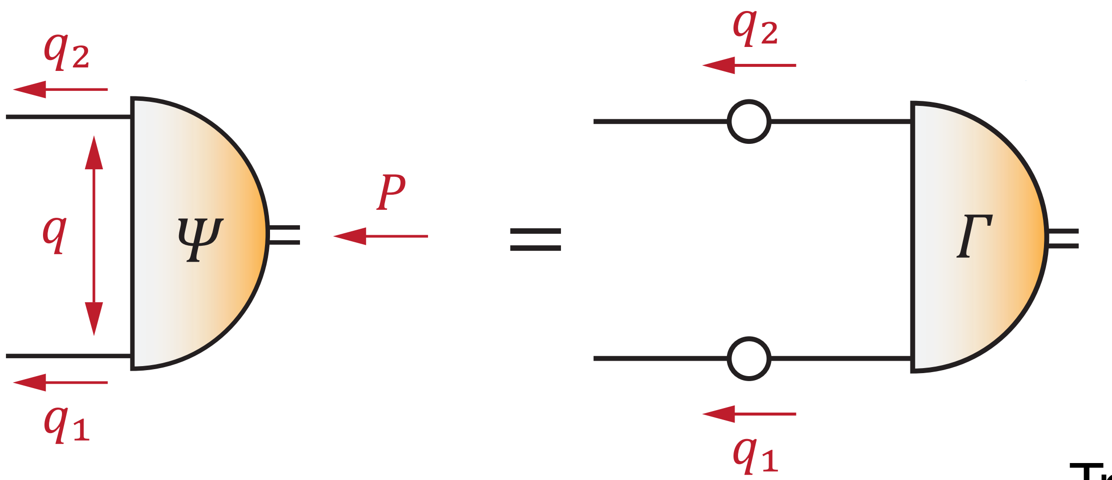


- Hadron structure described in terms of parton distribution functions (PDF’s, GPD’s, TMD’s, …)
- Parton distributions defined on the light front
- Functional methods: usually in momentum space (Euclidean or Minkowski)
- Various techniques for going to the light front (e.g. Nakanishi representation)
- We are exploring a new technique based on contour deformations
  Eichmann, Ferreira, Stadler, PRD **105**, 034009 (2022)

Bethe-Salpeter Wave Function

Bethe-Salpeter wave function

For two scalar particles



$$q_1 = q + P/2$$

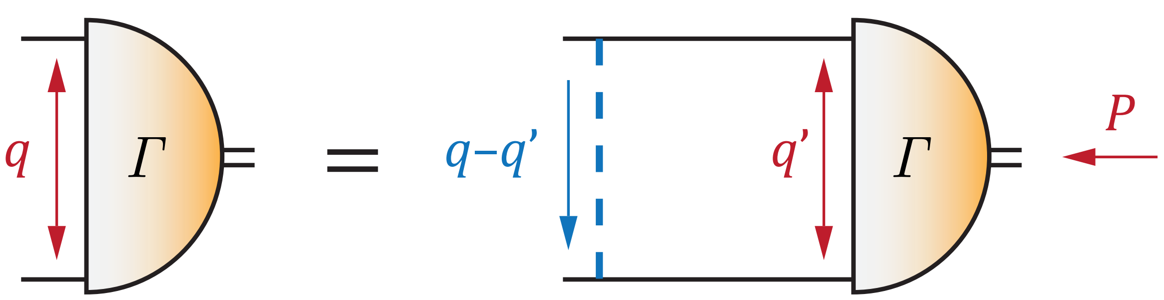
$$q_2 = -q + P/2$$

Tree-level propagators ($m_1 = m_2 = m$)

$$G_0(q,P) = \frac{i}{q_1^2 - m^2 + i\epsilon} \, \frac{i}{q_2^2 - m^2 + i\epsilon}$$

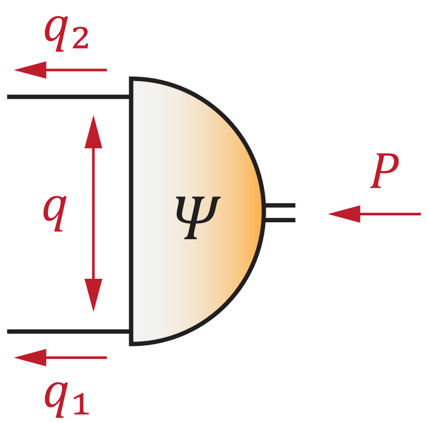
$$\Psi(q,P) = G_0(q,P)\, i\Gamma(q,P)$$

Bethe-Salpeter amplitude satisfies the Bethe-Salpeter equation



$$\Gamma(q,P) = \int \frac{d^4q'}{(2\pi)^4}\, K(q,q',P)\, G_0(q',P)\, \Gamma(q',P)$$

Light-front wave functions



LFWF = BSWF integrated over q^-

$$\psi(q^+, \boldsymbol{q}_\perp) = \mathcal{N} P^+ \int_{-\infty}^{\infty} \frac{dq^-}{2\pi} \Psi(q, P)$$

Light-front variables:

$$q^\pm = q^0 \pm q^3$$

$$\int d^4 q = \tfrac{1}{2} \int d^2 \boldsymbol{q}_\perp \int dq^+ \int dq^-$$

Introduce a momentum partitioning parameter $\alpha \in [-1, +1]$

$q = k + \frac{\alpha}{2} P$ with $k^+ = 0$

such that
$$q_1 = k + \tfrac{1+\alpha}{2} P$$
$$q_2 = -k + \tfrac{1-\alpha}{2} P$$

$\Rightarrow$ α plays role of longitudinal momentum fraction:

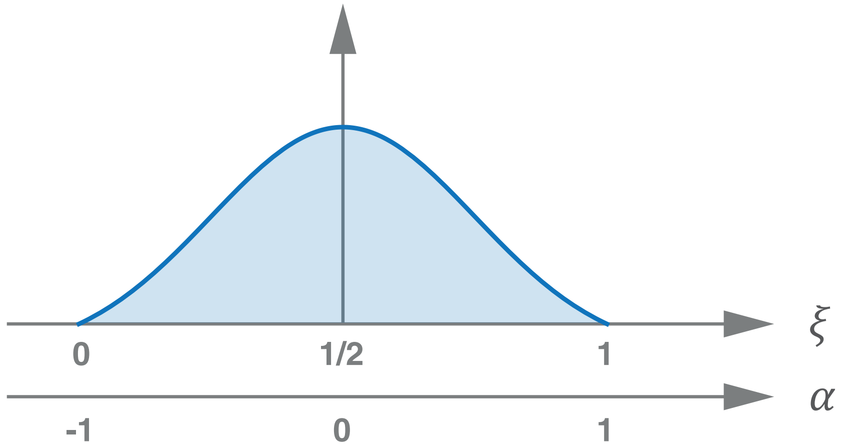
$$q_1^+ = \xi P^+ \qquad \xi = \frac{1+\alpha}{2}$$

$$q_2^+ = (1-\xi) P^+ \qquad \xi \in [0, +1]$$

Then:

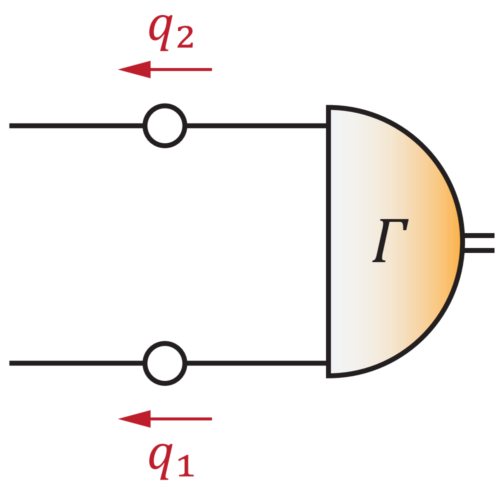
LFWF: $\psi(\alpha, \boldsymbol{k}_\perp) = \mathcal{N} P^+ \int_{-\infty}^{\infty} \frac{dq^-}{2\pi} \Psi(q, P)\Big|_{q^+ = \frac{\alpha}{2} P^+,\, \boldsymbol{q}_\perp = \boldsymbol{k}_\perp}$

PDA: $\phi(\alpha) = \frac{1}{16\pi^3 f} \int d^2 \boldsymbol{k}_\perp \, \psi(\alpha, \boldsymbol{k}_\perp)$



Example: monopole Bethe-Salpeter amplitude

Bethe-Salpeter Wave Function



Choose monopole BSA

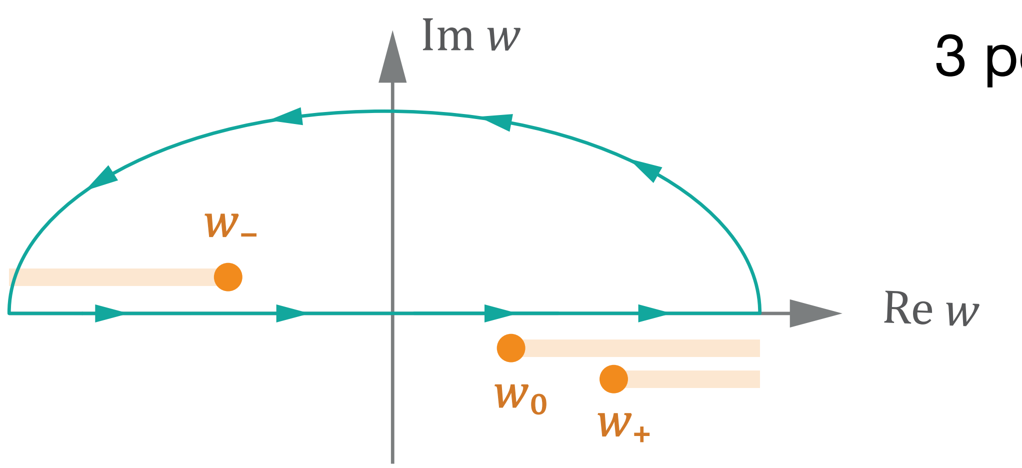
$$\Gamma(q,P) = -\frac{m^2}{q^2 - m^2\gamma + i\epsilon}$$

