Colour structure of
Multiparton Scattering Amplitudes

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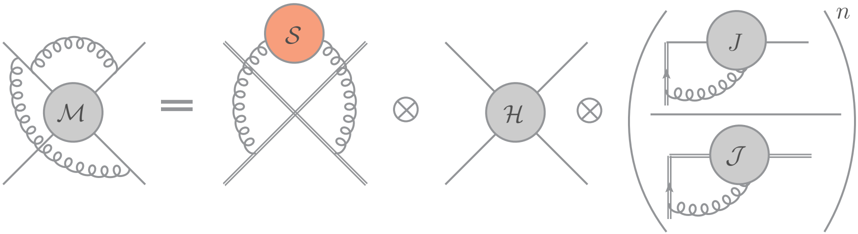
Preview

- Introduction
- Soft Function and Cwebs
- Building blocks of Cwebs
- Summary

Introduction

Factorization

In massless gauge theory amplitude factorizes as...

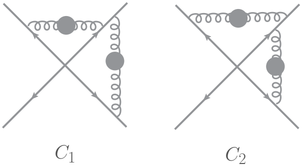


Cwebs

$\mathcal{S}$ is an exponentiation of Feynman diagrams called Cwebs [1] $\mathcal{W}$.

$$\mathcal{S} \; = \; \exp\left[\, \mathcal{W} \,\right]$$

The set of diagrams closed under the shuffle of gluon attachments on Wilson line.



$$\mathcal{W} = \sum_{D} F(D)\, \widetilde{C}(D)\,.$$

$$\widetilde{C}(D) = \sum_{D'} R_{DD'} C(D')$$

Symmetry factor : Number of ways each correlator can be shrunk independently to origin

$$S = \{s(C_1), s(C_2)\} = \{1, 1\}$$

[1] [Agarwal et.al. 2020]

Properties of R

- Web mixing matrix follow zero row sum.

- It satisfy weighted column sum.[1]

$$\sum_d s(d)R(d,d') = 0 \qquad \forall\, d'$$

- Idempotence $R^2 = R$ implies $\mathrm{Tr}(R) = \mathsf{r}(R)$

[1] [Gardi et.al. 2011]

Building Blocks of Cwebs

[JHEP06(2022)020 : Agarwal, Pal, AS, Tripathi]

Diagonal blocks of Mixing matrix

[1] [JHEP06(2022)020 : Agarwal, Pal, AS, Tripathi]

Diagonal blocks of Mixing matrix

Classify diagrams[1]

R

Normal Ordering[1]

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$$R$$

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$$\begin{pmatrix} A & B \\ O & D \end{pmatrix}$$

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Structure of D

Structure of A

[1] [JHEP06(2022)020 : Agarwal, Pal, AS, Tripathi]

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$S \leftrightarrow R$ [1]

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Structure of D

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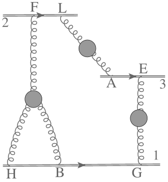
Verified known results and Predictions

[1] [JHEP06(2022)020 : Agarwal, Pal, AS, Tripathi]

Classify diagrams

Reducible diagrams [1]
$s \neq 0$



Irreducible diagrams [1]
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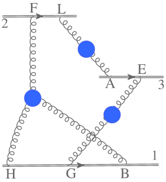

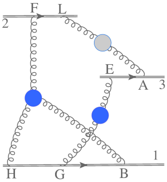

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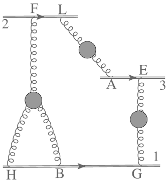

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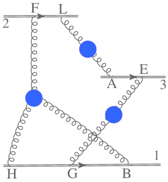

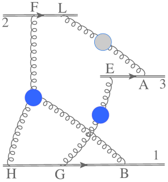

Entangled piece

Completely Entangled [1]

Partially Entangled [1]

[1] [JHEP06(2022)020 : Agarwal, Pal, AS, Tripathi]

Normal ordering

Cweb with n diagrams: l Irreducible and m reducible.

l diagrams $s = 0$
m diagrams $s \neq 0$

$$R = \begin{pmatrix} A_{l\times l} & B_{l\times m} \\ O_{m\times l} & D_{m\times m} \end{pmatrix}$$

k Completely entangled
$(l-k)$ partially entangled

$$A = \begin{pmatrix} I_{k\times k} & (A_U)_{k\times(l-k)} \\ O_{(l-k)\times k} & (A_L)_{(l-k)\times(l-k)} \end{pmatrix}$$

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A : Irreducible diagrams $\qquad D$: Reducible diagrams

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A : Irreducible diagrams $\qquad$ D : Reducible diagrams

Properties of $R \Rightarrow D$ is a mixing matrix with $S = \{s_{l+1}, \ldots, s_{l+m}\}$ [1]

