Study of Ising model and three state Pott's model cumulants in 2D lattice

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*** WORK IN PROGRESS

Outline:

- Motivation
- Ising and Pott's model
- Observables
- Analysis details
- Results
- Summary and outlook

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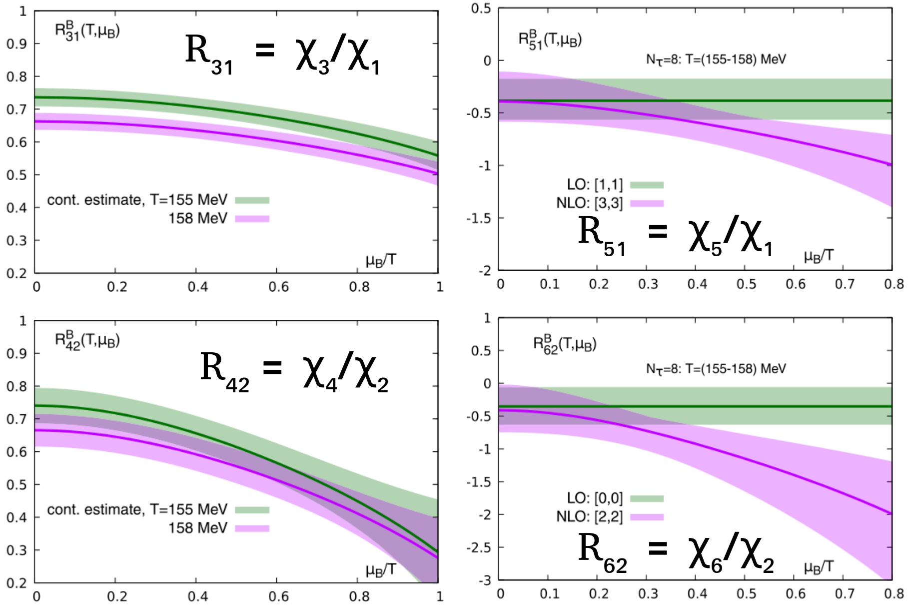
Motivation

First principle LQCD predicts, for small μ_B, near T_C

$$\frac{\chi_6^B}{\chi_2^B} < \frac{\chi_5^B}{\chi_1^B} < \frac{\chi_4^B}{\chi_2^B} < \frac{\chi_3^B}{\chi_1^B}$$



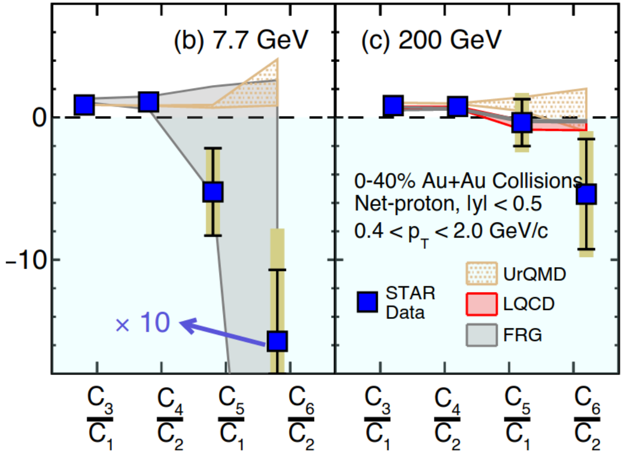
HotQCD: PRD 101,074502 (2020)

Experiment

Ordering of net baryon ratios at STAR

$$\frac{C_6}{C_2} < \frac{C_5}{C_1} < \frac{C_4}{C_2} < \frac{C_3}{C_1}$$

$$\frac{\chi_m}{\chi_n} = \frac{C_m}{C_n}$$



STAR: arXiv:2207.09837

We want to understand the ordering of susceptibility χ using simple models having phase-transition at (i) high temperature (ii) low temperature and its (iii) volume dependence

Overview of models

Ising model

- For a d-dimensional lattice, each lattice site having spin σ_k = {+1,-1}, energy of the lattice configuration

$$\mathcal{H} = -\mathcal{J} \sum_{\langle i,j \rangle} \sigma_i \sigma_j$$

Stephen G. Brush, Rev. Mod. Phys. 39, 883

- Order-parameter m where N is the number of lattice site in a lattice configuration

$$m = \frac{\sum_{i=1}^{N} \sigma_i}{N}$$

- Critical temperature T_c of Ising model in d = 2 lattice

$$T_C = \frac{2}{\ln(1+\sqrt{2})} = 2.269 \ \mathrm{J}/k_\mathrm{B}$$

Lars Onsager, Phys. Rev. 65, 117

q=3 state Pott's model

- For a d-dimensional lattice, each lattice site having spin σ_k = {0,1,2}, energy of the lattice configuration

$$\mathcal{H} = -\mathcal{J} \sum_{\langle i,j \rangle} \delta(\sigma_i, \sigma_j)$$

R.B. Potts, Proc. Cambridge Philos. Soc. 48 (1952) 106

- Order-parameter m where N is the number of lattice site in a lattice configuration