Tree-level propagators ($m_1 = m_2 = m$)

$$G_0(q,P) = \frac{i}{q_1^2 - m^2 + i\epsilon}\,\frac{i}{q_2^2 - m^2 + i\epsilon}$$

Use dimensionless Lorentz-invariants:

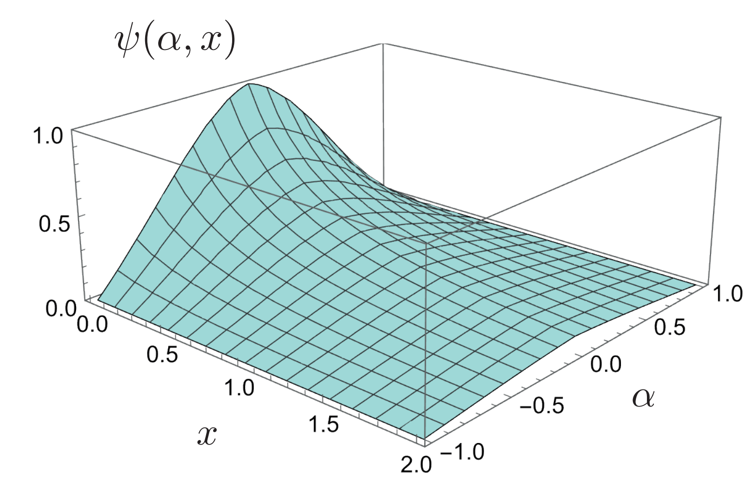
$$t = -\frac{M^2}{4m^2} \in [-1,0] \qquad w = \frac{M}{2m^2}q^- \in \mathbb{R} \qquad x = \frac{\boldsymbol{q}_\perp^2}{m^2} > 0$$



3 poles in complex w plane

$$w_\pm = \pm\left(t + \frac{1+x-i\epsilon}{1\pm\alpha}\right)$$

$$w_0 = \frac{x+\gamma-i\epsilon}{\alpha}$$

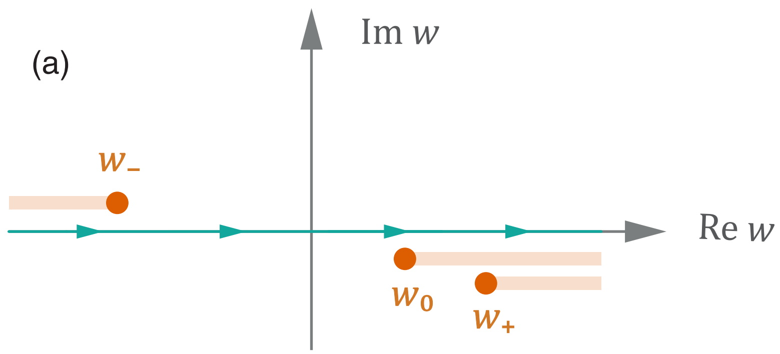


Result:

$$\psi(\alpha,x) = \frac{\mathcal{N}}{m^2}\,\frac{1}{x+A}\,\frac{1-|\alpha|}{x+A+(1-|\alpha|)\,B}\,, \qquad A = 1+(1-\alpha^2)\,t \qquad B = \gamma - 1 - t$$

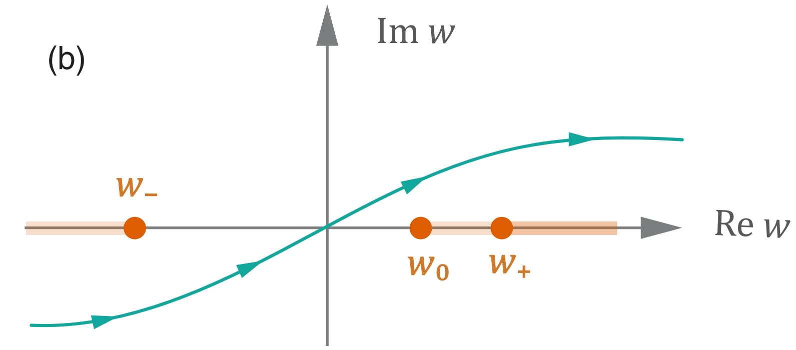
- vanishes at endpoints $\alpha = \pm 1$
- falls off like $1/x^2$
- support only for $-1 < \alpha < 1$ (not an analytic function!)

Example: monopole Bethe-Salpeter amplitude



$$\int_{-\infty}^{\infty} dw$$

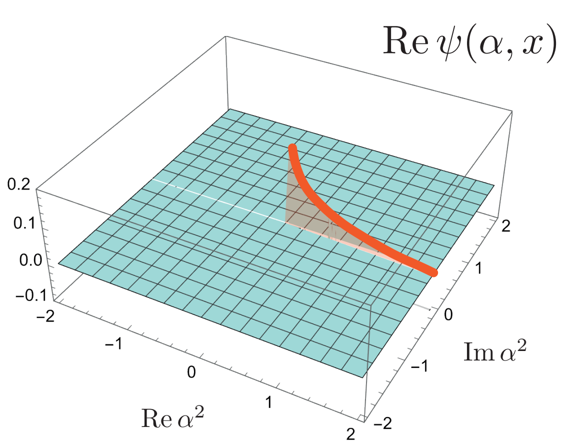
⇒ support only for $-1 < \alpha < 1$,
not an analytic function,
x, t ∈ $\mathbb{R}$

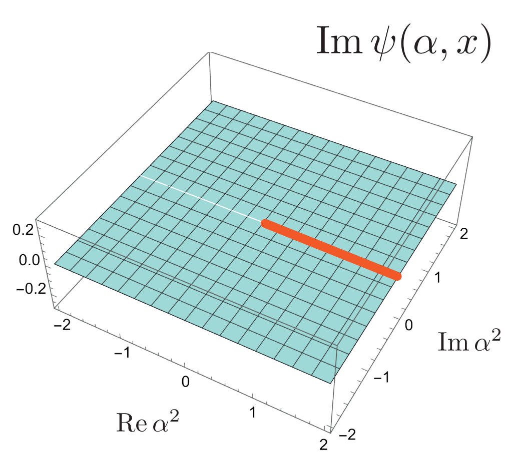


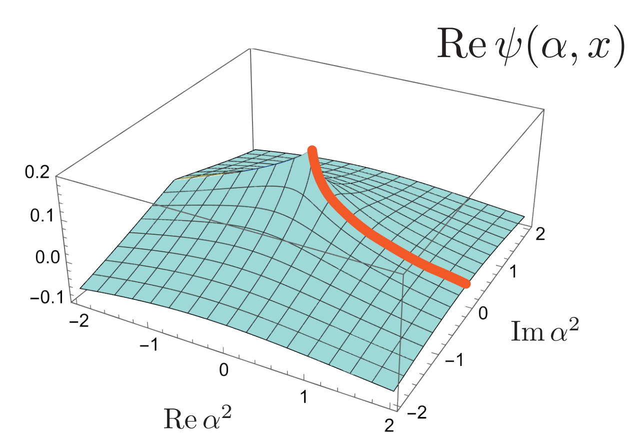
$$\int_{-\infty\,(1+i\epsilon)}^{\infty\,(1+i\epsilon)} dw$$

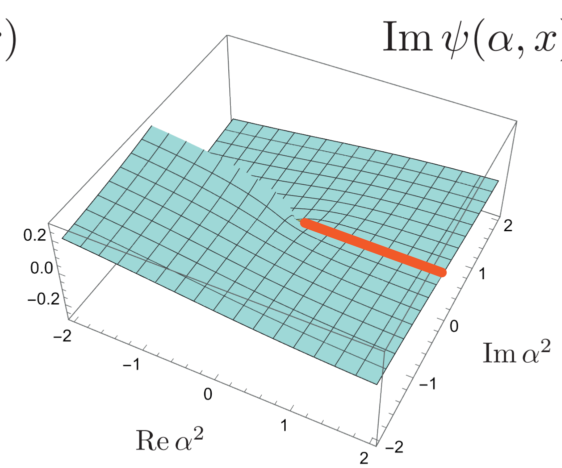
⇒ analytic function in α^2 for any x, t ∈ $\mathbb{C}$,
for $-1 < \alpha < 1$ result is the same