Structure of A

Irreducible Diagrams

Entangled piece $\rightarrow$ Fused correlator

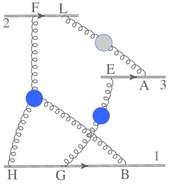
Fused diagram

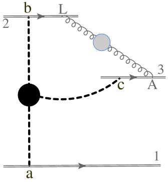
Structure of A

Irreducible Diagrams

Entangled piece $\rightarrow$ Fused correlator

Fused diagram



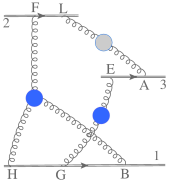


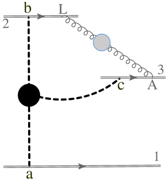
Structure of A

Irreducible Diagrams

Entangled piece $\rightarrow$ Fused correlator

Fused diagram





Shuffle

Diagrams with $s \neq 0$

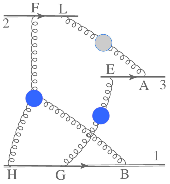
Fused-Web

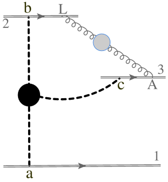
Structure of A

Irreducible Diagrams

Entangled piece $\rightarrow$ Fused correlator

Fused diagram





Shuffle

Diagrams with $s \neq 0$

Fused-Web

Uniqueness theorem gives $R_F \Rightarrow A = \begin{pmatrix} R_F^1 & & \\ & R_F^2 & \\ & & \ddots \end{pmatrix}$

Uniqueness theorem

For a given column weight vector S with all $s \neq 0$
The mixing matrix is unique.

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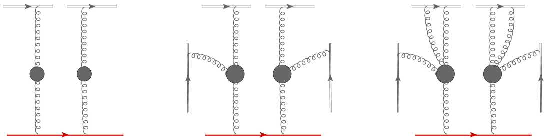


(d) (e) (f)

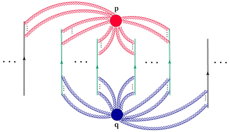
$$S = \{1, 1\} \quad \Leftrightarrow \quad R = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

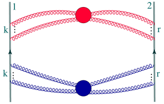
Predictions !!!

Two all order classes

Fused-Web

Direct construction [1]





Cweb	R	rank r(R)
$W_4^{(1,0,1)}(1,1,1,3)$	$R(0_1, 1_2)$	2
$W_4^{(0,2)}(1,1,1,3)$		
$W_{3,\text{I}}^{(0,2)}(1,2,3)$		
$W_3^{(1,0,1)}(1,2,3)$		
$W_{4,\text{I}}^{(1,0,1)}(1,1,2,2)$	$R(0_2, 1_2)$	3
$W_{4,\text{II}}^{(0,2)}(1,1,2,2)$		
$W_3^{(1,0,1)}(2,2,2)$		
$W_3^{(1,0,1)}(1,1,4)$		

Cweb	R	rank r(R)
$W_2^{(0,2)}(2,4)$	$R(0_4, 1_2)$	5
$W_{3,\text{I}}^{(1,0,1)}(1,2,3)$		
$W_{3,\text{II}}^{(1,0,1)}(1,2,3)$		
$W_3^{(0,2)}(1,1,4)$		
$W_2^{(1,0,1)}(2,4)$	$R(0_6, 1_2)$	7
$W_2^{(1,0,1)}(3,3)$	$R(0_7, 1_2)$	8
$W_2^{(0,2)}(3,3)$		

At four loops: Explicit R for 15 Cwebs

[1] [Agarwal et.al. 2021]

Predictions !!!

At four loops

The Uniqueness Theorem

Explicit R for 11 Cwebs

Cweb	S	R
$\mathsf{W}_5^{(0,2)}(1,1,1,1,2)$	$\{1_2\}$	$R(1_2)$
$\mathsf{W}_5^{(1,0,1)}(1,1,1,1,2)$	$\{1_2\}$	$R(1_2)$
$\mathsf{W}_{4,\mathrm{II}}^{(1,0,1)}(1,1,2,2)$	$\{1_2\}$	$R(1_2)$
$\mathsf{W}_{4,\mathrm{I}}^{(0,2)}(1,1,2,2)$	$\{1_2\}$	$R(1_2)$
$\mathsf{W}_{3,\mathrm{I}}^{(0,2)}(2,2,2)$	$\{1_2\}$	$R(1_2)$
$\mathsf{W}_{3,\mathrm{II}}^{(1,0,1)}(1,2,3)$	$\{1_2\}$	$R(1_2)$
$\mathsf{W}_{4,\mathrm{III}}^{(2,1)}(1,1,2,3)$	$\{1_6\}$	$R(1_6)$
$\mathsf{W}_5^{(2,1)}(1,1,1,1,3)$	$\{1_6\}$	$R(1_6)$
$\mathsf{W}_{4,\mathrm{I}}^{(2,1)}(1,2,2,2)$	$\{1_2, 2_2\}$	$R(1_2, 2_2)$
$\mathsf{W}_{5,\mathrm{I}}^{(2,1)}(1,1,1,2,2)$	$\{1_2, 2_2\}$	$R(1_2, 2_2)$
$\mathsf{W}_{5,\mathrm{II}}^{(2,1)}(1,1,1,2,2)$	$\{1_2, 2_2\}$	$R(1_2, 2_2)$

[JHEP06(2022)020 : Agarwal, Pal, AS, Tripathi]

Predictions !!!

Diagonal blocks at Four loops.