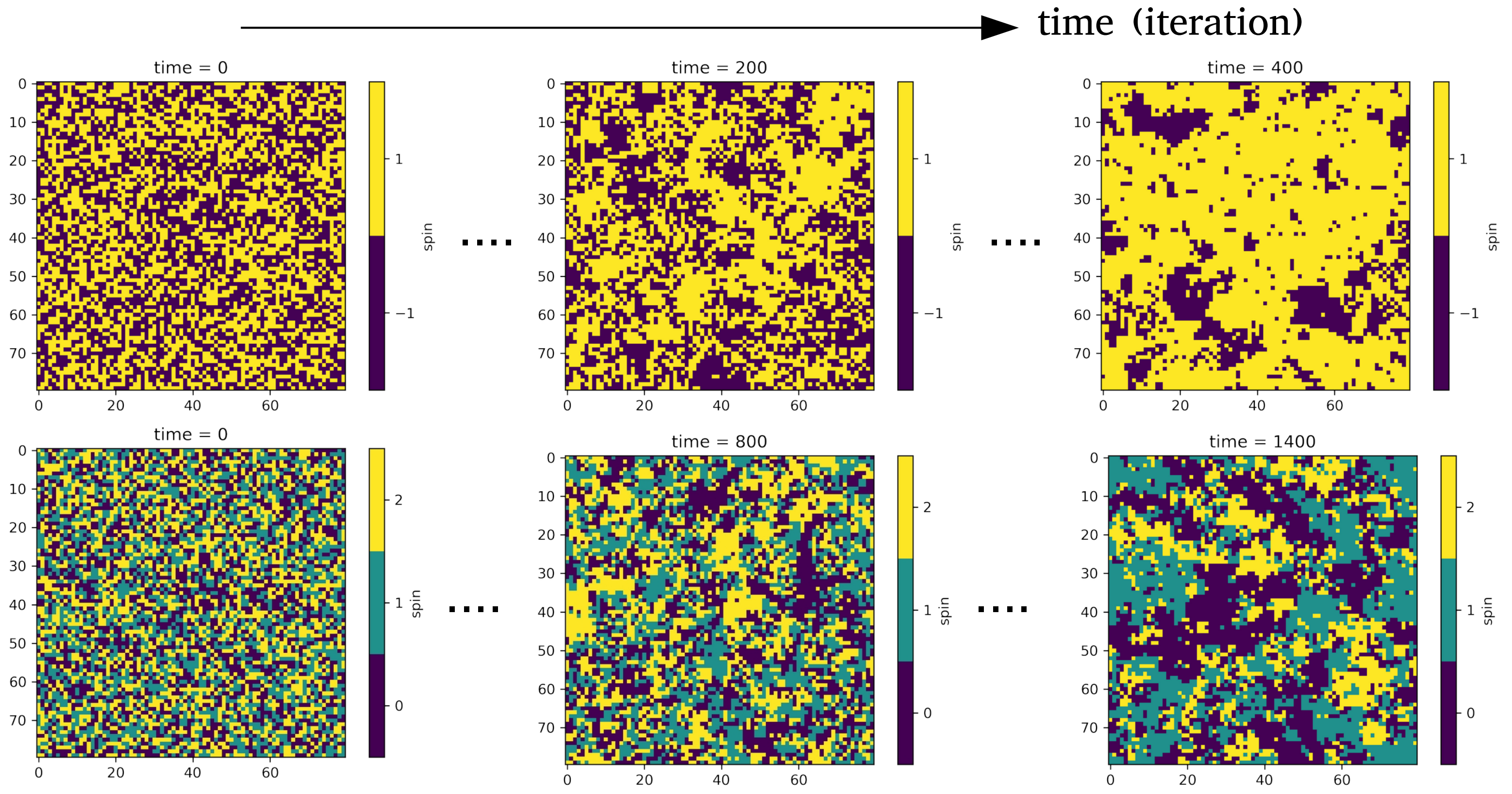
$$m = \frac{3 \cdot \max(N_0, N_1, N_2) - N}{2N}$$

F. Y. Wu, Rev. Mod. Phys. 54, 235

- Critical temperature T_c of q=3 state Pott's model in d = 2 lattice

$$T_C = \frac{1}{\ln(1+\sqrt{3})} = 0.995 \ \mathrm{J}/k_\mathrm{B}$$

Simulation in 2D lattice



- Ising model
  Temp = 2.26

- q=3 Pott's model
  Temp = 0.98

Order parameter m for each lattice configuration (after reaching equilibrium) is calculated at temperature T. Cumulants of m are further calculated for different T.

Observables

Cumulants of order parameter m :

$$C_1 = \langle m \rangle$$
$$C_2 = \mu_2 N$$
$$C_3 = \mu_3 N$$
$$C_4 = (\mu_4 - 3\mu_2^2)N$$
$$C_5 = (\mu_5 - 10\mu_3\mu_2)N$$
$$C_6 = (\mu_6 - 15\mu_4\mu_2 - 10\mu_3^2 + 30\mu_2^3)N$$

$$\langle m \rangle = \frac{1}{S_{\text{tot}}} \sum_{k=1}^{S_{\text{tot}}} m_k$$

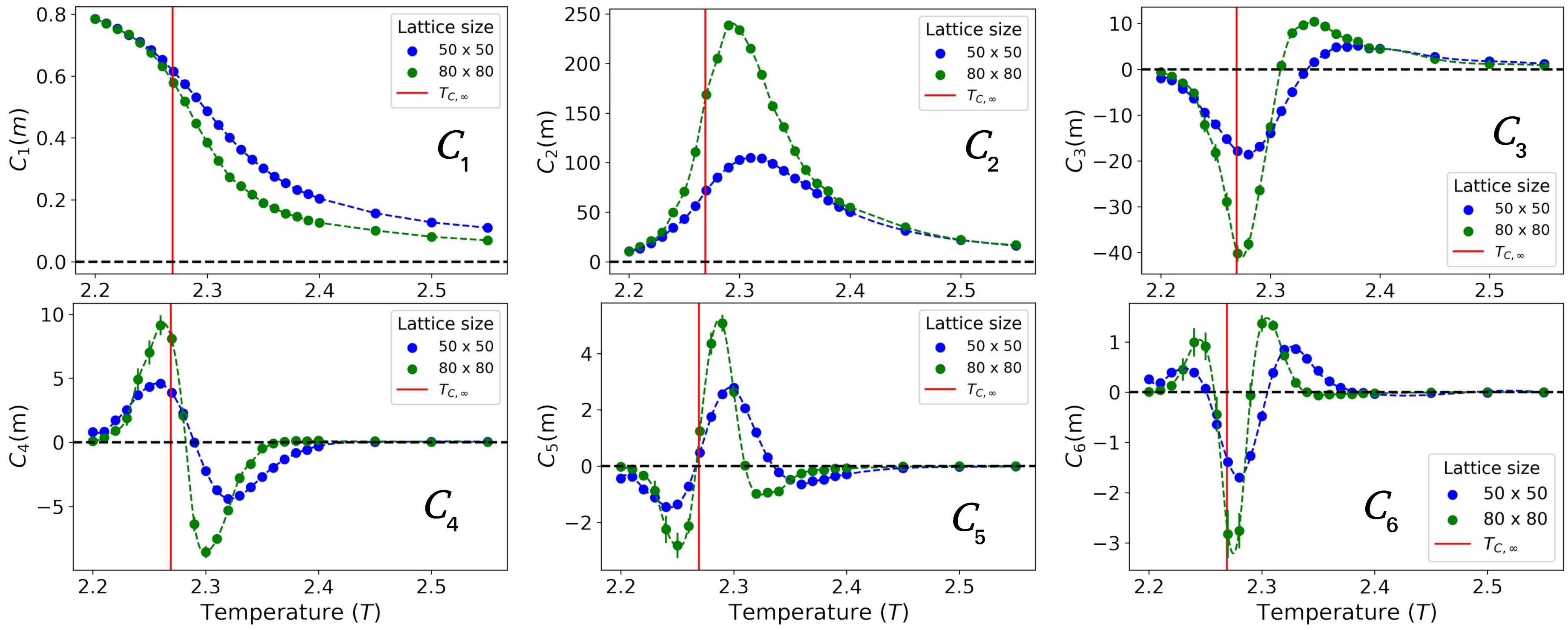
$$\mu_n = \langle (m - \langle m \rangle)^n \rangle$$

