An effective field theory of thermal QCD with higher dimensional gradient term

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Motivation

- Effective Field theories provide us very powerful ways to organizing the computation of low-energy effects in QFT.
- QCD at low temperature (T) is well described by the dynamics of pions.
- The long distance or low energy physics at very high temperatures are qualitatively well understood by an effective weak coupling expansion.
- This weak coupling expansion relies on the separation of hierarchy of scales as, $T >> gT >> g^2T$, where g is the gauge coupling at momentum scale T .
- This separation of scales break down when T is few hundred MeV, when $g \sim 1$.
- However, this is the range of temperature which is of great physical interest. The transition of a chiral symmetry broken hadronic state to a symmetry restored quark-gluon state occurs here.

Global Symmetries and the EFT

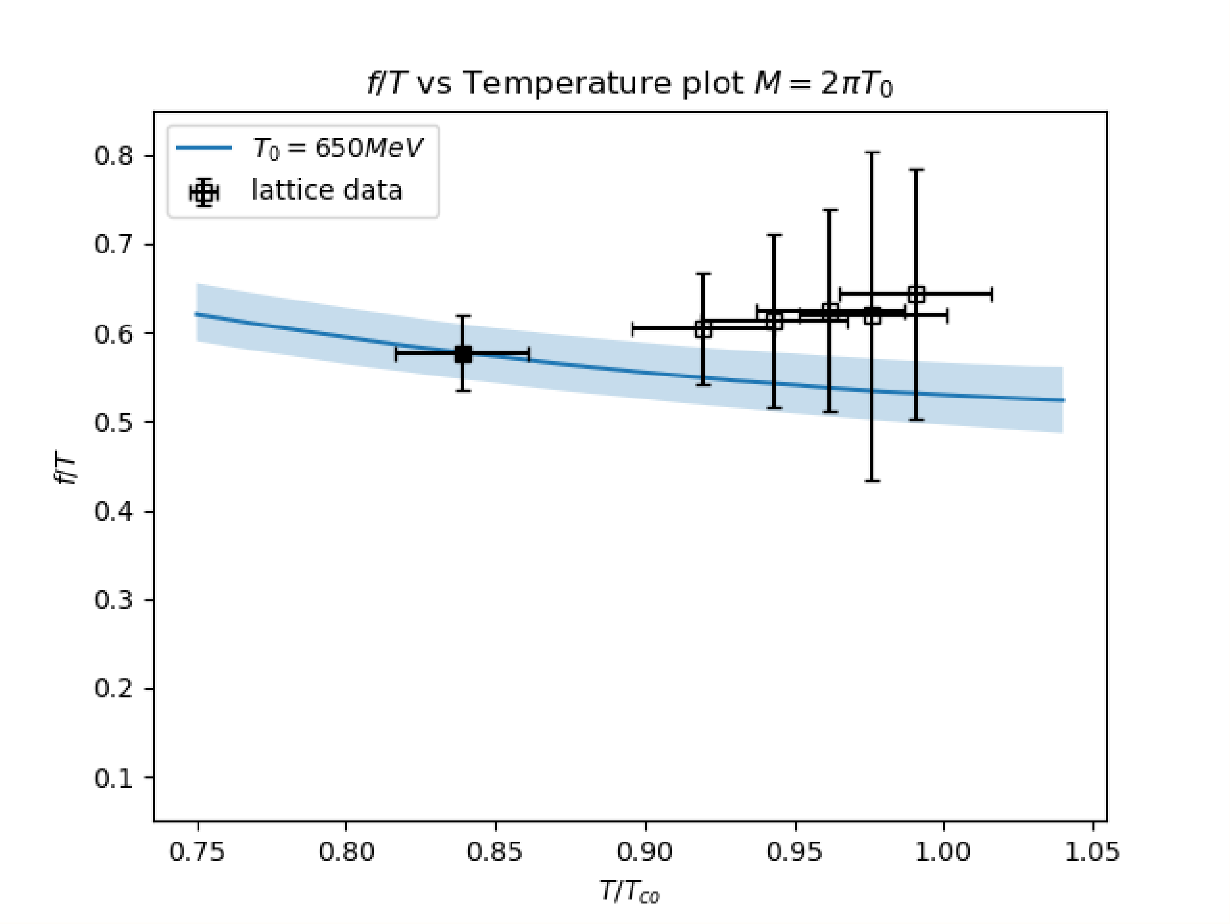
- This temperature range is also seems to be most relevant for experiments using heavy-ion collisions.
- Here we try to propose an Effective Field Theory designed to describe the physics of QCD around Cross-over temperature T_{co}
- We in this case proceed with the global symmetries of the QCD as the guiding principle and arrange the EFT in mass dimension of the relevant terms which obeys this.
- Particularly we use Vector (V) and Axial (A) symmetries of QCD for N_f number of flavours, namely $SU_V(N_f) \times SU_A(N_f)$ symmetry.
- We will also find that as our theory is at finite temperature the Lorentz group after Euclidization will be reduced to a rotation group with time reversal symmetry, i.e. a cylindrical symmetry $O(3) \times Z_2$.
- Discrete symmetries such as Charge conjugation (C), Parity (P) and Time reversal (T) and CPT will also be our guiding principle to construct relevant Lagrangian terms.