Using the Fused-Web Formalism

Mixing matrices of 9 Cweb

Cweb	D	$r(R)$
$\mathsf{W}_{3,\mathsf{I}}^{(2,1)}(2,2,3)$	$R(1_6)$	16
$\mathsf{W}_{3,\mathsf{I}}^{(2,1)}(1,3,3)$	$R(1_6)$	10
$\mathsf{W}_{3}^{(2,1)}(1,2,4)$	$R(1_6)$	15
$\mathsf{W}_{4}^{(2,1)}(1,1,1,4)$	$R(1_6)$	6
$\mathsf{W}_{4,\mathsf{III}}^{(2,1)}(1,2,2,2)$	$R(1_6)$	4
$\mathsf{W}_{3,\mathsf{III}}^{(2,1)}(2,2,3)$	$R(1_6)$	7
$\mathsf{W}_{4,\mathsf{I}}^{(2,1)}(1,1,2,3)$	$R(1_2,2_2)$	2
$\mathsf{W}_{4,\mathsf{II}}^{(2,1)}(1,2,2,2)$	$R(1_2,2_2)$	3
$\mathsf{W}_{4}^{(2,1)}(1,1,2,3)$	$R(1_2,2_2)$	2

[JHEP06(2022)020 : Agarwal, Pal, AS, Tripathi]

Results in nutshell

[JHEP06(2022)020 : Agarwal, Pal, AS, Tripathi]

At four loops

From Two all order classes	Explicit 15 Cwebs
Uniqueness theorem	Explicit 11 Cwebs
Fused-Web formalism	Diagonal blocks for 9 Cwebs

Results in nutshell

[JHEP06(2022)020 : Agarwal, Pal, AS, Tripathi]

At four loops

From Two all order classes	Explicit 15 Cwebs
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For 26 Cwebs explicit Form

$$26/60 \sim 43\%$$

For 35 Cwebs number of ECFs

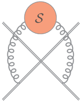
$$35/60 \sim 60\%$$

Summary

Scattering Amplitude

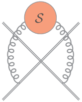
Scattering Amplitude

$\mathcal{S}\times\mathcal{H}\times\mathcal{J}$



$\mathcal{W} = FRC$

Scattering Amplitude



$\mathcal{S}\times\mathcal{H}\times\mathcal{J}$

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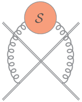
Replica Trick

R

Direct Construction

Scattering Amplitude



$\mathcal{S} \times \mathcal{H} \times \mathcal{J}$

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Classify diagrams

$$\begin{pmatrix} A & B \\ O & D \end{pmatrix}$$

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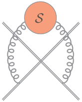

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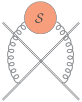
Structure of D

$S \leftrightarrow R$

Structure of A

Scattering Amplitude

$\mathcal{S}\times\mathcal{H}\times\mathcal{J}$



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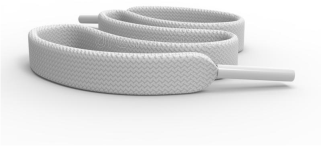
Structure of A

At four loops: 26 explicit, # ECFs for 35 Cwebs

Open Ends . . . !!!



Can this formalism be extended for off diagonal block B !

Upto Four loops : Only 4 basis Cwebs for determining diagonal blocks of any R.

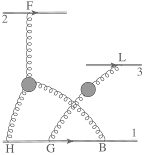
Are basis really the building blocks of Cweb : In terms of kinematics as well !

The effect of formalism on the anomalous dimension structure.

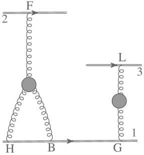
Thank you

Appendix

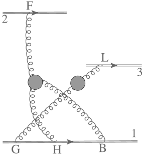
Framework $\mathrm{W}_3^{(1,1)}(1,1,3)$



$s=0$



$s=1$



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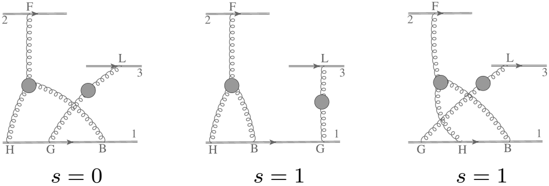
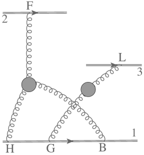


$$R = \begin{pmatrix} A & B \\ O & D \end{pmatrix}$$

$$S = \{0, 1, 1\}$$

$$D = R(1_2)$$

Framework $\mathrm{W}_3^{(1,1)}(1,1,3)$



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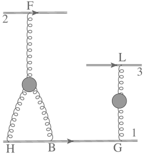
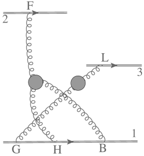


$s = 1$



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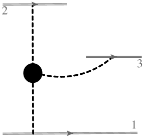

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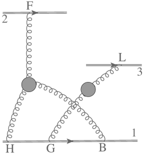
Framework $\mathrm{W}_3^{(1,1)}(1,1,3)$



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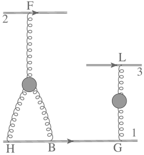

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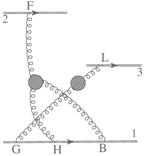

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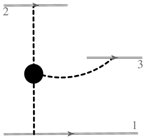


$s = 1$

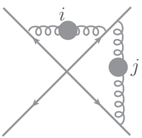
$A = R_F$ for $S_F = \{1_1\}$ the matrix is $R_F(1) = 1$

$$R = \begin{pmatrix} 1 & B \\ O & R(1_2) \end{pmatrix}, \qquad r(R) = 1 + \mathrm{Tr}(R(1_2)) = 2$$

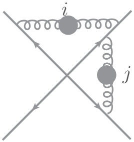
Two independent exponentiated colour factors

Mixing matrix

Mixing matrix R is determined through replica trick.