- m_k is order parameter in the k^{th} state of lattice at a temp (T)
- S_{tot} = total no of states
- N = total no of lattice sites

Model	Ising model		q = 3 state Pott's model	
Lattice size (L)	50 x 50	80 x 80	50 x 50	80 x 80
States (iteration)	2×10^5	2×10^5	2×10^5	2×10^5
Independent states	9000	9000	3600	3600

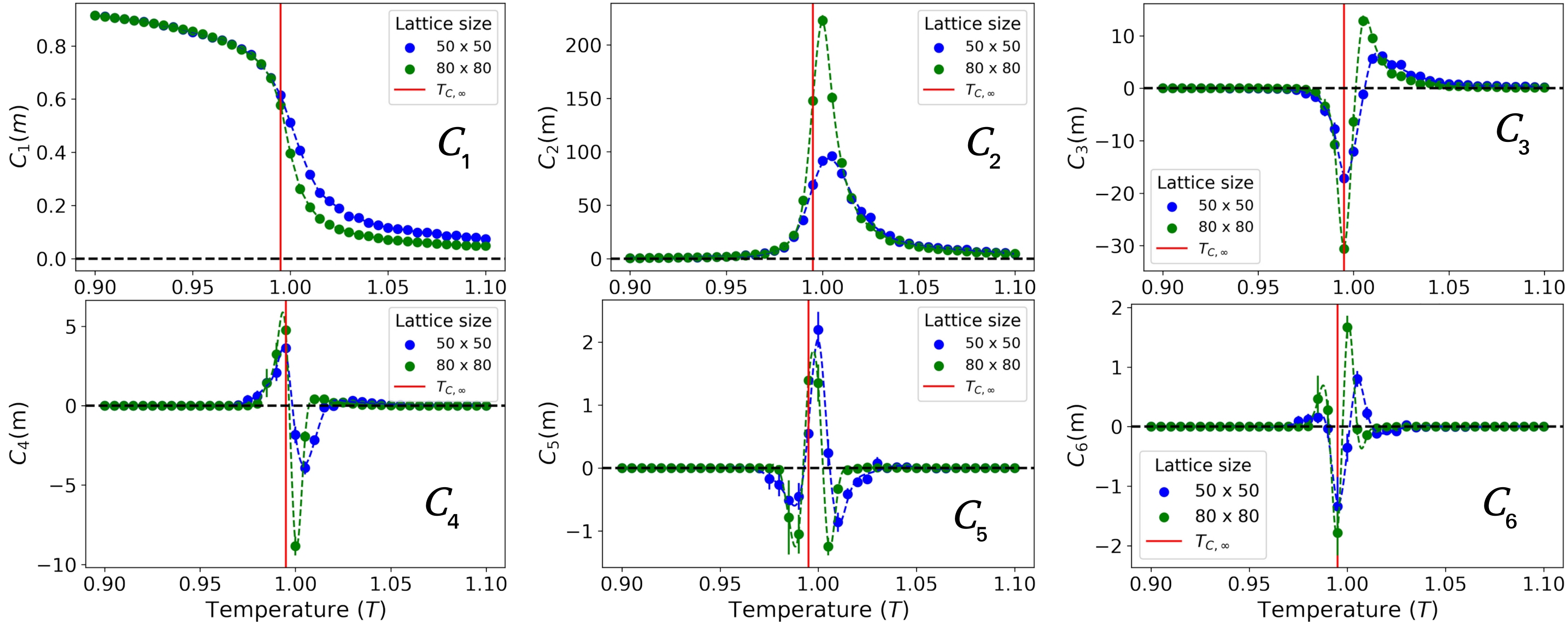
For each lattice, simulation performed for different T near critical temperature T_C .
(i) for Ising model: T in [2.2, 2.8] J/k_B, (ii) for q=3 state Pott's model: T in [0.9, 1.1] J/k_B

Cumulants in Ising model



- C_2 and higher order cumulants show non-monotonic variation with temperature.
- FWHM decreases with lattice size (for C_2, 37% reduction from $L = 50$ to $L = 80$). For all higher-order cumulants as well, the region of critical behaviour narrows with lattice size.

Cumulants in q=3 state Pott's model



→ q=3 state Pott's model also shows non-monotonic variation with *T*.