Result from before with $|\alpha| \to \sqrt{\alpha^2}$

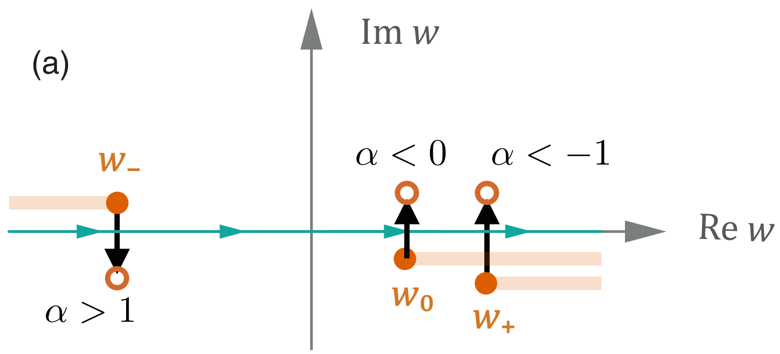




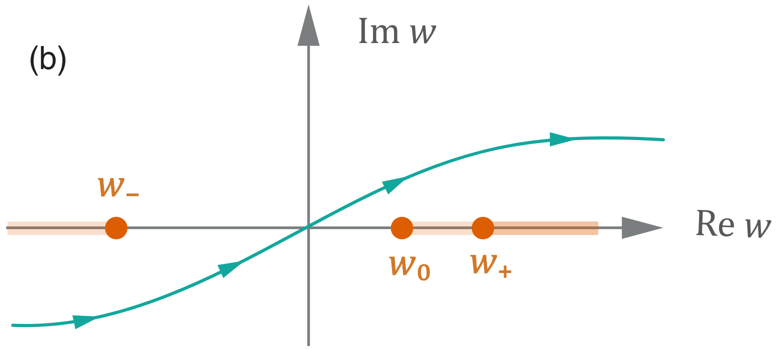




Example: monopole Bethe-Salpeter amplitude



$$\int_{-\infty}^{\infty} dw$$



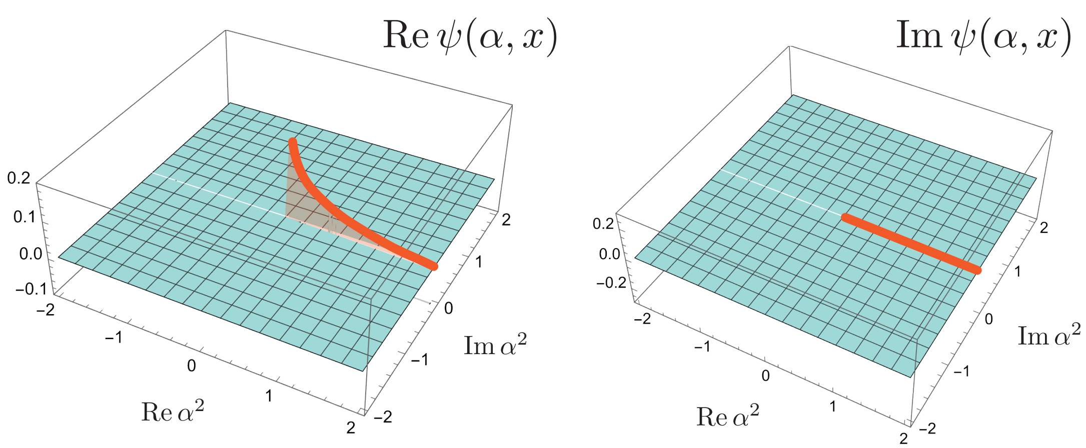
$i\epsilon$ in the path:
(from path integral)

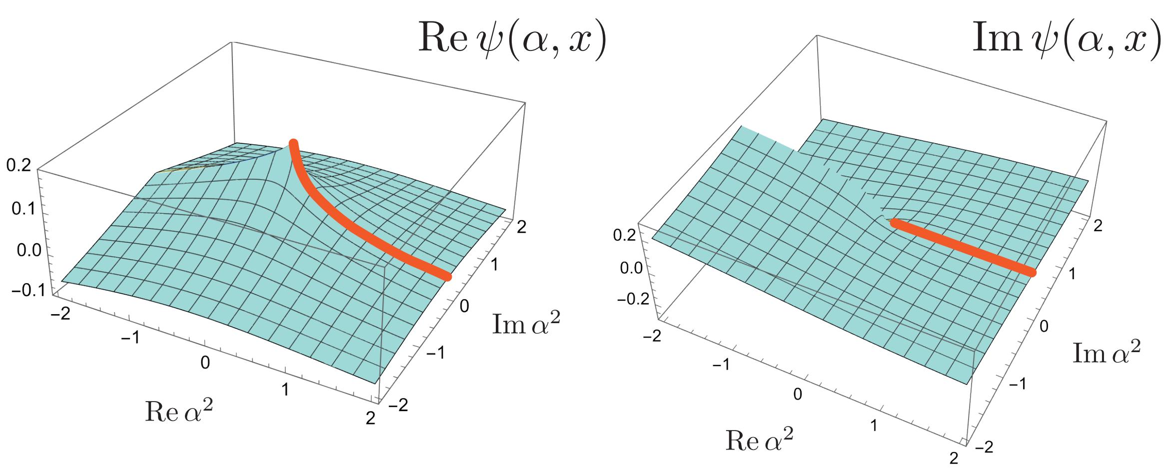
$$\int_{-\infty\,(1+i\epsilon)}^{\infty\,(1+i\epsilon)} dw$$

$\Rightarrow$ support only for $-1 < \alpha < 1$,
not an analytic function,
x, t ∈ ℝ

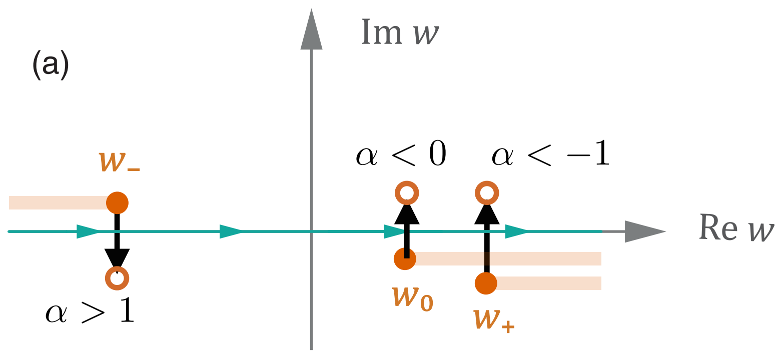
$\Rightarrow$ analytic function in α^2 for any x, t ∈ ℂ,
for $-1 < \alpha < 1$ result is the same

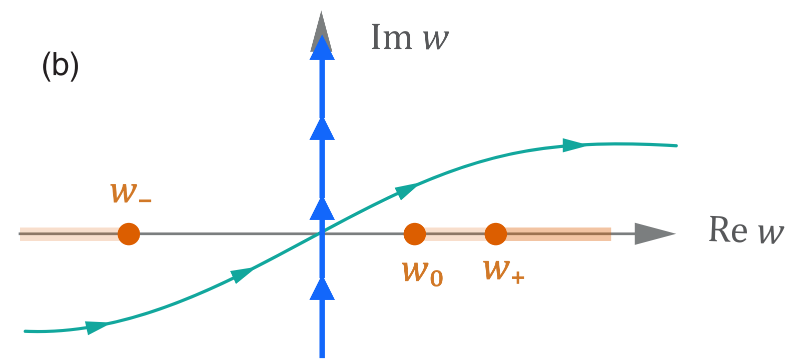
Result from before with $|\alpha| \to \sqrt{\alpha^2}$





Example: monopole Bethe-Salpeter amplitude





$$\int_{-\infty}^{\infty} dw$$

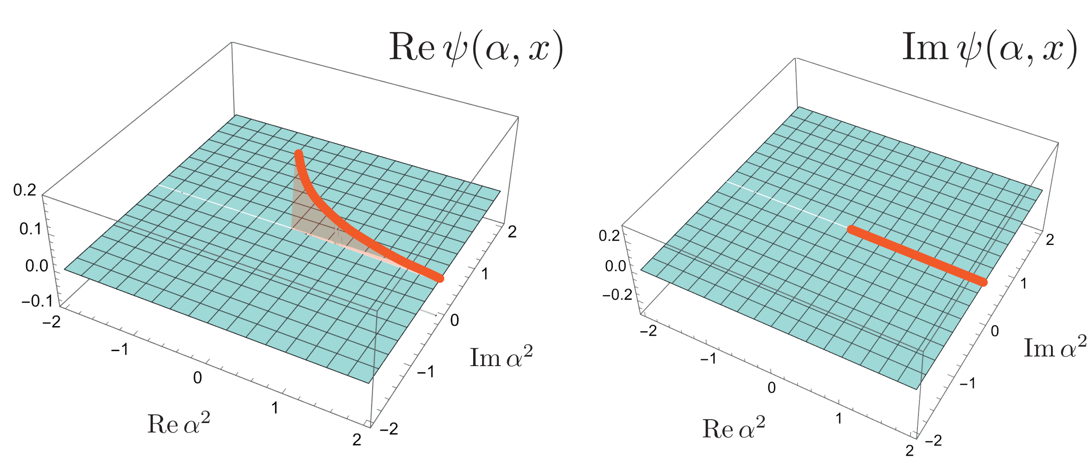
$i\epsilon$ in the path:
(from path integral)

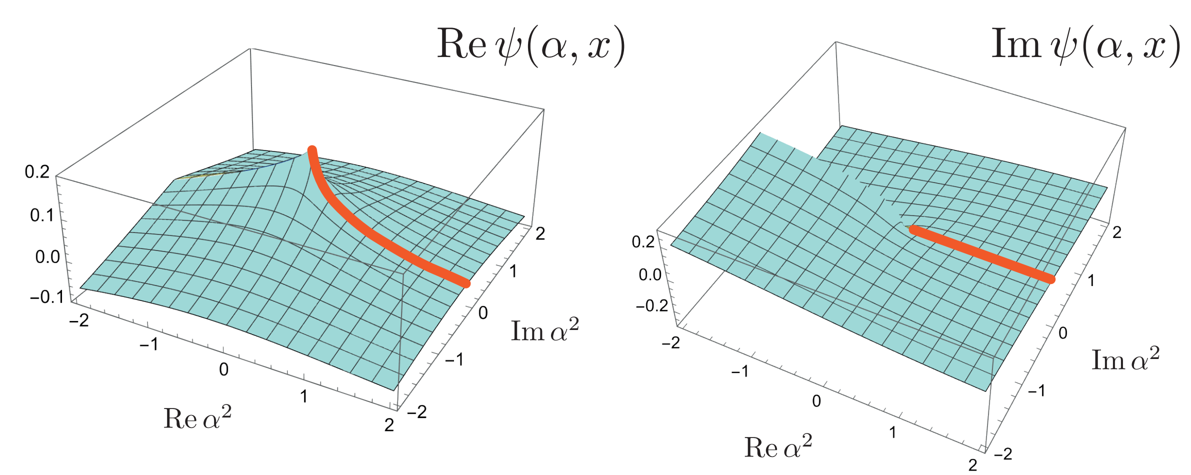
$$\int_{-\infty\,(1+i\epsilon)}^{\infty\,(1+i\epsilon)} dw \longrightarrow \int_{-i\infty}^{+i\infty} dw$$

$\Rightarrow$ support only for $-1 < \alpha < 1$,
not an analytic function,
x, t ∈ ℝ

$\Rightarrow$ analytic function in α^2 for any x, t ∈ ℂ,
for $-1 < \alpha < 1$ result is the same