C_1

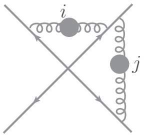


C_2

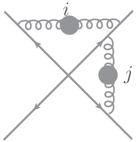
h	$\mathcal{R}[C_1]$	M_{N_r}	$\mathcal{O}(N_r)$
$i = j$	C_1	${}^{N_r}C_1$	1
$i < j$	C_1	${}^{N_r}C_2$	$-1/2$
$i > j$	C_2	${}^{N_r}C_2$	$-1/2$

Mixing matrix

Mixing matrix R is determined through replica trick.



C_1



C_2

h	$\mathcal{R}[C_1]$	M_{N_r}	$\mathcal{O}(N_r)$
$i = j$	C_1	${}^{N_r}C_1$	1
$i < j$	C_1	${}^{N_r}C_2$	$-1/2$
$i > j$	C_2	${}^{N_r}C_2$	$-1/2$

The exponentiated color factor is

$$\widetilde{C}_1 = C_1 - \frac{1}{2}C_2 - \frac{1}{2}C_1 = \frac{1}{2}(C_1 - C_2); \qquad \widetilde{C}_2 = \frac{1}{2}(C_2 - C_1)$$

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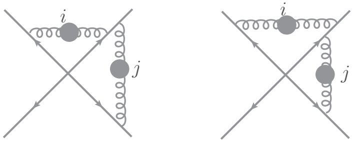

C_1 C_2

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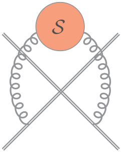
Mixing matrix of the web with the order of diagrams

$$\begin{pmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \qquad R = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Soft Function



Replacing all hard partons by lightlike Wilson lines.

$$\mathcal{S} = \langle 0|\Phi_{\beta_1}(\infty, 0) \otimes .... \otimes \Phi_{\beta_n}(\infty, 0)|0\rangle$$

Wilson line operator and Soft function exponentiate

Connected subdiagrams sit in the exponent.

Renormalized $\mathcal{S}$

One can renormalize the UV poles of $\mathcal{S}$ as

$$\mathcal{S}_{\mathsf{ren}}(\alpha_s, \mu) = \mathcal{S}_{\mathsf{UV+IR}}(\alpha_s, \epsilon)\ \mathcal{Z}_{\mathsf{UV}}(\alpha_s, \epsilon, \mu)$$

$\mathcal{S}_{\mathsf{ren}}$ has only IR poles

However $\mathcal{S}_{\mathsf{UV+IR}} = 1$ in dimensional regularization (Scaleless integrals).

The IR poles are the UV poles of renormalization factor $\mathcal{Z}$

Thus in these context $\mathcal{Z}$ and $\mathcal{S}$ refer to same soft function.

Webs and Γ

$\mathcal{S}$ is an exponentiation of Feynman diagrams called webs $\mathcal{W}$

$$\mathcal{S} \;=\; \exp\left[\, w \,\right] = \exp\left[\int \frac{d\lambda^2}{\lambda^2}\Gamma(\alpha_s(\lambda^2,\epsilon))\right]$$

w has perturbative expansion.

$$w = w^{(1)}\alpha_s^1 + w^{(2)}\alpha_s^2 + w^{(3)}\alpha_s^3 + \cdots$$

For a given order in perturbation

$$w^{(n)}\alpha_s^n = \sum_{n_1..n_L} W^{(n)}_{n_1,n_2,\ldots n_L}$$

Wilson Line & Soft emission

- Gluon emissions form parton line

$$\mathcal{M} \;=\; (-g_s)^2 \bar{u}_l(p_i) \left[\mathbf{T}_i^a \mathbf{T}_i^b \left(\frac{p_i^\mu}{p_i.k_1} \right) \left(\frac{p_i^\nu}{p_i.(k_1+k_2)} \right) \right] \mathcal{M}_0 \epsilon_\mu(k_1) \epsilon_\nu(k_2)$$

- For soft emission : no momentum change, only colour & phase of parton field p_i will change.
- Parton field undergoes a gauge transformation due to soft emissions.

- Wilson lines in gauge theories capture info of gauge transformation, defined as

$$\Phi_n(\lambda_2, \lambda_1) \; = \; \mathcal{P} \exp \left[i g_s \mathbf{T}_a \int_{\lambda_1}^{\lambda_2} d\lambda \; n.A^a(\lambda n) \right]$$

- For single gluon emission

$$\begin{aligned} \Phi_n(\infty, 0) &= 1 + i g_s \mathbf{T}_a \int_0^{\infty} d\lambda \; n.A^a(\lambda n) \\ &= 1 + i g_s \mathbf{T}_a \int \frac{d^d k}{(2\pi)^d} \int_0^{\infty} d\lambda e^{-ik.\lambda n} \; n \cdot \tilde{A}^a(k) \\ &= 1 + i g_s \mathbf{T}_a \int \frac{d^d k}{(2\pi)^d} \frac{n.\tilde{A}^a(k)}{i k.n} \end{aligned}$$