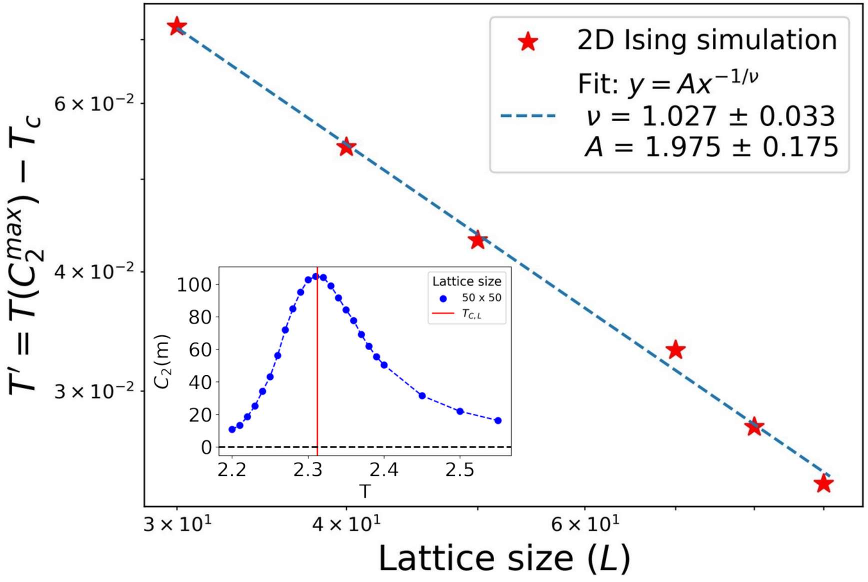
Scaling

Scaling theory: $T_{C,L} - T_{C,\infty} \propto L^{\frac{-1}{\nu}}$

M E. Fisher, 1967 Rep. Prog. Phys. 30 615
M E. Fisher et.al, Phys. Rev. Lett. 28, 1516

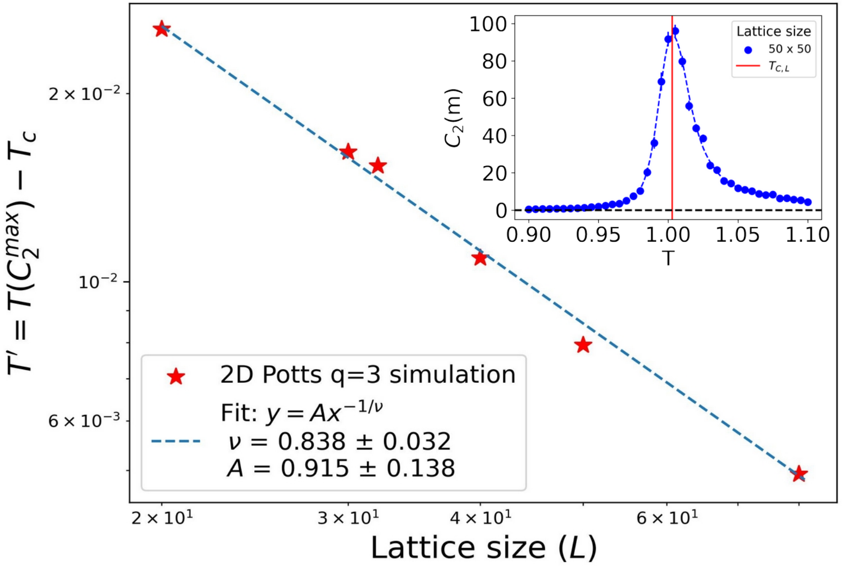
ν is the critical exponent
- ν = 1 (Ising)
- ν = 5/6 ~0.833 (q=3 Pott's)