Outline of the work

- Taking the global symmetries of QCD as our guiding principle we write here an EFT near cross-over temperature of QCD.
- Up-to dimension-6 apart from having current-current interactions we also include dimension-6 gradient operators in our theory.
- We treat the theory in Mean field Approximation to get free energy and gap equation. We also find a second solution of critical temperature at chiral limit, which can be then used to limit the coupling strength of dimension-6 gradient term.
- We proceed to continue with pionic fluctuations, which after comparing with lattice data fixes all the LECs of EFT.
- We compute the predictions of the EFT after fixing the LECs, the results are found to be pleasing.

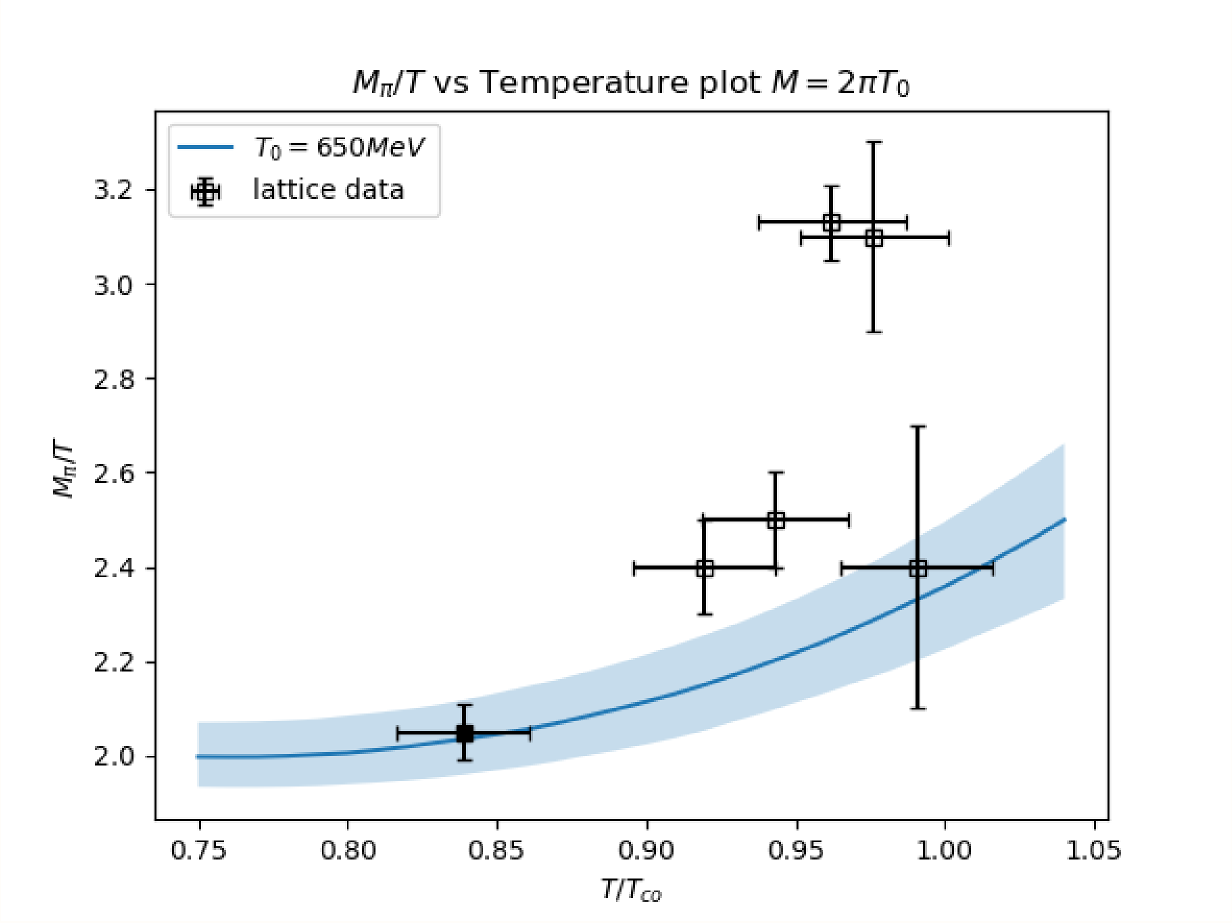
Predictions of the EFT

- We at first present the predictions of the EFT here, for f/T of pions,



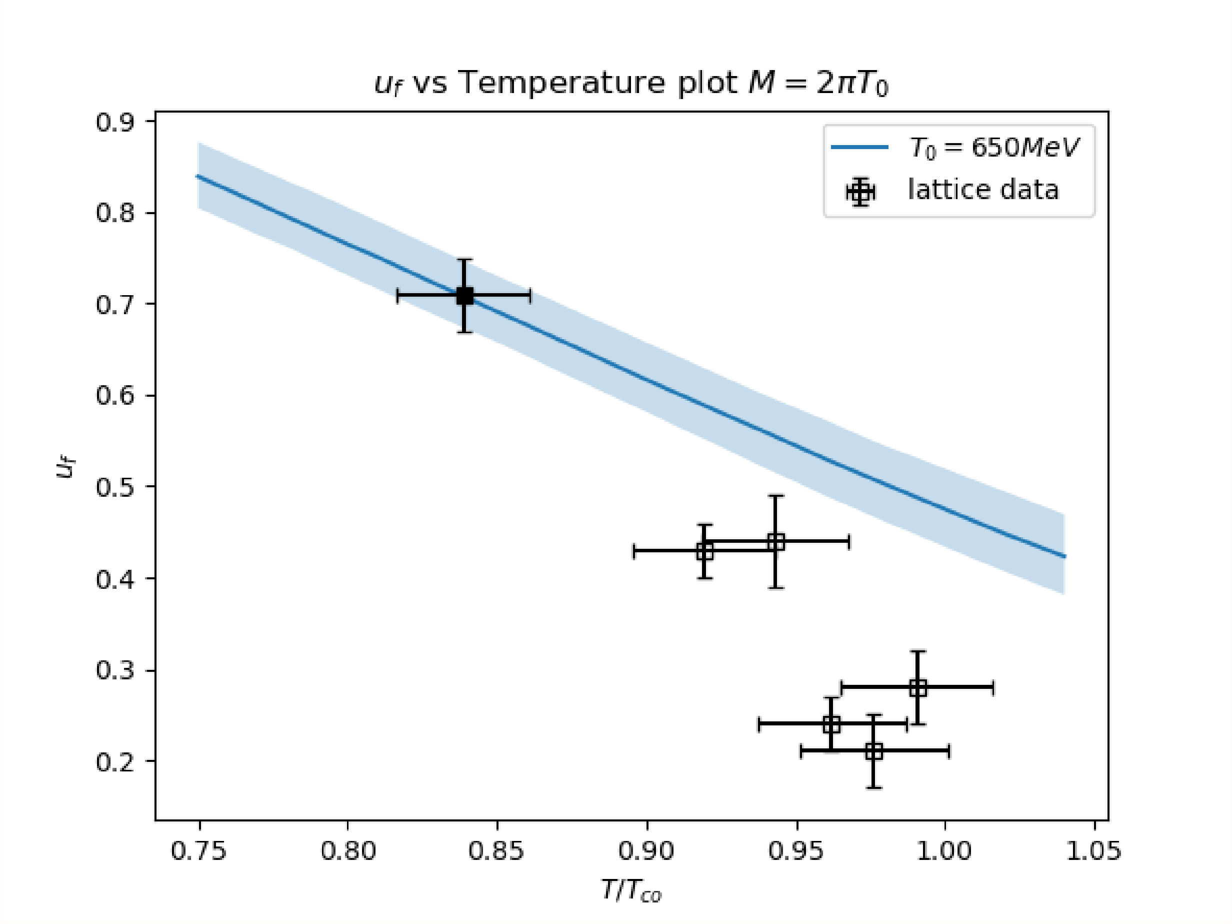
Predictions of the EFT

- For M_π/T of pions the predictions are as follows,



Predictions of the EFT

- For u_f of pions the predictions of EFT are as follows,



The EFT Lagrangian