Result from before with $|\alpha| \to \sqrt{\alpha^2}$



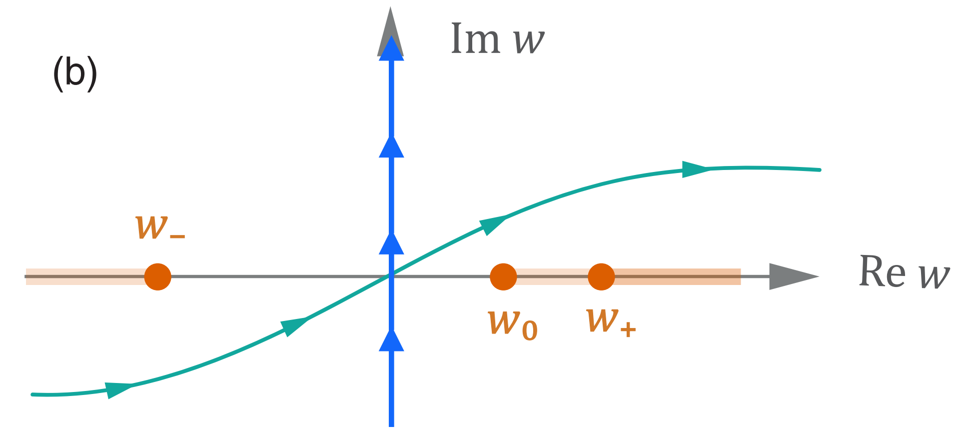


Example: monopole Bethe-Salpeter amplitude

After integrating over w,
poles in w become **branch cuts** in x:

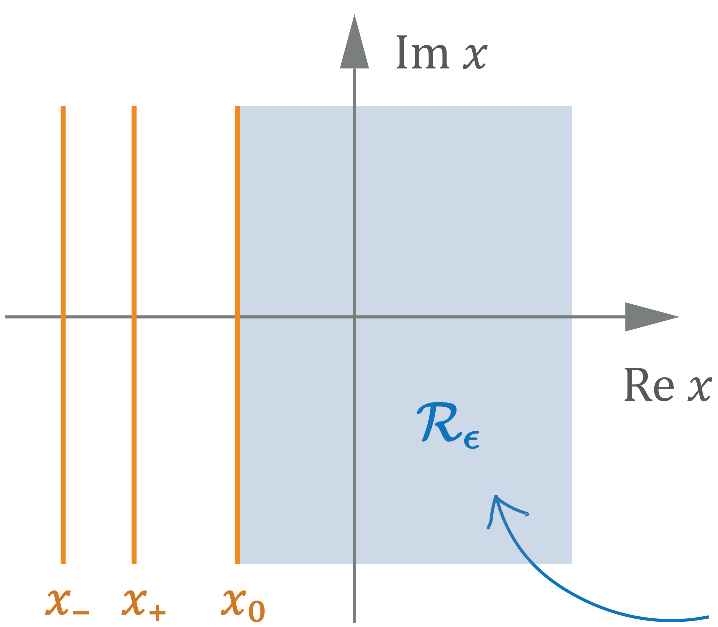
$$w_\pm = \pm\left(t + \frac{1+x}{1\pm\alpha}\right) \qquad w_0 = \frac{x+\gamma}{\alpha} \quad \Rightarrow \quad x_\pm = (1\pm\alpha)(\pm w - t) - 1 \qquad x_0 = \alpha w - \gamma$$

$w = -i\infty \ldots i\infty$



$$\int_{-i\infty}^{+i\infty} dw$$

These separate different **regions**
in complex x plane:



As long as we stay
inside this region,
we will get correct
result for the LFWF

Euclidean variables

Now, go to **Euclidean metric:**

$$k_E = \begin{bmatrix} \boldsymbol{k} \\ k_4 \end{bmatrix} = \begin{bmatrix} \boldsymbol{k} \\ ik^0 \end{bmatrix} \qquad k_3 = \frac{k^+ - k^-}{2}, \qquad k_4 = i\,\frac{k^+ + k^-}{2} \qquad k_E \cdot p_E = -k \cdot p = \boldsymbol{k}_\perp \cdot \boldsymbol{p}_\perp - \tfrac{1}{2}\left(k^- p^+ + k^+ p^-\right)$$

Drop index E.
BSWF depends on two four-vectors k, P :

$$\Psi(q, P) = \Psi(x, \omega, t, \alpha) \qquad q = k + \alpha\,\frac{P}{2}$$

Three Lorentz invariants:

$$x = \frac{k^2}{m^2} = \frac{\boldsymbol{k}_\perp^2}{m^2}$$

$$\omega = \hat{k} \cdot \hat{P} = -\frac{w + \alpha t}{2\sqrt{xt}} \quad (q^-)$$

$$t = \frac{P^2}{4m^2} = -\frac{M^2}{4m^2}$$

⇒ **LFWF & PDA:**

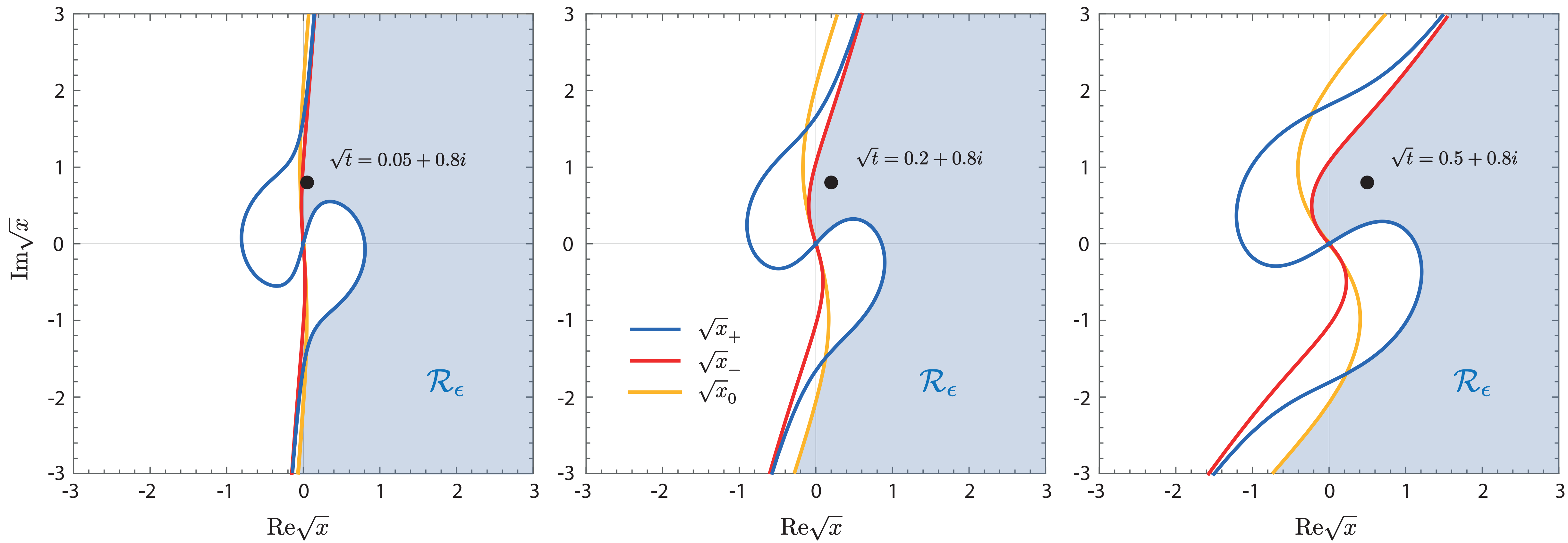
$$\psi(\alpha, x, t) = \frac{\mathcal{N} m^2}{i\pi}\, 2\sqrt{xt} \int_{-\infty}^{\infty} d\omega\, \Psi(x, \omega, t, \alpha)$$

$$\phi(\alpha) = \frac{m^2}{(4\pi)^2 f} \int_0^{\infty} dx\, \psi(\alpha, x, t)$$

- Can be integrated **numerically** (check for monopole: same result)
- BSWF is **Lorentz-invariant**, can be calculated in any frame (also rest frame)
- But now the **branch cuts** in complex x plane will look different, need to stay inside $\mathcal{R}_\epsilon$

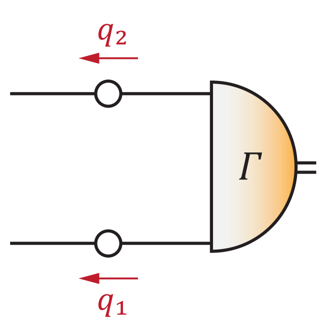
Euclidean variables

Branch cuts in complex $\sqrt{x}$ plane:



Propagator poles

Pole in BS amplitude



- Correct result for LFWF inside $\mathcal{R}_\epsilon$
- Physical region 0 < M < 2m is imaginary axis: $\sqrt{t} = \frac{iM}{2m}$
  Not possible ⇒ must calculate for complex $\sqrt{t}$
- If cuts cross real $\sqrt{x}$ axis, must deform contour to calculate PDA