Ising model



Simulation: $\nu = 1.027 \pm 0.033$

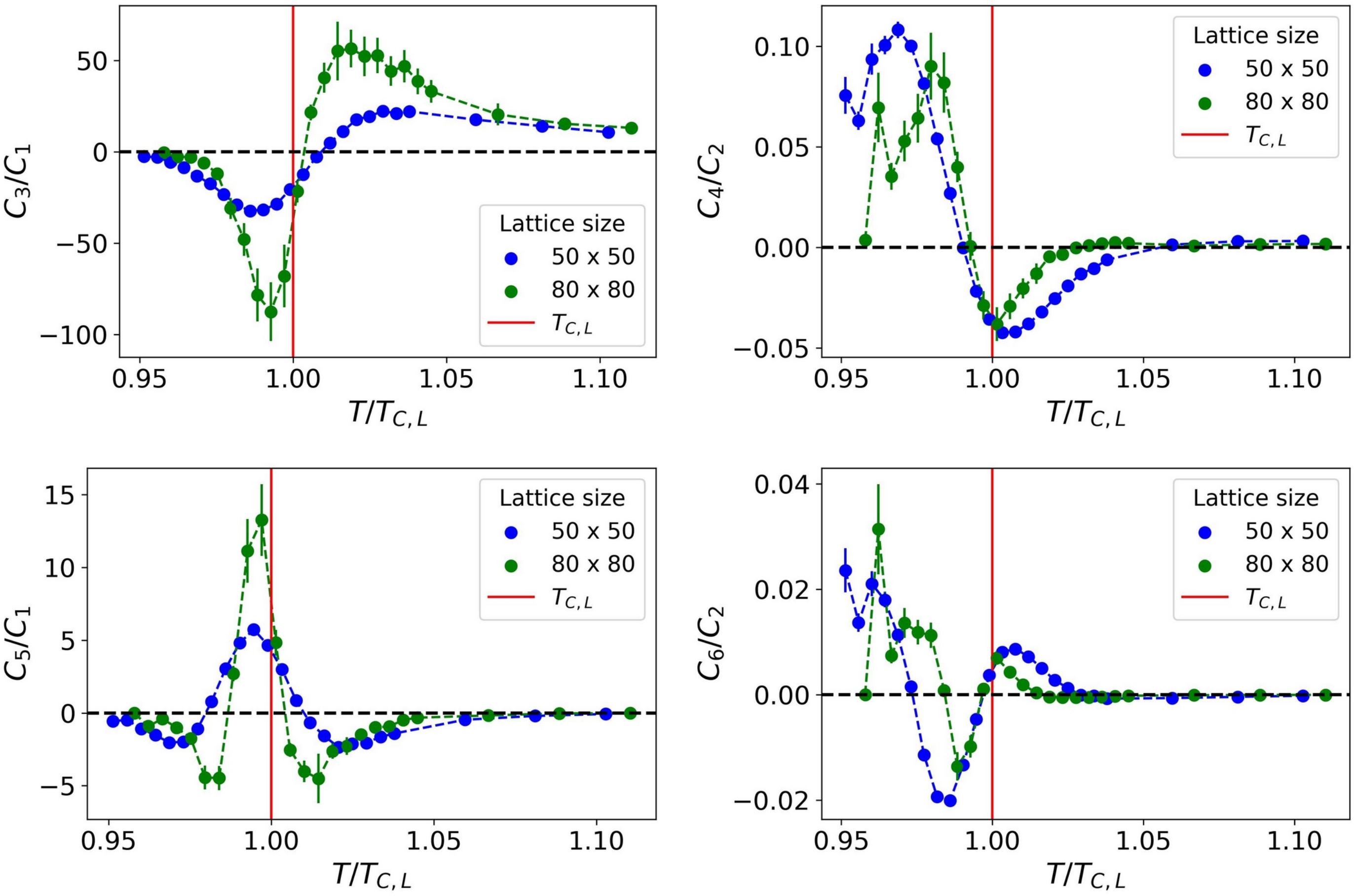
q=3 state Pott's model



Simulation: $\nu = 0.838 \pm 0.032$

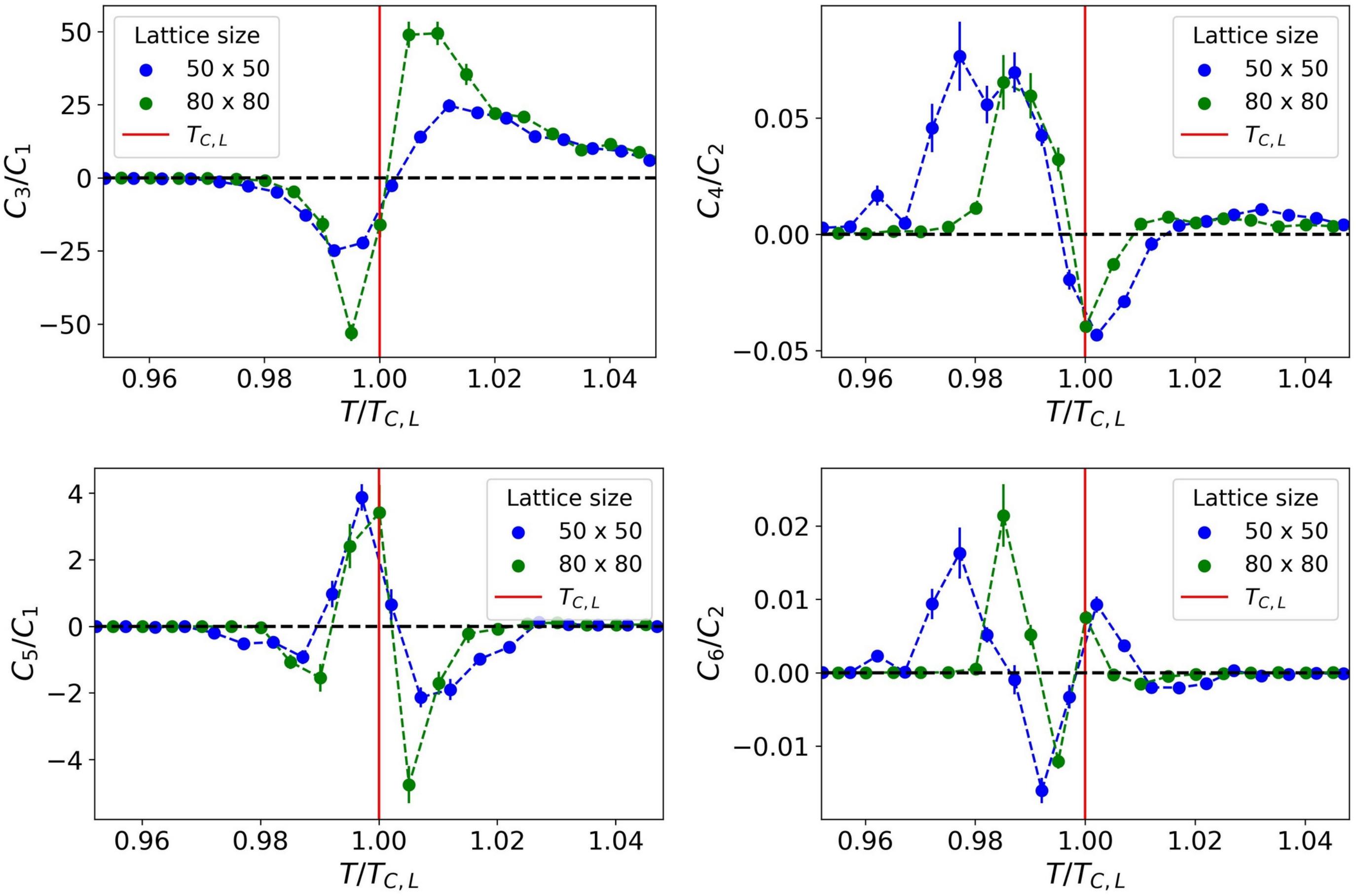
→ Critical exponents are in **good agreement** with theory. $T_{C,L}$ obtained for lattices in both models.

Ratio of cumulants in Ising model



- Cumulant ratios show a non-monotonic variation with $T/T_{C,L}$.
- The critical region is volume dependent, it narrows with increase in lattice size.
- Uncertainities are statistical (computed using sub-sampling method)