- Similarly for two gluon emission

$$\begin{aligned} &= (i g_s)^2 \mathbf{T}^a \mathbf{T}^b \int_0^{\infty} d\lambda_1 \int_0^{\lambda_1} d\lambda_2 n.A^b(\lambda_2 n) n.A^a(\lambda_1 n) \\ &= g_s^2 \mathbf{T}^a \mathbf{T}^b \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \; \frac{n.\tilde{A}^a(k_1) \; n.\tilde{A}^b(k_2)}{(n.k_1) n.(k_1 + k_2)} \end{aligned}$$

Scaleless Integrals

- These subdiagrams are essentially made up of scaleless integrals
- Scaleless integrals, defined as, in dimensional regularization vanish.

$$\int_0^\infty \frac{d^dk}{(k^2)^\alpha} \to \Omega_d \int_0^\infty k_E^{d-1-2\alpha} dk_E = 0 \quad \forall \;\; \alpha > 0$$

- Have UV and IR poles for the $d > 2\alpha$ and $d < 2\alpha$ respectively.

$$\int_0^\infty \frac{d^dk}{(k^2)^\alpha} \;\; = \;\; \int_0^\Lambda \frac{d^dk}{(k^2)^\alpha} + \int_\Lambda^\infty \frac{d^dk}{(k^2)^\alpha}$$

In $d = 4 - \epsilon$ for IR $\epsilon = \epsilon_{IR} < 0$ and for UV $\epsilon = \epsilon_{UV} > 0$
Integrating over allowed regions

$$\begin{aligned} \int_0^\infty \frac{d^d k}{(k^2)^\alpha} &= \Omega_d \int_0^\Lambda k_E^{d-1-2\alpha} dk_E + \Omega_d \int_\Lambda^\infty k_E^{d-1-2\alpha} dk_E \\ &= \Omega_d \left(-\frac{\Lambda^{-\epsilon_{IR}}}{\epsilon_{IR}} + \frac{\Lambda^{-\epsilon_{UV}}}{\epsilon_{UV}} \right) \quad ; \alpha = 2 \end{aligned}$$

- Integral should not depend on the regulator : can put $\epsilon_{UV} = \epsilon_{IR}$
- IR poles are cancelled by UV poles.

$$\int \frac{d^4 k}{k^4} \to \int \frac{d^d k}{(k^2)^2} = 0$$

- Having no dimensionfull parameter it vanishes.

Single Poles

- RG evolution equation

$$\frac{d\mathcal{Z}(\alpha_s,\epsilon,\mu)}{d\ln\mu} \;=\; -\Gamma(\alpha_s)\;\mathcal{Z}(\alpha_s,\epsilon,\mu)$$

- Form of soft function in colour singlet case, the solution will be

$$\mathcal{Z} \;=\; \exp\left[\frac{1}{2}\int_{\lambda^2}^{\infty}\frac{d\lambda^2}{\lambda^2}\Gamma(\alpha_s(\lambda^2,\epsilon))\right]$$

- Writing the β function and Γ in the order of α_s as

$$\frac{d\alpha_s}{d\ln\mu^2} = -\epsilon\alpha_s + \sum_n \alpha_s^{n+2}b_n \quad ; \quad \Gamma \;=\; \sum_{n=1}^{\infty}\Gamma^{(n)}\alpha_s^n$$

- For conformal case $b_i = 0$

$$\frac{d\ln\alpha_s}{d\ln\lambda^2} = -\epsilon \quad \Rightarrow \quad d\lambda^2 = -\frac{1}{\epsilon}\alpha_s^{-(1+1/\epsilon)}d\alpha_s$$

- Then using (1) we will have

$$\begin{aligned}\mathcal{Z} &= \exp\left[\frac{1}{2}\int_{\alpha_s}^{0} -\frac{1}{\epsilon}\sum_n \frac{\alpha_s^{-(1+1/\epsilon)}d\alpha_s}{(\alpha_s)^{-1/\epsilon}}\Gamma^{(n)}\alpha_s^n\right] \\ &= \exp\left[\frac{1}{2\epsilon}\sum_n \frac{1}{n}\Gamma^{(n)}\alpha_s^n\right]\end{aligned}$$

- In conformal case only simple poles arise to all order in α_s

Structure of D

$$R = \begin{pmatrix} A_{l\times l} & B_{l\times m} \\ O_{m\times l} & D_{m\times m} \end{pmatrix}, \quad S = \{0, 0, \ldots, l\,\text{times}, s(d_{l+1}), \ldots, s(d_{l+m})\}$$

Row sum over R

Row sum over D

Column sum of R

Column sum of D

Idempotence of R

$A^2 = A, \qquad D^2 = D$

[1] [JHEP06(2022)020 : Agarwal, Pal, AS, Tripathi]

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Row sum over R

Row sum over D

Column sum of R

Column sum of D

Idempotence of R

$A^2 = A, \qquad D^2 = D$

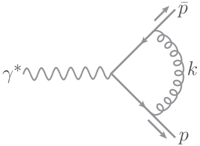
D is a mixing matrix with $S = \{s_{l+1}, \ldots, s_{l+m}\}$ [1]

$$r(R) = \text{Tr}(A) + \text{Tr}(D)$$ [1]

[1] [JHEP06(2022)020 : Agarwal, Pal, AS, Tripathi]