- We will Work with Euclidean Dirac matrices which are known to be Hermitian such that, $\gamma_4 = -i\gamma_0$ and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ with the generators, $S_{\mu\nu} = -i[\gamma_\mu, \gamma_\nu]/4$, and $\bar{\psi} = \psi^\dagger \gamma_4$.
- The most general Lagrangian which can be written with these symmetries up-to dimesion-6 is,

$$\mathcal{L} = d^3 T_0 \bar{\psi}\psi + \bar{\psi}\partial\!\!\!/_4 \psi + d^4 \bar{\psi}\nabla\!\!\!\!/\,\psi + \mathcal{L}_6$$

- Where, $\mathcal{L}_6 = \mathcal{L}_6^{\text{current}} + \mathcal{L}_6^{\text{gradient}}$, with ,

$$\begin{aligned}\mathcal{L}_6^{\text{current}} = \; & + \frac{d^{61}}{T_0^2}[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau^a\psi)^2] + \frac{d^{62}}{T_0^2}[(\bar{\psi}\tau^a\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2] \\ & + \frac{d^{63}}{T_0^2}(\bar{\psi}\gamma_4\psi)^2 + \frac{d^{64}}{T_0^2}(\bar{\psi}i\gamma_i\psi)^2 + \frac{d^{65}}{T_0^2}(\bar{\psi}\gamma_5\gamma_4\psi)^2 + \frac{d^{66}}{T_0^2}(\bar{\psi}i\gamma_5\gamma_i\psi)^2 \\ & + \frac{d^{67}}{T_0^2}[(\bar{\psi}\gamma_4\tau^a\psi)^2 + (\bar{\psi}\gamma_5\gamma_4\tau^a\psi)^2] + \frac{d^{68}}{T_0^2}[(\bar{\psi}i\gamma_i\tau^a\psi)^2 + (\bar{\psi}i\gamma_5\gamma_i\tau^a\psi)^2] \\ & + \frac{d^{69}}{T_0^2}[(\bar{\psi}iS_{i4}\psi)^2 + (\bar{\psi}S_{ij}\tau^a\psi)^2] + \frac{d^{60}}{T_0^2}[(\bar{\psi}iS_{i4}\tau^a\psi)^2 + (\bar{\psi}S_{ij}\psi)^2]\end{aligned}$$

- And, $\mathcal{L}_6^{\text{gradient}} = \frac{\tilde{d}^6}{T_0^2}\bar{\psi}\nabla\!\!\!\!/\,\nabla\!\!\!\!/\,\nabla\!\!\!\!/\,\psi$

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NJL Model

- And, $\mathcal{L}_6^{\text{gradient}} = \frac{\tilde{d}^6}{T_0^2}\bar{\psi}\not{\nabla}\not{\nabla}\not{\nabla}\psi$

Mean Field Theory (MFT)

- We now proceed to obtain a fermionic mean field approximation to evaluate the thermodynamic properties of this EFT.
- We use the operator Identity,

$$\bar{\psi}_\alpha \psi_\beta = \delta_{\alpha\beta}\langle\bar{\psi}\psi\rangle + \left(\bar{\psi}_\alpha\psi_\beta - \delta_{\alpha\beta}\langle\bar{\psi}\psi\rangle\right)$$

where α and β represents combined spinor-colour-flavor indices.