Ratio of cumulants in q=3 state Pott's model



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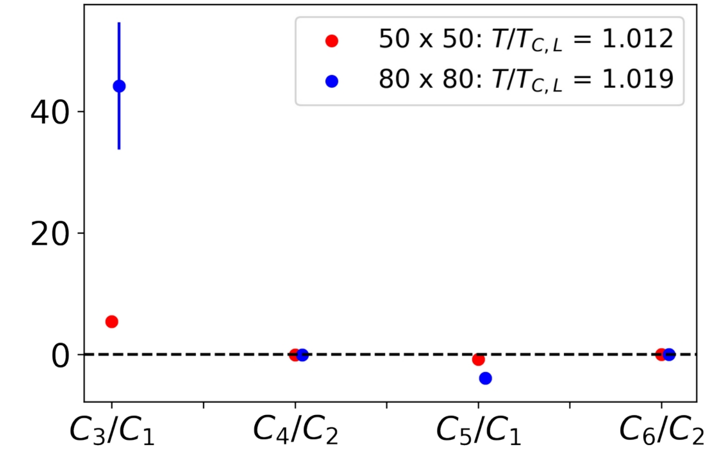
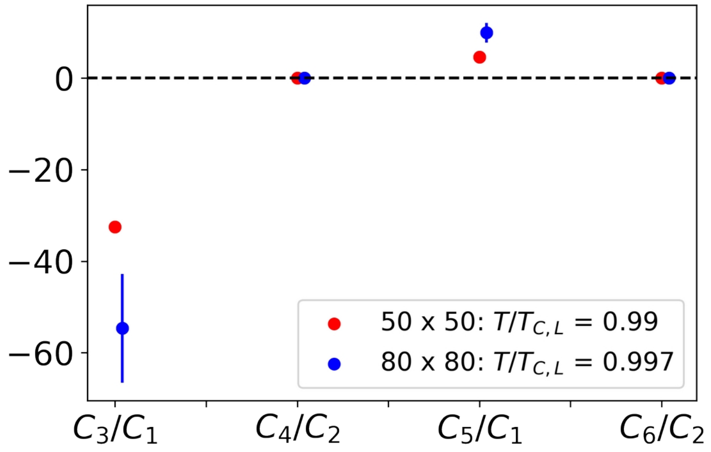
Ordering in Ising model

(above $\sim T_C$)



(below $\sim T_C$)



LQCD prediction ➡ $C_6/C_2 < C_5/C_1 < C_4/C_2 < C_3/C_1$

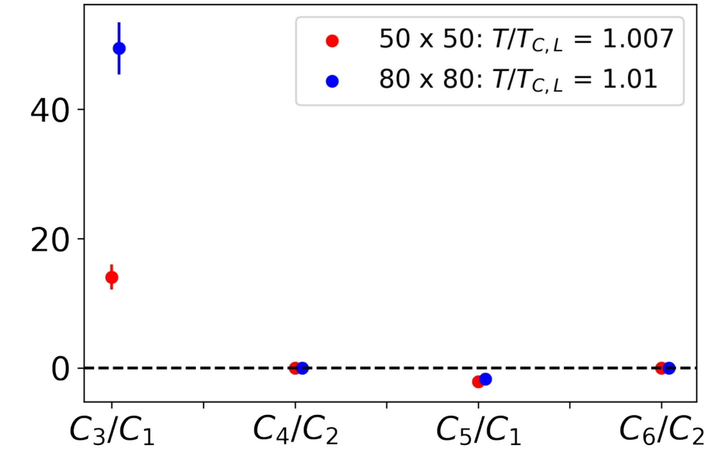
$T_{C,L}$ = 2.312 (50x50)

$T_{C,L}$ = 2.297 (80x80)

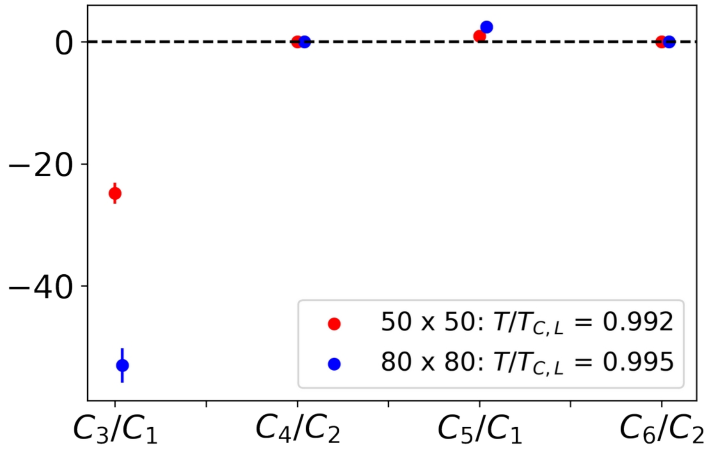
- C_6/C_2 does not follow the ordering of cumulant ratios.
- Ordering is observed from C_3/C_1 to C_5/C_1 in a narrow region close to critical temperature ($T_{C,L}$). For $T > T_{C,L}$ - $C_5/C_1 < C_4/C_2 < C_3/C_1$
- For $T < T_{C,L}$, inequality of the ordering is reversed, - $C_5/C_1 > C_4/C_2 > C_3/C_1$
- Temperature at which inequality is reversed varies with lattice size.

Ordering in q=3 state Pott's model

(above $\sim T_C$)



(below $\sim T_C$)



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- Temperature at which inequality is reversed varies with lattice size.

Summary

- Lattice QCD predicted an ordering of baryon susceptibility χ_B ratios which STAR experiment seems to have observed. QCD critical point in T-μ plane is expected to be Ising class with μ playing the role of magnetic field. Hence, we tried to study the ordering using cumulants of order parameter in spin models.

- Our study shows that the **ordering of cumulant ratios is followed from C_3/C_1 to C_5/C_1 only** above critical temperature. Below that, inequality of the ordering is reversed. Temperature at which inequality is reversed seems to be volume dependent.

- A more realistic simulation would be q=3 state Pott's model in 3-dimensional lattice with magnetic field *B*, where *B* will correspond to chemical potential μ_B. This is a **work in progress.**

Thank You

Backup

Algorithm of simulation

Simulation performed using Wolf-Cluster algorithm Phys. Rev. Lett. 62, 361

(1) Select a random spin configuration of 2d-lattice

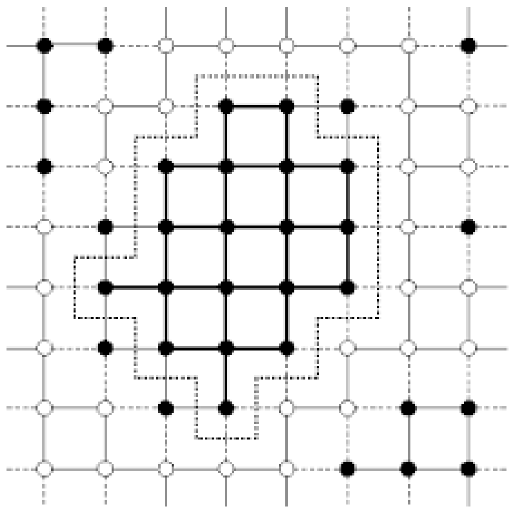
(2) Using random number generator, select a lattice site i randomly

(3) Get the nearest neighbour of i^{th} lattice site, i.e j. For each neighbour j, if spins of i and j are parallel, and the bond is not considered before, then j is added to the cluster with probability $1-\exp(-2\beta J)$

(4) After all i's neighbours, j have been considered, we repeat the same process from step 2 for the j sites that are added to the cluster.