Factorisation in QED

- IR factorisation in one loop



In soft limit $k \to 0$ for massless partons, amplitude

$$\begin{aligned}\mathcal{M} &= [\bar{u}(p)\gamma^\mu v(\bar{p})] \left(\frac{-p^\alpha}{p.k}\right) \left(\frac{\bar{p}^\beta}{\bar{p}.k}\right) \mathcal{D}_{\alpha\beta}(k) \\ &= \mathcal{M}_0 \left(\frac{p.\bar{p}}{(p.k)(\bar{p}.k)}\right)\end{aligned}$$

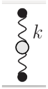
- Eikonal Feynman Rules :

  For each emission of photon k from parton p_i we get

$$\frac{p_i^\mu}{(p_i.k)} \tag{1}$$

- Independence from spin of emitting parton (large λ_c).
- Factorisation happened due to separate energy scales.
- Sum of all n-loop diagrams is given by exponentiating one PI.

$$\mathcal{M} \;=\; \mathcal{M}_0 \exp \tag{2}$$



- Thus soft function exponentiate with photon subdiagrams.

Replica method

- Soft function, Wilson line correlators

$$\mathcal{S}_n = \langle 0|\Phi_{\beta_1}(\infty,0)\otimes ..\otimes \Phi_{\beta_n}(\infty,0)|0\rangle = \exp[\mathcal{W}_n]$$

- Replicated theory : N_r non-interacting replica

$$\mathcal{S}_n^{rep} = \langle 0|\prod_{k=1}^{n}\prod_{i=1}^{N_r}\Phi^i_{\beta_k}(\infty,0)|0\rangle$$

$$\mathcal{S}_n^{rep} = [\mathcal{S}_n]^{N_r} = \exp[N_r\,\mathcal{W}] = 1 + N_r\mathcal{W} + \mathcal{O}(N^2)$$

- Calculating the coefficient of N_r

Replica method

- $\mathcal{R}$ operator : replicated and normal theory

$$\prod_{i=1}^{N_r} \Phi^i(\gamma) = \prod_{i=1}^{N_r} \mathcal{P} \exp \left[ig \int_\gamma dx A^i(x) \right] = \mathcal{R}\mathcal{P} \sum_{i=1}^{N_r} \exp \left[ig \int dx A^i(x) \right]$$

- Two loop Web

h	$\mathcal{R}[C_1]$	MN_r	$\mathcal{O}(N_r)$
$i = j$	C_1	${}^{N_r}C_1$	1
$i < j$	C_2	${}^{N_r}C_2$	$-1/2$
$i > j$	C_1	${}^{N_r}C_2$	$-1/2$

Hierarchy	Effective Hierarchy	Multiplicity
$i = j = k$	i	${}^{N}C_1$
$i = j, k > i$ $i = j, k < i$	$k > i$ $k < i$	${}^{N}C_2$
$i = k, j > i$ $i = k, j < i$	$j > i$ $j < i$	${}^{N}C_2$
$j = k, i > j$ $j = k, i < j$	$i > j$ $i < j$	${}^{N}C_2$
$i < j < k$ $i < k < j$ $j < i < k$ $j < k < i$ $k < i < j$ $k < j < i$	$-$ $-$ $-$ $-$ $-$ $-$	${}^{N}C_3$

Table: Replica trick analysis for three replica variables.

Replica method

- ECF of C_1 is $\tilde{C}_1 = 1/2\,[C_1 - C_2]$
- Mixing matrix for two loop

$$R = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

- Rank one matrix has only one independent ECF

Expression for $\gamma_{\mathcal{J}_i}$

- From RG equation of eikonal jet

$$\mathcal{J}_i(\omega_i, \alpha_s(\mu^2), \epsilon) = \exp\left[-\frac{1}{2}\int_0^{\mu^2} \frac{d\zeta^2}{\zeta^2}\gamma_{\mathcal{J}_i}\left(\omega_i, \alpha_s(\zeta^2, \epsilon), \epsilon\right)\right]$$

- $\gamma_{\mathcal{J}_i}$ should split in terms with kinematic and singular cusp $\gamma_K^{(i)}$.

$$\gamma_{\mathcal{J}_i}(\omega_i, \alpha_s(\mu^2), \epsilon) = -\frac{1}{2}G_{\mathcal{J}_i}(\omega_i, \alpha_s(\mu^2)) + \frac{1}{4}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\gamma_K^{(i)}(\alpha_s(\lambda^2, \epsilon))$$

$$\boxed{\gamma_{\mathcal{J}_i} = \frac{1}{4}\gamma_K^{(i)}(\alpha_s)\ln\omega_i - \frac{1}{2}\delta_{\mathcal{J}_i}(\alpha_s) + \frac{1}{4}\int_0^{\mu^2} \frac{d\zeta^2}{\zeta^2}\gamma_K^{(i)}(\alpha_s(\zeta^2, \epsilon))} \qquad (3)$$

Structure of $\Gamma^{\mathcal{S}}$

- From relations of anomalous dimensions and $\gamma_{\mathcal{J}}$

$$\Gamma^{\bar{S}}_{IJ}(\rho_{ij}, \alpha_s) = \Gamma^{S}_{IJ}(\beta_i.\beta_j, \alpha_s, \epsilon) - \delta_{IJ} \sum_{k=1}^{n} \gamma_{\mathcal{J}_k}(\omega_k, \alpha_s, \epsilon)$$