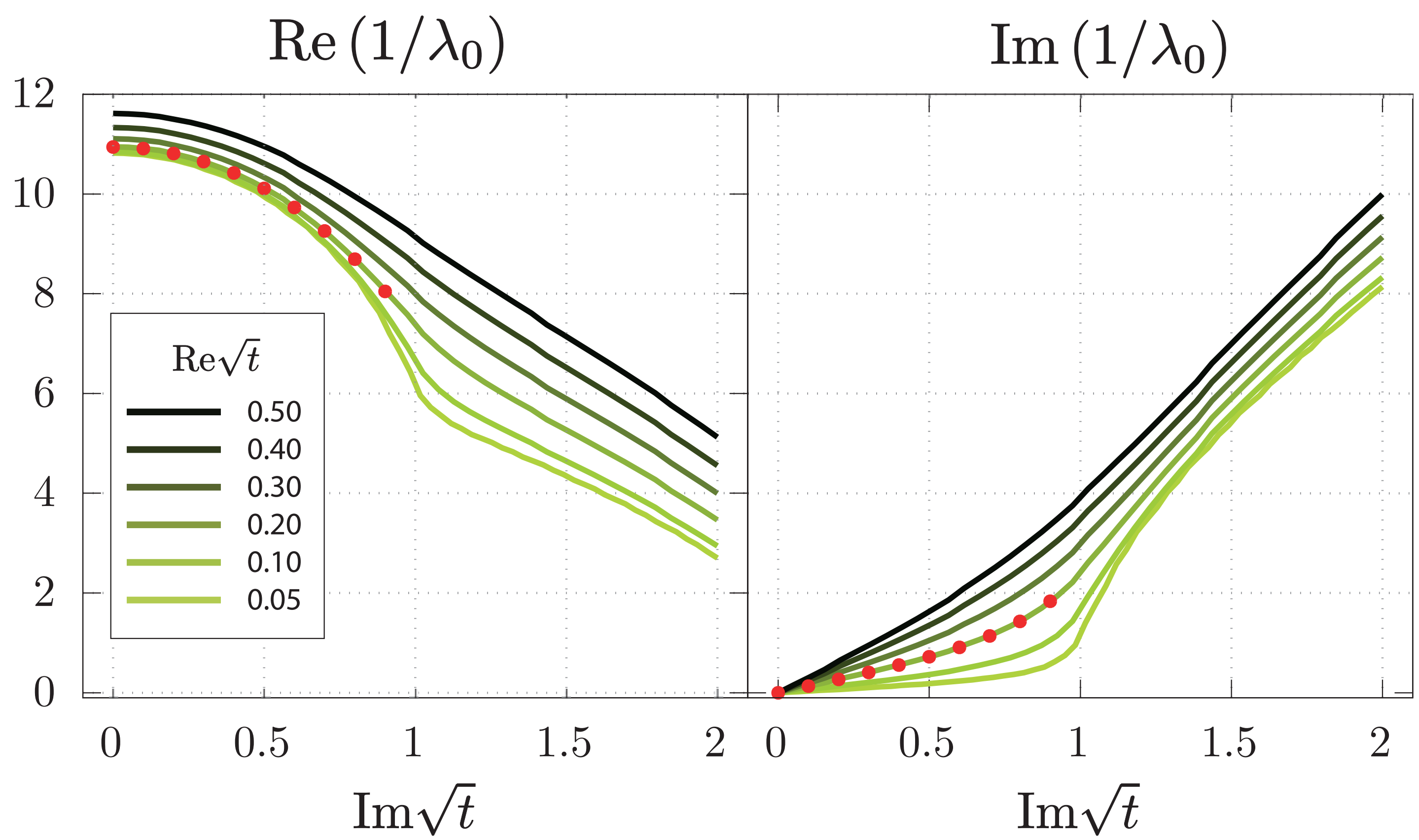
Bethe-Salpeter equation

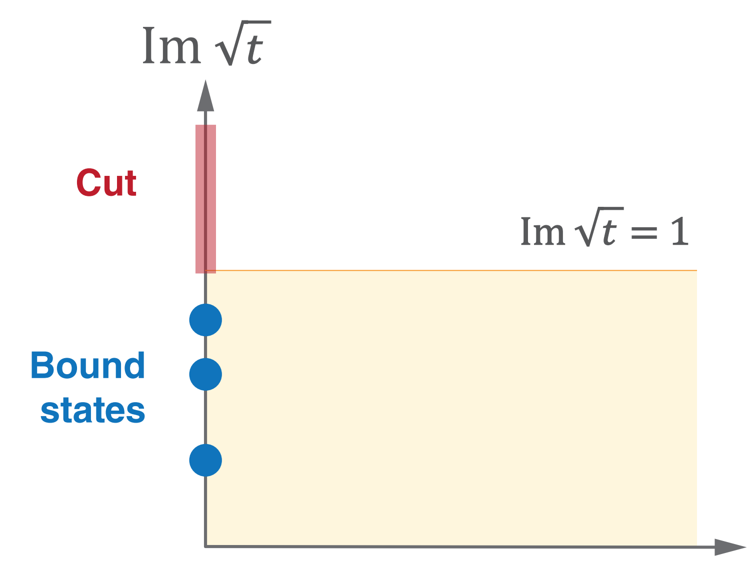
Ultimately, we want to compute BSWF dynamically from its **Bethe-Salpeter equation.**
Here: massive Wick-Cutkosky model (scalar ladder exchange)

Wick 1954, Cutkosky 1954, Nakanishi 1969

$$\Gamma(q,P) = \int \frac{d^4q'}{(2\pi)^4}\, K(q,q',P)\, G_0(q',P)\, \Gamma(q',P) \qquad K(q,q',P) = \frac{g^2}{(q-q')^2+\mu^2} \qquad c = \frac{g^2}{(4\pi m)^2}$$

- Two parameters: overall coupling strength c, mass ratio $\beta = \mu/m$
- Compute eigenvalue spectrum for complex $\sqrt{t} = \frac{iM}{2m}$, bound states if $\frac{1}{\lambda(t)} \overset{!}{=} c$





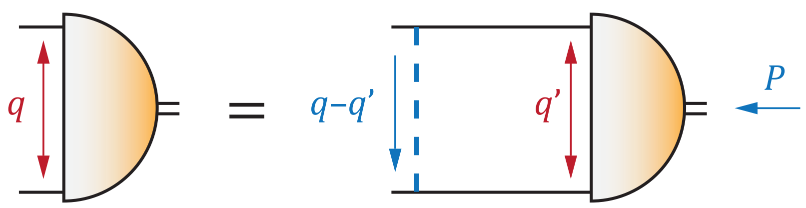
Lines: direct result using contour deformations

Red dots: Nakanishi method

Virtual state if c too small, tachyon if c too large

Eichmann, Duarte, Peña, Stadler, PRD100 (2021)

Singularities in the BSE

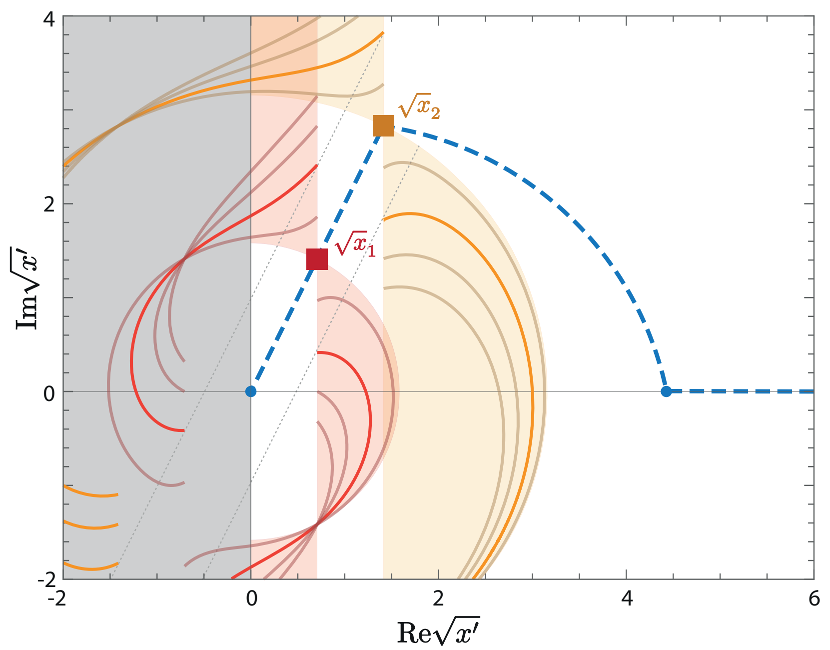


$$\Gamma(x,\omega,t,\alpha) = \int_0^\infty dx' \int_{-1}^{1} d\omega' \int_{-1}^{1} dy\, K(x,x',\Omega)\, G_0(x',\omega',t,\alpha)\, \Gamma(x',\omega',t,\alpha)$$

$$\Omega = \omega\omega' + y\sqrt{1-\omega^2}\sqrt{1-\omega'^2}$$

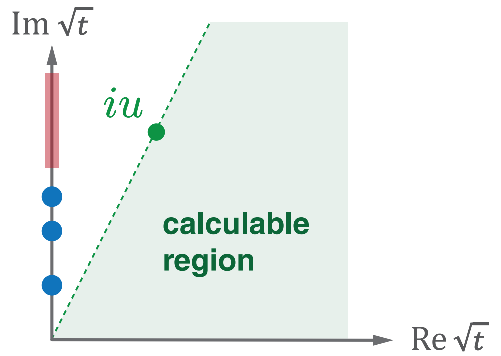
- BSE must be solved for $\sqrt{x} \in \mathcal{R}_\epsilon$
- BSE is **integral equation** ⇒ must be solved along path $\sqrt{x}$ that coincides with **integration path** $\sqrt{x'}$, must lie inside $\mathcal{R}_\epsilon$ ⇒ need contour deformations

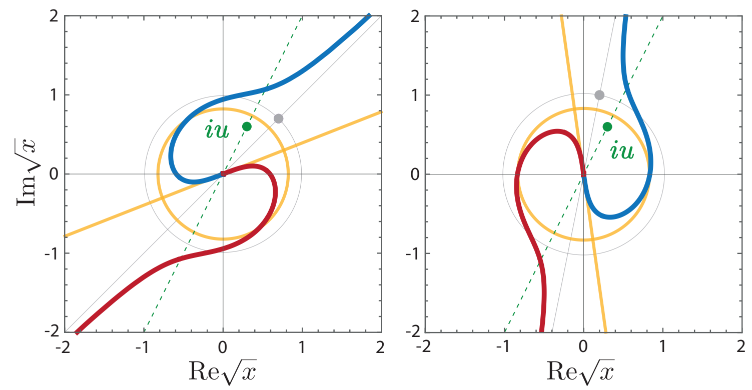
- **Kernel** has pole ⇒ after integrating over y and ω', becomes branch cut in $\sqrt{x'}$, automatically avoided if Re$\sqrt{x'}$ and $|\sqrt{x'}|$ increase along integration path