- With this identity our current-current operators in the MFT limit becomes, $\left(\bar{\psi}\Gamma\psi\right)^2 = 2\langle\bar{\psi}\psi\rangle\left[\mathrm{Tr}(\Gamma)(\bar{\psi}\Gamma\psi) - (\bar{\psi}\Gamma\Gamma\psi)\right] - \langle\bar{\psi}\psi\rangle^2\left[(\mathrm{Tr}(\Gamma))^2 - \mathrm{Tr}(\Gamma\Gamma)\right]$
- Hence our MFT Lagrangian takes the form,

$$\begin{aligned}\mathcal{L}^{MFT} &= \bar{\psi}\partial\!\!\!/_4\psi + d^4\bar{\psi}\nabla\!\!\!\!/\,\psi + d^3T_0\bar{\psi}\psi \\ &\quad - \mathcal{N}\frac{T_0^2}{4\lambda}\Sigma^2 + \Sigma\bar{\psi}\psi + \frac{\tilde{d}^6}{T_0^2}\bar{\psi}\nabla\!\!\!\!/\,\nabla\!\!\!\!/\,\nabla\!\!\!\!/\,\psi \\ &= \bar{\psi}\partial\!\!\!/_4\psi + d^4\bar{\psi}\nabla\!\!\!\!/\,\psi + m\bar{\psi}\psi \\ &\quad - \mathcal{N}\frac{T_0^2}{4\lambda}\Sigma^2 + \frac{\tilde{d}^6}{T_0^2}\bar{\psi}\nabla\!\!\!\!/\,\nabla\!\!\!\!/\,\nabla\!\!\!\!/\,\psi\end{aligned}$$

Where, $\Sigma = \frac{2\lambda}{T_0^2}\langle\bar{\psi}\psi\rangle$

And, $m = d^3T_0 + \Sigma$

With, $\mathcal{N} = 4N_fN_c$ ($N_f \to 2$, $N_c \to 3$)

- With, $\lambda = (\mathcal{N}+2)d^{61} - 2d^{62} - d^{63} + 3d^{64} + d^{65} - 3d^{66} - \frac{3}{2}d^{69} + \frac{3}{2}d^{60}$

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Four LECs

Where, $\Sigma = \dfrac{2\lambda}{T_0^2}\langle\bar{\psi}\psi\rangle$

And, $m = d^3T_0 + \Sigma$

With, $\mathcal{N} = 4N_fN_c$ ($N_f \to 2$, $N_c \to 3$)

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Pionic fluctuations around mean field theory

- The pionic theory can be realized by writing fluctuations about condensate in axial direction as local isospin waves parametrized by,

$$\psi \to e^{i\pi^a \tau^a \gamma_5/(2f)}\psi \quad \text{, and} \quad \bar{\psi} \to \bar{\psi} e^{i\pi^a \tau^a \gamma_5/(2f)}$$

- Integrating out fermions up-to one loop we get the general for of the pionic effective theory to look like,

$$\mathcal{L}_f^\pi = \frac{1}{2}\left[(\partial_4 \pi)^2 + c^4 (\nabla \pi)^2 + c^2 T_0^2 \pi^2\right] + \mathcal{L}_6^\pi$$

- Matching two point functions in the original MFT and pionic theory we get,

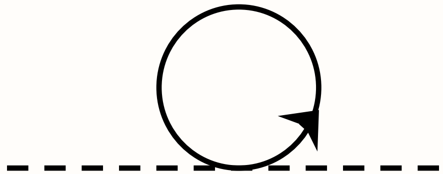
$$f^2 = -\frac{\mathcal{N}}{4}\mathcal{I}_{44}^{(1)}(0) = -\frac{\mathcal{N}}{4}\left[\mathcal{I}_{44}^{(0)}(0) + \Delta\mathcal{I}_{44}^{(1,0)}(0)\right]$$

$$c^2 T_0^2 = -\frac{4\mathcal{I}^{(1)}(0)}{\mathcal{I}_{44}^{(1)}(0)} = -4\frac{\mathcal{I}^{(0)}(0) + \Delta\mathcal{I}^{(1,0)}(0)}{\mathcal{I}_{44}^{(0)}(0) + \Delta\mathcal{I}_{44}^{(1,0)}(0)} \quad , \quad c^4 = \frac{\mathcal{I}_{ii}^{(1)}(0)}{\mathcal{I}_{44}^{(1)}(0)} = \frac{\mathcal{I}_{ii}^{(0)}(0) + \Delta\mathcal{I}_{ii}^{(1,0)}(0) + \Delta\mathcal{I}_{ii}^{(0,1)}(0)}{\mathcal{I}_{44}^{(0)}(0) + \Delta\mathcal{I}_{44}^{(1,0)}(0)}$$