- Off diagonal terms of $\Gamma^{\mathcal{S}}$ are non-singular.
- Singular terms came from integral : finite $\Gamma^{\bar{\mathcal{S}}}$

$$\Gamma^{\mathcal{S}}_{IJ}(\beta_i.\beta_j, \epsilon) = \delta_{IJ} \sum_i \frac{1}{4} \int_0^{\mu^2} \frac{d\zeta^2}{\zeta^2} \gamma_K^{(i)}(\alpha_s(\zeta^2, \epsilon))$$

Constraints on $\Gamma^{\bar{\mathcal{S}}}$

- $\Gamma^{\bar{\mathcal{S}}}$ depends on $\gamma_{\mathcal{J}}$ and which inturn depends on $\ln \omega_k$

$$\frac{\partial}{\partial \ln \omega_i} \Gamma^{\bar{\mathcal{S}}}_{MN}(\rho_{ij}, \alpha_s) = \frac{1}{4} \gamma_K^{(i)}(\alpha_s) \delta_{MN}$$

$$\boxed{\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\bar{\mathcal{S}}}_{MN}(\rho_{ij}, \alpha_s) = \frac{1}{4} \gamma_K^{(i)}(\alpha_s) \delta_{MN}} \tag{4}$$

- Ignoring higher order Casimir's contributions to cusp.

$$\gamma_K^{(i)} = C_i \hat{\gamma}_K(\alpha_s)$$

Colour Conservation

- Emission of a gluon from Multiparton process
- Eikonal Feynman rule

$$j^\mu \quad = \quad g_s \sum_i \frac{\mathbf{T}_i^a \, p_i^\mu}{(p_i.k)}$$

- For a colour conserving scattering

$$\partial_\mu j^\mu = 0 \;\longrightarrow\; k_\mu j^\mu = 0$$

Illustration

- Colour Conservation

$$\boxed{\sum_i \mathbf{T}_i^a = 0}$$

- For the process

$$\sum_{i=1}^{3} \mathbf{T}_i \; = \; \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 = 0$$

- Can also be used as

$$\sum_{i\neq 2} \mathbf{T}_i \; = \; -\mathbf{T}_2 \quad ; \quad \sum_{i\neq j} \mathbf{T}_i = -\mathbf{T}_j$$

Colour Symmetrization

- Two gluon emission

- From eikonal Feynman rules

$$\sum_i g_s^2 n_i^\mu \, n_i^\nu \left[\frac{\mathbf{T}_i^b \mathbf{T}_i^a}{n_i \cdot (k_1 + k_2)(n_i \cdot k_2)} + \frac{\mathbf{T}_i^a \mathbf{T}_i^b}{n_i \cdot (k_1 + k_2)(n_i \cdot k_1)} \right]$$

$$\rightarrow \quad \sum_i \frac{g_s^2}{2} n_i^\mu \, n_i^\nu \left[\frac{\{\mathbf{T}_i^a, \mathbf{T}_i^b\}}{(n_i \cdot k_1)(n_i \cdot k_2)} \right]$$

Colour Symmetrization

- For general emission

$$\left(\mathbf{T}_{i_1}^{a_1}\cdots\mathbf{T}_{i_n}^{a_n}\right)_+ = \frac{1}{n!}\sum_{\pi}\mathbf{T}_{i_{\pi_1}}^{a_{\pi_1}}\cdots\mathbf{T}_{i_{\pi_n}}^{a_{\pi_n}}$$

- Generators in different partons commute.

$$\left(\mathbf{T}_i^a\,\mathbf{T}_i^b\,\mathbf{T}_j^c\,\mathbf{T}_i^d\,\mathbf{T}_j^e\,\mathbf{T}_k^f\right)_+ = \left(\mathbf{T}_i^a\mathbf{T}_i^b\mathbf{T}_i^d\right)_+\left(\mathbf{T}_j^c\mathbf{T}_j^e\right)_+\mathbf{T}_k^f$$

Factorization

- Soft function captures all the soft singularities (has double poles)
- Jet function associated with each external parton has the collinear poles.
- Double counting of poles are pulled out by Eikonal jets.

Colour space

- More than one possible colour flow

$$\mathcal{M} = \sum_{L=1}^{m} \mathcal{M}_L \, c_L$$

- Two possible flow for $q_a \bar{q}_b \to q_c \bar{q}_d$

- Two color flow basis vectors $c_1 = \delta_{ac}\delta_{bd}$ and $c_2 = \delta_{ab}\delta_{cd}$

$$\mathcal{M} = \mathcal{M}_1 \, c_1 + \mathcal{M}_2 \, c_2 \tag{5}$$

Amplitudes

Physical observable relate to Scattering amplitude

$$\sigma \propto \int d\Pi \, |\mathcal{M}|^2$$

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Feynman diagrams: Visualizing maths of scattering

$$e^+ \; + e^- \; \longrightarrow \; \bar{q} \; + \; q$$

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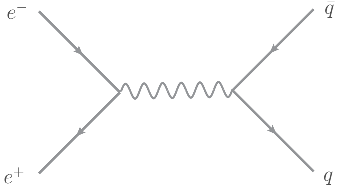

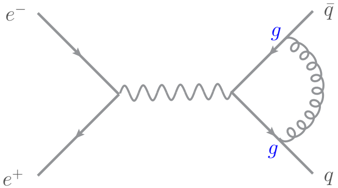

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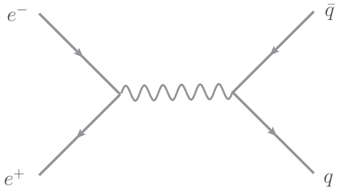
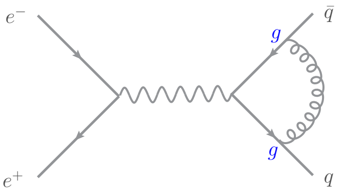