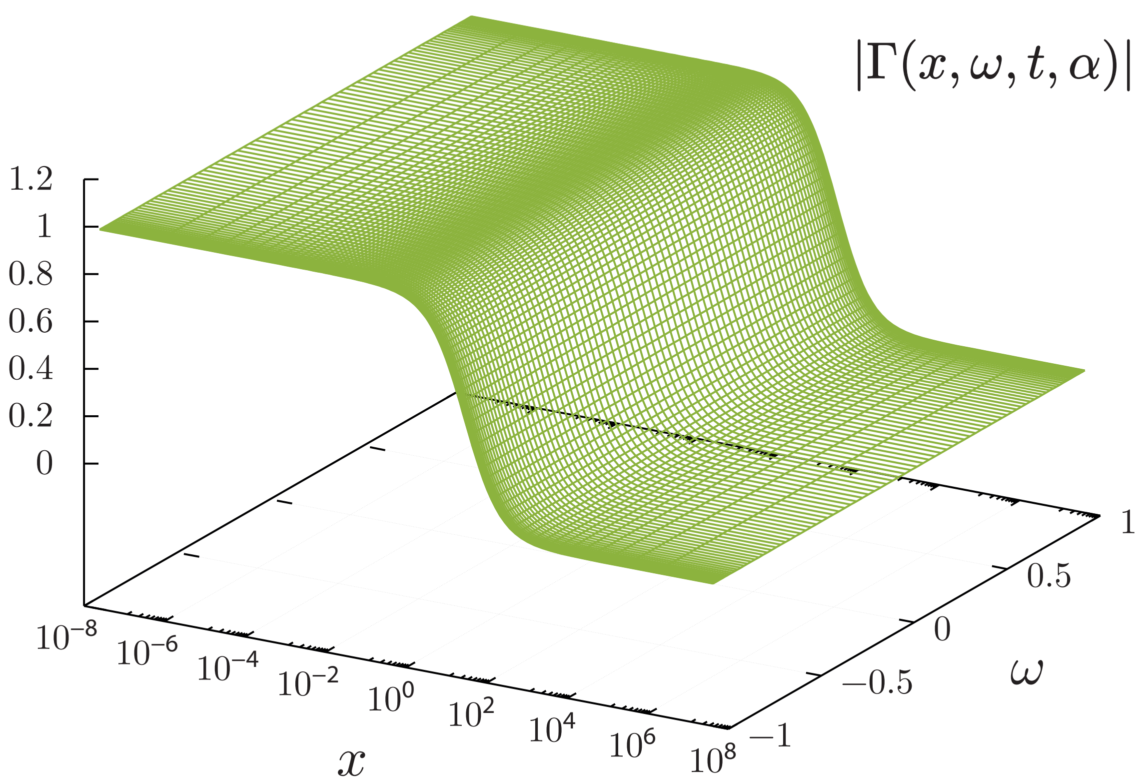
- **Propagators** have poles ⇒ branch cuts from before, automatically avoided if integration path connects $\sqrt{x'}$ = 0 ... $(1+|\alpha|)\sqrt{t}$ and goes back to real axis

- **BS amplitude** may dynamically generate singularities at $q^2/m^2 = -u^2$ ⇒ avoided as long as arg $\sqrt{t}$ < arg(iu)





Bethe-Salpeter amplitude



$\alpha = 0.6$,
$\sqrt{t} = 0.2 + 0.2i$

- BS amplitude falls off like 1/x
- only weak dependence on ω and α
  $\Rightarrow$ α dependence in LFWF comes from propagators:

$$\psi(\alpha,x,t) \propto \int_{-\infty}^{\infty} d\omega\, G_0(x,\omega,t,\alpha)\,\Gamma(x,\omega,t,\alpha)$$

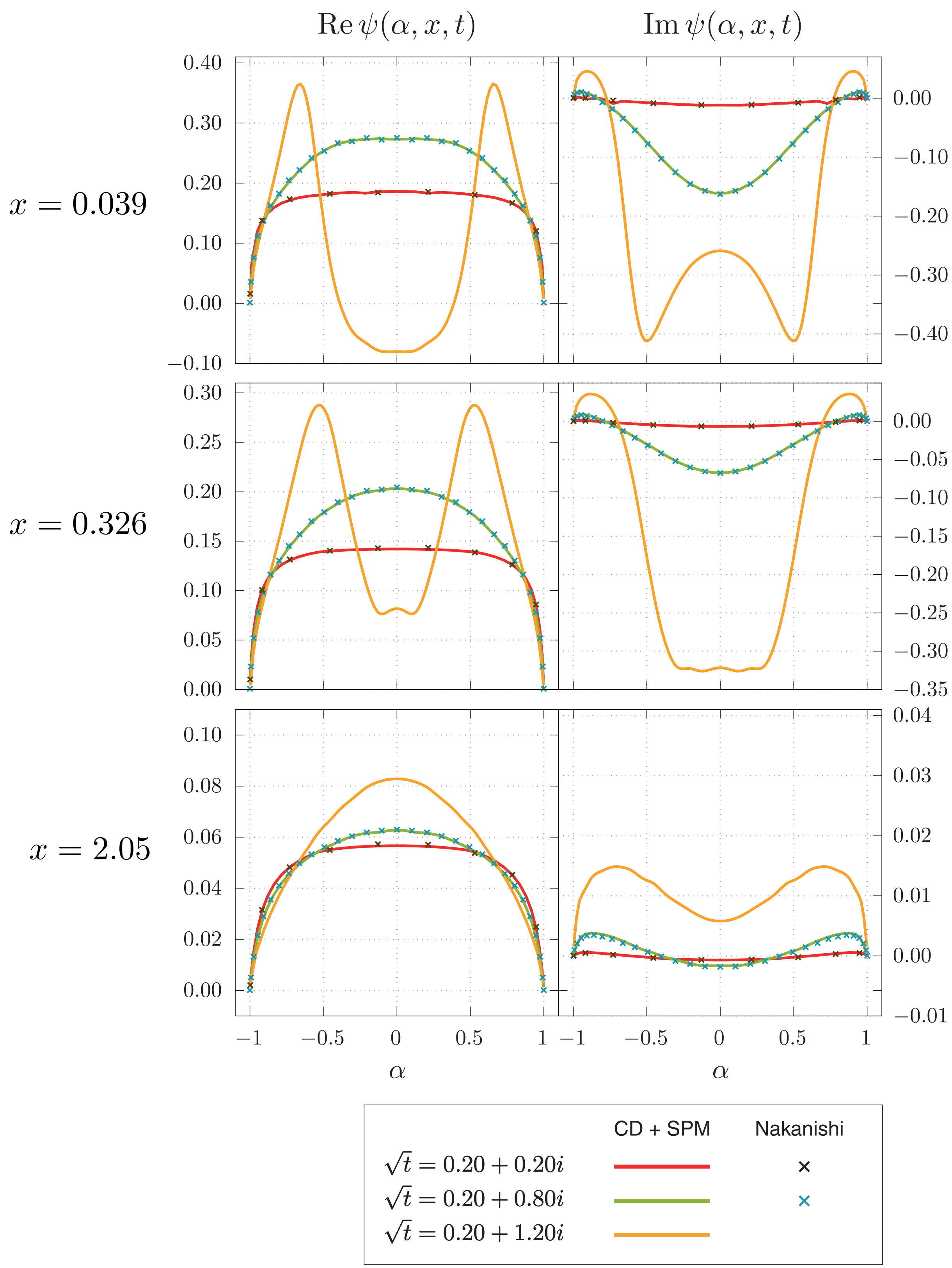
- LFWF needs Γ for $\omega \in (-\infty,\infty)$, but
  BSE solution only known for $\omega \in [-1,1]$

  Use **Schlessinger point method** (SPM)
  for analytic continuation
  Schlessinger, Phys. Rev. 167 (1968)

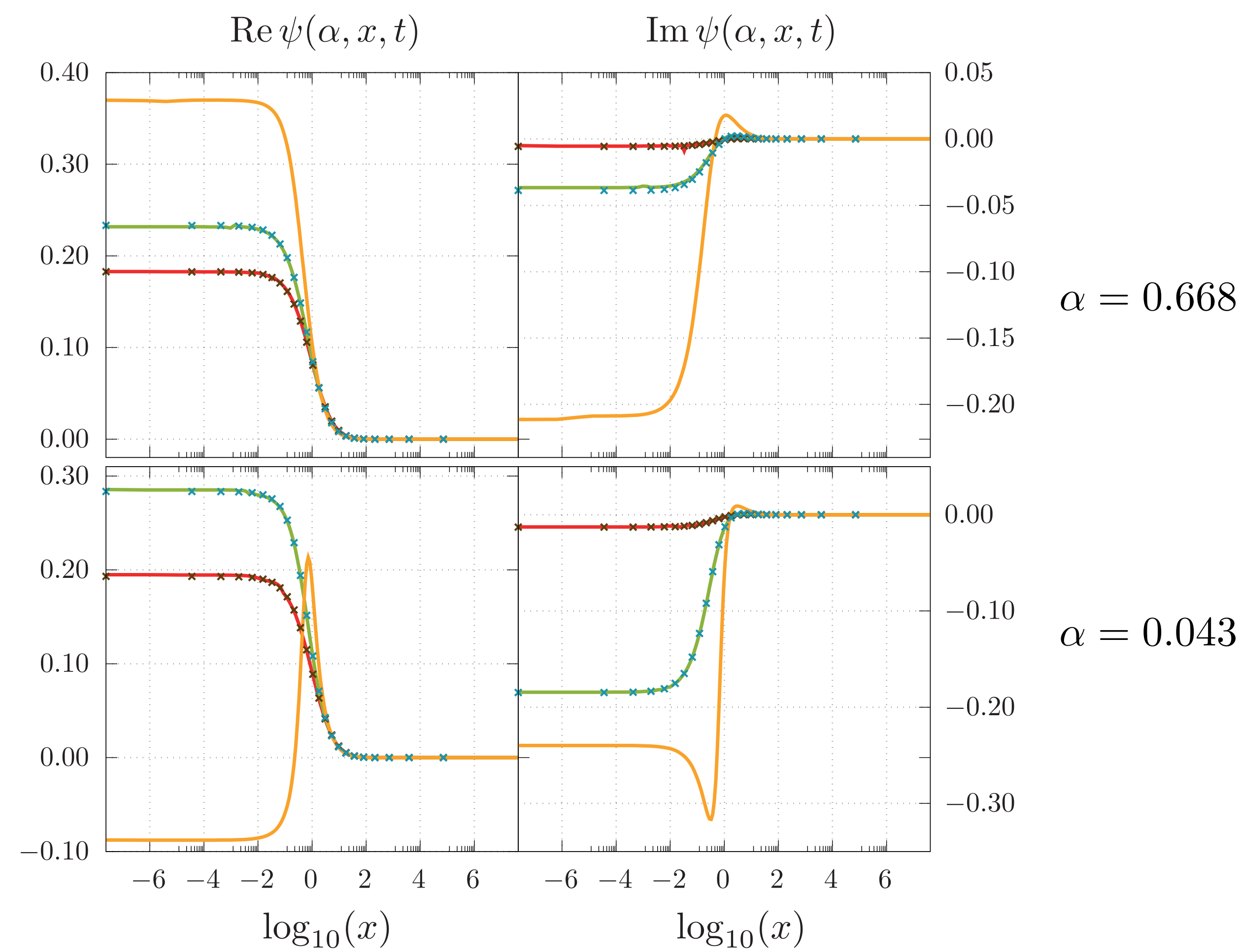
$$f(\omega) = \cfrac{c_1}{1 + \cfrac{c_2\,(\omega-\omega_1)}{1 + \cfrac{c_3\,(\omega-\omega_2)}{1 + \cfrac{c_4\,(\omega-\omega_3)}{\cdots}}}}$$