(5) The process is repeated until the cluster grows as much as possible.

(6) The spins of the cluster is inverted.



Statistical error estimation

Errors are estimated using sub-sampling method

- Total available statistics is divided into smaller groups. Each group is a sub-sample. $N = 25$ samples are created in our case.
- C_n is calculated for each sub-sample.
- Statistical error on C_n is given by

$$\sigma_{C_n} = \sqrt{\frac{\sum_{i=1}^{N}(C_{n,i}-C_n)^2}{N(N-1)}}$$

where $C_{n,i}$ is the value of C_n calculated for the ith sub-sample

Auto-correlation time t_r

- Time auto-correlation function is defined by $\varphi(t)$ where $M(0)$ and $M(t)$ are the order parameters a t'=0 and t'=t time

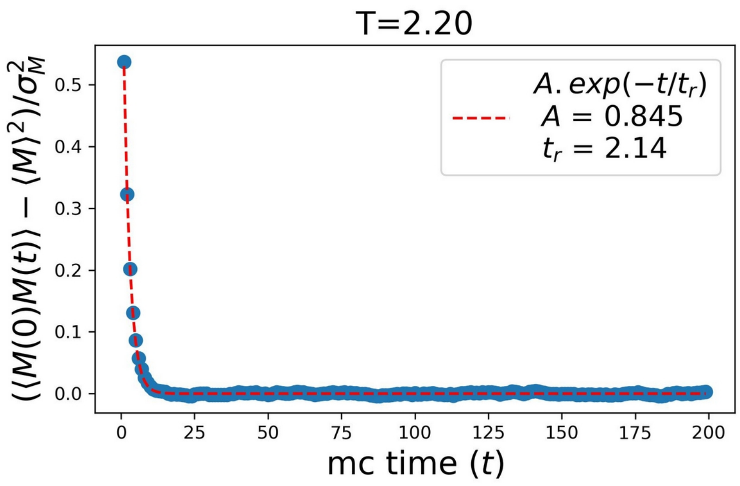
$$\phi(t) = \frac{\langle M(0)M(t)\rangle - \langle M\rangle^2}{\langle M^2\rangle - \langle M\rangle^2}$$

- In discrete case, if we have n number of states produced at different mc-times, M_s and M_{s+i} are the magnetization per spin of states at s and $s+i^{th}$ mc time, the function can be written as

$$\phi_i = \frac{\frac{1}{n-i}\sum_{s=1}^{n-i} M_s M_{s+i} - \langle M\rangle^2}{\langle M^2\rangle - \langle M\rangle^2}$$

- The auto-correlation time can be extracted using the relation

$$\phi(t) = exp\left(-\frac{t}{t_r}\right)$$



Finite size scaling and critical exponents

Characteristic feature of a second-order phase transition is the divergence of the correlation length at a critical temperature $T_C = T_C(\infty)$

$$\xi(T) = \xi_0 + |1 - T/T_C|^{\nu} + ...$$

This leads to the singularities of the specific heat, magnetization ($T < T_C$), susceptibility parameterized by the critical exponents (2D Ising model: $\nu = 1$, $\alpha = 0$, $\gamma = 7/4$, $\beta = 1/8$

$$C(T) = C' + C_0|1 - T/T_C|^{-\alpha} + ... \qquad m(T) = m_0|1 - T/T_C|^{\beta} + ...$$

$$\xi(T) = \xi_0|1 - T/T_C|^{-\gamma} + ...$$

In any numerical simulation the system size is finite, and hence near T_C the role of ξ is taken over by the linear system size L

$$|1 - T/T_C(\infty)| \propto \xi(T)^{-1/\nu} \rightarrow |1 - T/T_C(\infty)| \propto L^{-1/\nu}$$

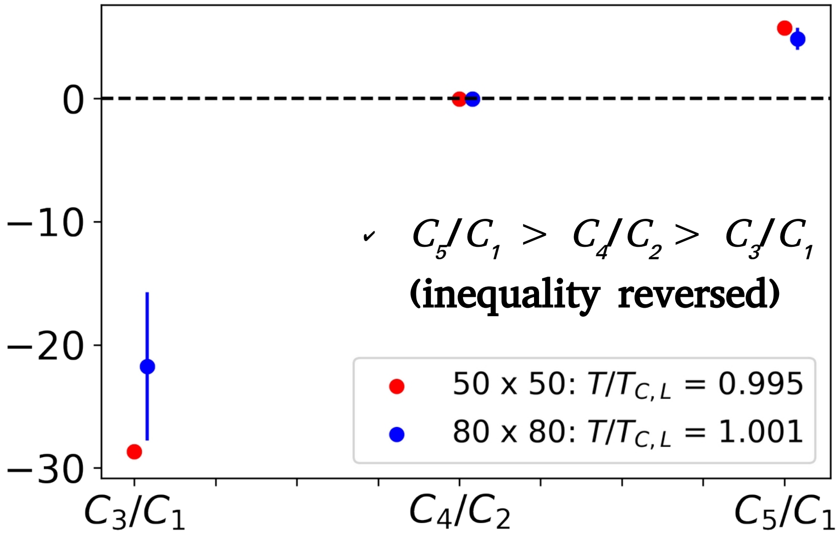
As a critical temperature of the finite lattice T_C(L) we take the location of the specific-heat peak (or susceptibility). This leads to