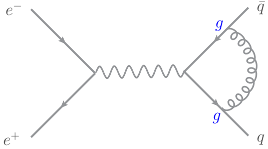
Leading Order (α_s^0)



Next-to-leading order ($\alpha_s = g^2$)

Amplitudes

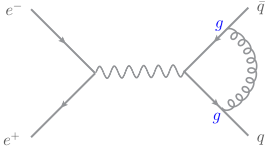
For the process...



$$\sigma = \alpha_s^0 \sigma_{\text{LO}} + \alpha_s^1 \sigma_{\text{NLO}} + \cdots$$

Amplitudes

For the process...



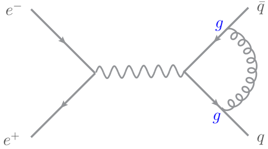
$$\sigma = \alpha_s^0 \sigma_{\mathsf{LO}} + \alpha_s^1 \sigma_{\mathsf{NLO}} + \cdots$$

Typical integrals for NLO and beyond

$$\int_0^\infty \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^\alpha} \qquad ; \qquad \alpha > 0$$

Amplitudes

For the process...



$$\sigma = \alpha_s^0 \sigma_{\mathsf{LO}} + \alpha_s^1 \sigma_{\mathsf{NLO}} + \cdots$$

Typical integrals for NLO and beyond

$$\int_0^\infty \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^\alpha} \qquad ; \qquad \alpha > 0$$

Lead to UV divergences for $k \to \infty$

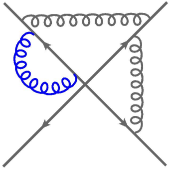
For $k \to 0$, IR divergences appear.

Redefinitions of fields and couplings : Renormalization

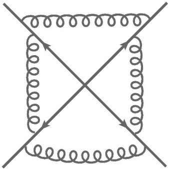
In massive case

Nine boomerang webs at three loops

Kinematics and colour of five MGEW boomerang is completed recently [2].



Four loops webs . . .



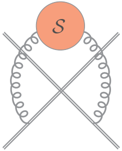
Massless case four loops webs are enumerated.[2]

Cweb is the set of diagrams made with the connected gluon correlators.

Diagrams are related by the shuffle of gluon attachments.

[2] [Agrawal et.al.2020]

Soft Function

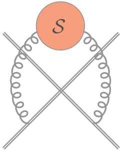


$$\mathcal{S} = \langle 0|\Phi_{\beta_1}(\infty,0) \otimes .... \otimes \Phi_{\beta_n}(\infty,0)|0\rangle$$

One can renormalize the UV poles of $\mathcal{S}$ as

$$\mathcal{S}_{\mathsf{ren}}(\alpha_s,\mu) = \mathcal{S}_{\mathsf{UV+IR}}(\alpha_s,\epsilon)\ \mathcal{Z}_{\mathsf{UV}}(\alpha_s,\epsilon,\mu)$$

Soft Function



$$\mathcal{S} = \langle 0|\Phi_{\beta_1}(\infty,0) \otimes .... \otimes \Phi_{\beta_n}(\infty,0)|0\rangle$$

One can renormalize the UV poles of $\mathcal{S}$ as

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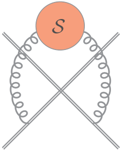

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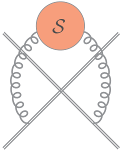

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The IR poles are the UV poles of renormalization factor $\mathcal{Z}$

$$\mathcal{S} \leftrightarrow \mathcal{Z}$$

RG evolution equation

$$\frac{d\mathcal{Z}(\alpha_s, \epsilon, \mu)}{d \ln \mu} = -\,\Gamma(\alpha_s, \epsilon)\ \mathcal{Z}(\alpha_s, \epsilon, \mu)$$

Form of soft function in colour singlet case, the solution will be

$$\mathcal{Z} = \exp\left[\frac{1}{2}\int_{\lambda^2}^{\infty} \frac{d\lambda^2}{\lambda^2}\Gamma(\alpha_s(\lambda^2, \epsilon))\right]$$

Scale integral gives

$$\mathcal{Z} = \exp\left[\frac{1}{2\epsilon}\sum_n \frac{1}{n}\Gamma^{(n)}\alpha_s^n\right]$$

Only simple poles arise in exponent to all order in α_s

Expanding the exponent of soft function

$$\mathcal{S} = \exp\left[\frac{1}{2\epsilon}\left(\Gamma^{(1)}\alpha_s^1 + \frac{1}{2}\Gamma^{(2)}\alpha_s^2 + \cdots\right)\right]$$

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Knowing only $\Gamma^{(1)}$ gives

$$\mathcal{S} = 1 + \left(\frac{1}{2\epsilon}\Gamma^{(1)}\right)\alpha_s^1 + \frac{1}{2!}\left(\frac{1}{2\epsilon}\Gamma^{(1)}\right)^2\alpha_s^2 + \mathcal{O}(\alpha_s^3) + \cdots$$

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Knowledge of $\Gamma^{(r)}$: Certain all order IR poles