Light-front wave function

For fixed x



For fixed α

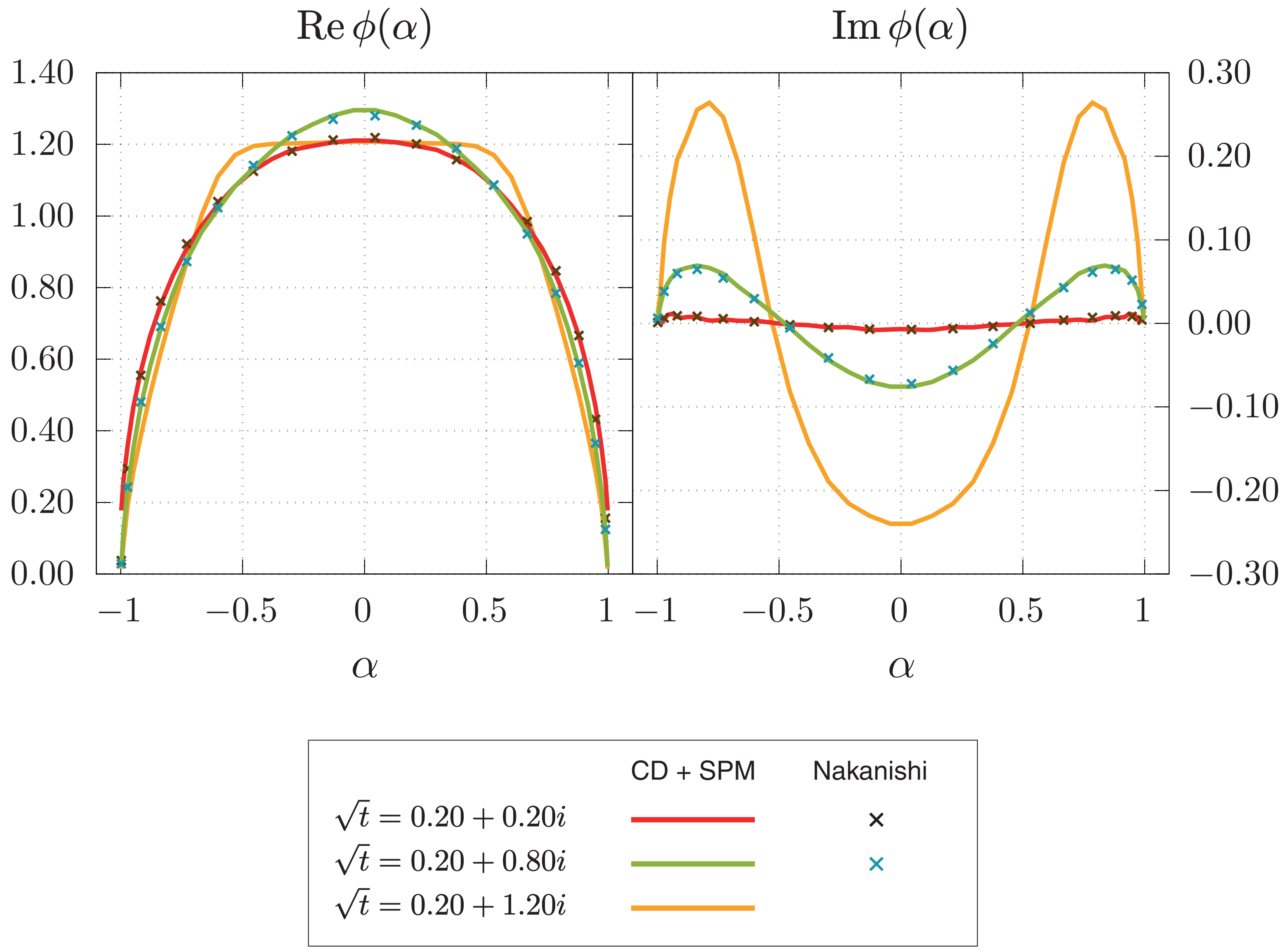


- Results agree with Nakanishi method
- LFWF vanishes at endpoints $\alpha = \pm 1$
- No expansion in moments involved, plain numerical result
- Also works above threshold (unphysical, no poles on 1st sheet, no resonances either)

Distribution amplitude

$$\phi(\alpha) = \frac{m^2}{(4\pi)^2 f} \int_0^\infty dx\, \psi(\alpha, x, t)$$

integrate along
previous contour



Unequal masses

Straightforward to implement:

$$G_0(q,P) = \frac{1}{q_1^2 + m_1^2}\,\frac{1}{q_2^2 + m_2^2} \qquad\qquad m_1 = m\,(1+\varepsilon)\,, \quad m_2 = m\,(1-\varepsilon)$$

Results depend on mass difference ε,
contour deformation works in same way

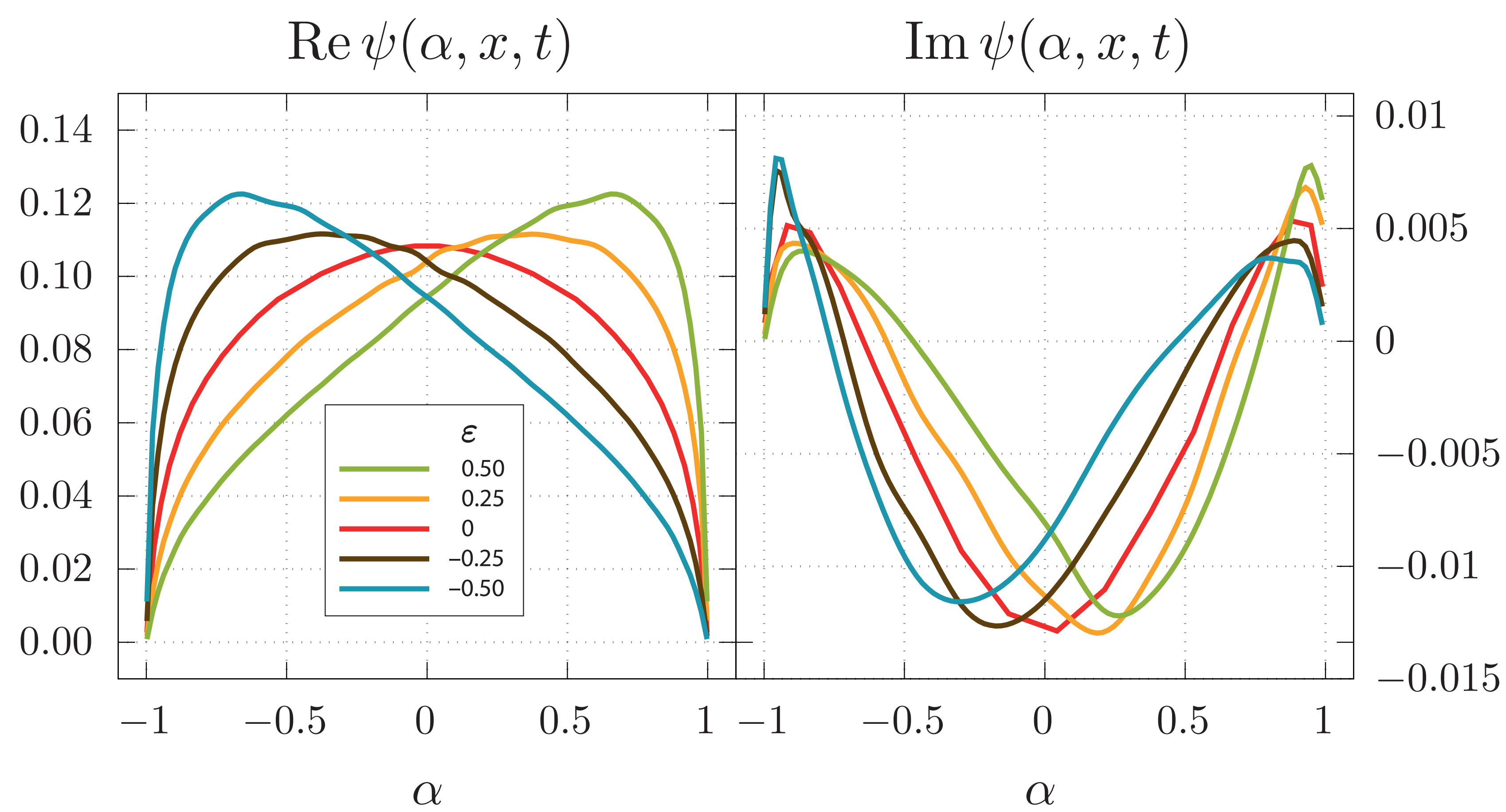
LFWF is no longer symmetric in α:

For:

$x = 1.22$

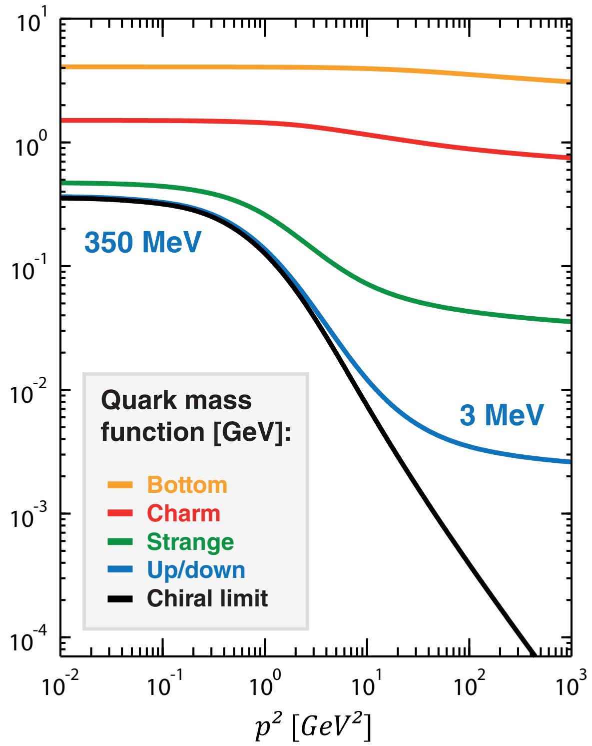
$\sqrt{t} = 0.2 + 0.8i$

$\beta = 4$



Complex conjugate poles

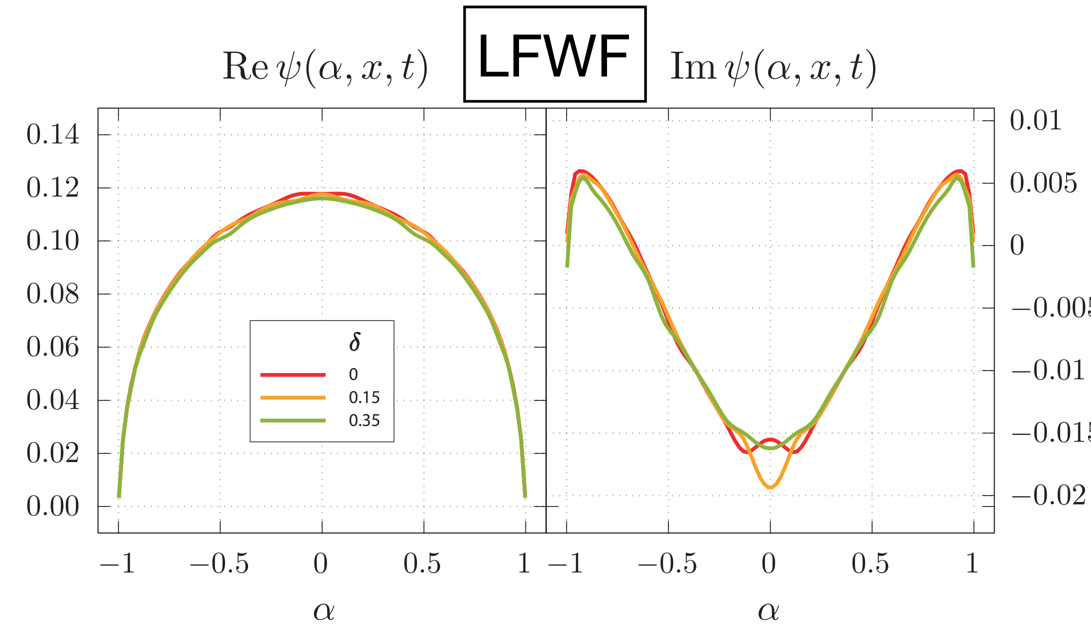
Complex conjugate poles in quarks and gluon propagators (typical in QCD)



Straightforward to implement in scalar model, contour deformations work in same way

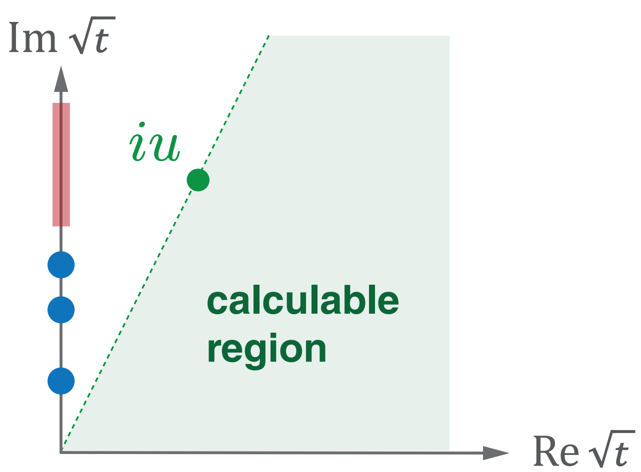
$$D(q^2) = \frac{1}{2}\left(\frac{1}{q^2 + m^2\,(1 + i\delta)} + \frac{1}{q^2 + m^2\,(1 - i\delta)}\right)$$

$$= \frac{q^2 + m^2}{(q^2 + m^2)^2 + m^4\,\delta^2}$$



One does not even need to know pole positions

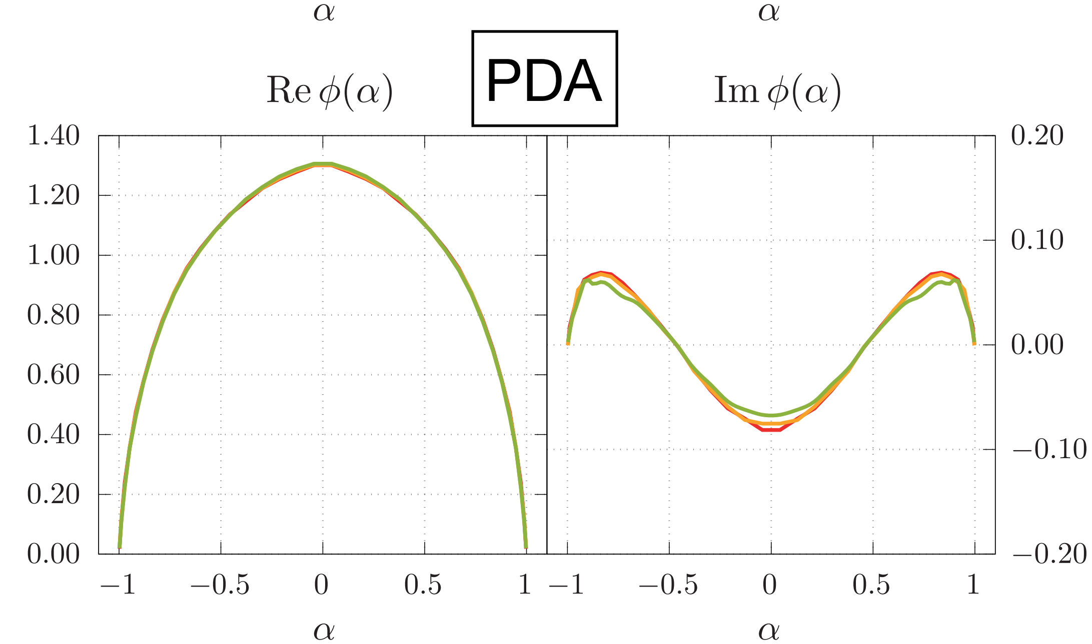
$q^2_{1,2}/m^2 = -u^2$ $\Rightarrow$ avoided as long as arg $\sqrt{t}$ < arg(iu)



works for general singularities in n-point functions!

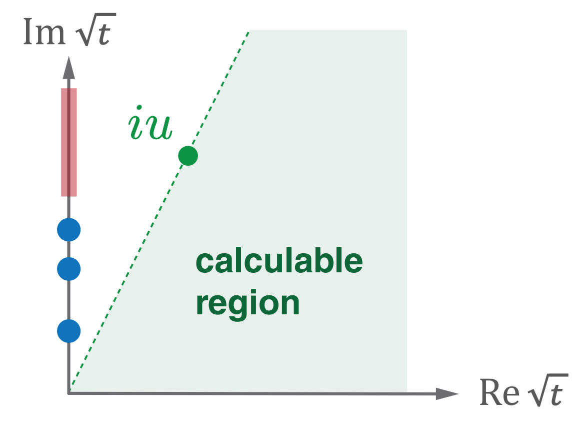
$\sqrt{t} = 0.2 + 0.8i$

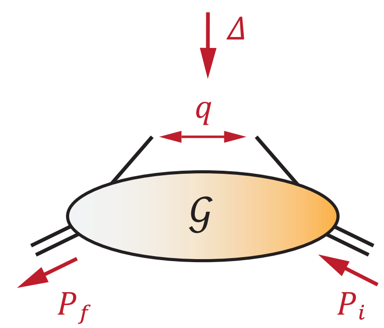
$\beta = 4 \quad x = 0.95$

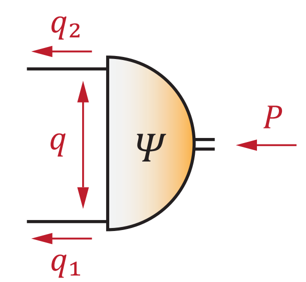


Summary

- We explored a new method to calculate LFWF and PDA, based on contour deformations
  Eichmann, Ferreira, Stadler, PRD **105**, 034009 (2022)
- Numerical integration is possible without knowledge of the exact location of singularities in the integrand (as long as confined to a certain region)
- The method works well in practice, also for unequal masses and complex conjugate propagator poles
- Proof of concept for scalar model, but should also work for more general cases







$\mathcal{G}(z,P,\Delta) = \langle P_f |\, \mathsf{T}\,\Phi(z)\,\mathcal{O}\,\Phi(0)\,| P_i\rangle$

$\Psi(z,P) = \langle 0 | \mathsf{T}\,\Phi(z)\,\Phi(0) | P\rangle$

	$\mathcal{G}(q,P,\Delta=0)$	$\mathcal{G}(q,P,\Delta)$	$\Psi(q,P)$
$\int dq^-$	TMD	GTMD	LFWF
$\int d^2\boldsymbol{q}_\perp \int dq^-$	PDF	GPD	PDA