$$T_{C,L} - T_{C,\infty} \propto L^{\frac{-1}{\nu}}$$

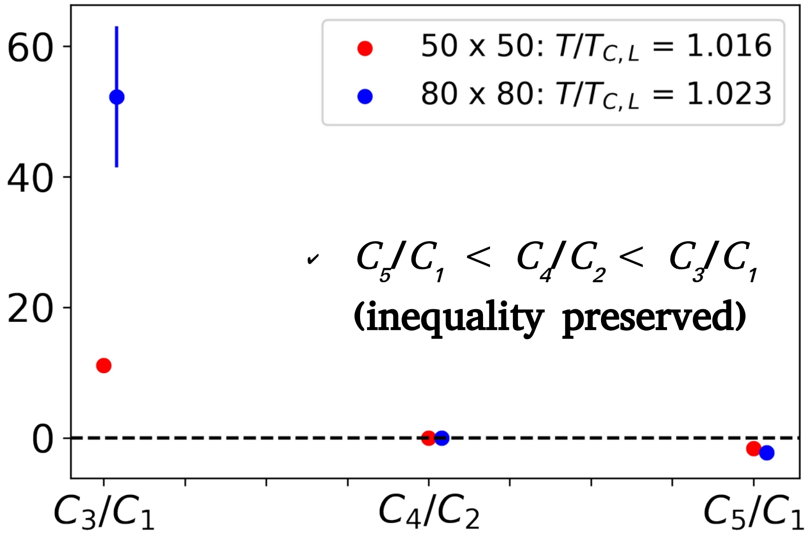
Ordering in Ising model

$T_{C,L}$ = 2.312 (50x50)
$T_{C,L}$ = 2.967 (80x80)

(below critical temperature)
$T/T_{C,L} \leq 1.00$



(above critical temperature)
$T/T_{C,L} \geq 1.01$

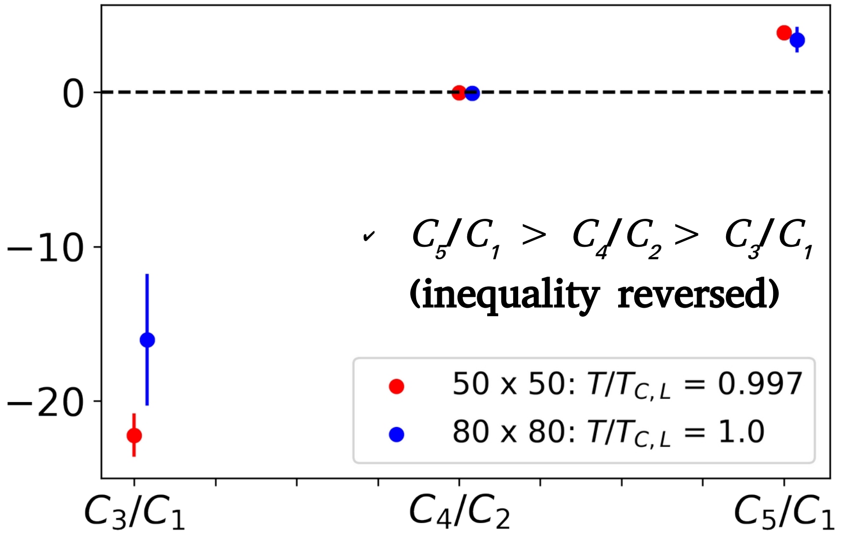


- L = 50 -----> $T/T_{C,L} \leq 1.008$, inequality reversed ($T/T_{C,L} \geq 1.012$, inequality maintained); the boundary line is between (1.008, 1.012), $\sim T = 1.01 T_{C,L}$
- L = 80 -----> $T/T_{C,L} \leq 1.001$, inequality reversed ($T/T_{C,L} \geq 1.006$, inequality maintained); the boundary line is between (1.001, 1.006), $\sim T = 1.003 T_{C,L}$

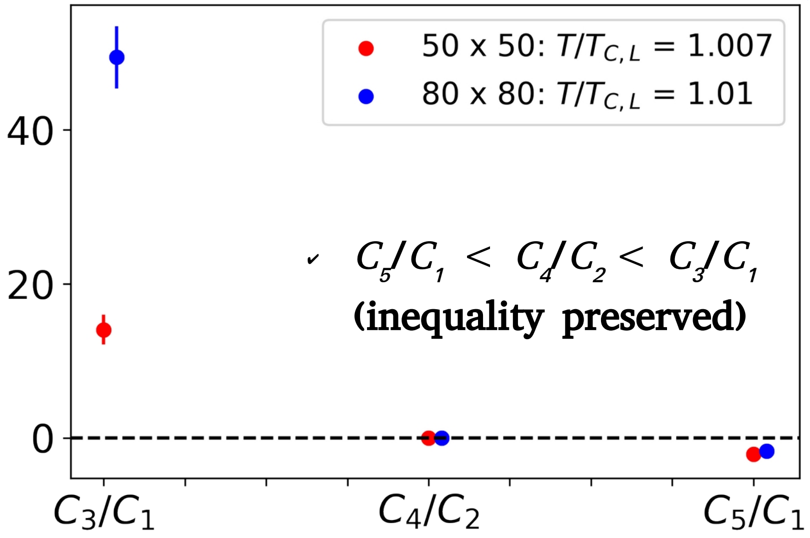
Ordering in q = 3 state Pott's model

$T_{C,L}$ = 1.003 (50x50)
$T_{C,L}$ = 1.000 (80x80)

(below critical temperature)
$T/T_{C,L} \leq 1.00$



(above critical temperature)
$T/T_{C,L} \geq 1.01$



- L = 50 -----> $T/T_{C,L} \leq 1.002$, inequality reversed ($T/T_{C,L} \geq 1.007$ inequality maintained); the boundary line is between (1.002, 1.007), ~ $T = 1.005\,T_{C,L}$
- L = 80 -----> $T/T_{C,L} \leq 1.00$, inequality reversed ($T/T_{C,L} \geq 1.005$, inequality maintained); the boundary line is between (1.0, 1.005), ~ $T = 1.003\,T_{C,L}$