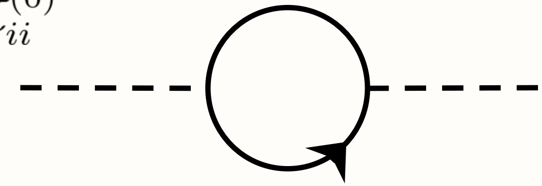
Pionic fluctuations around mean field theory

- Where,

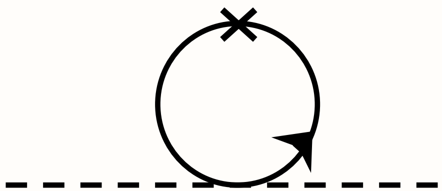
$\mathcal{I}^{(0)}$



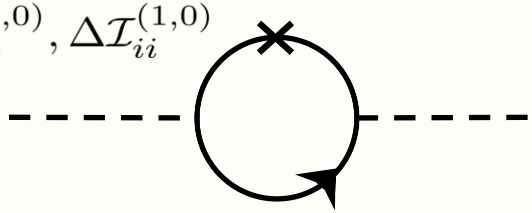
$\mathcal{I}^{(0)}, \mathcal{I}^{(0)}_{44}, \mathcal{I}^{(0)}_{ii}$

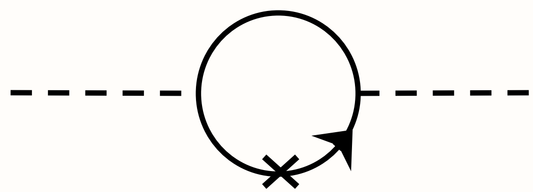


$\Delta\mathcal{I}^{(1,0)}$

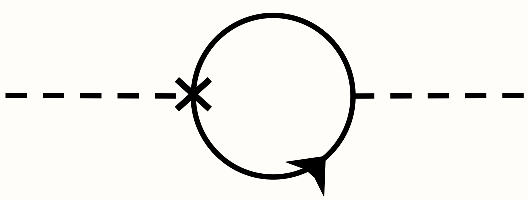


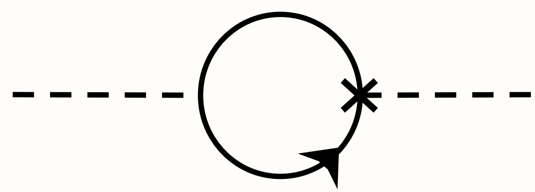
$\Delta\mathcal{I}^{(1,0)}, \Delta\mathcal{I}^{(1,0)}_{44}, \Delta\mathcal{I}^{(1,0)}_{ii}$

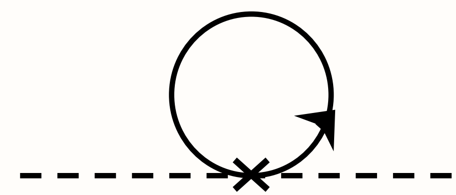




$\Delta\mathcal{I}^{(0,1)}_{ii}$







Matching with lattice results

- To obtain the predictions from our EFT, we have to fix the LECs of the theory.
- We fix the LECs in our theory by fitting our parameters of pionic theory against the lattice results.
- We particularly use the lattice results of Brandt *et al.,* Phys. Rev. D **90** (2014) no.5, 054509.
- Their definitions of pionic theory constants (u_f, f_π, m_π) are related to our definitions by,

$$u_f = \sqrt{c^4} \quad , \quad f_\pi = f\sqrt{c^4} \quad , \quad m_\pi = T_0\sqrt{c^2/c^4}$$

- We use the lattice data set C1, at T=177 MeV and chi square fit $u_f, f_\pi/T, m_\pi/T$ and T_{co} to get the best-fit values of LECs, and then proceed to evaluate the errors associated with both dependent and independent variable using bootstrap method.

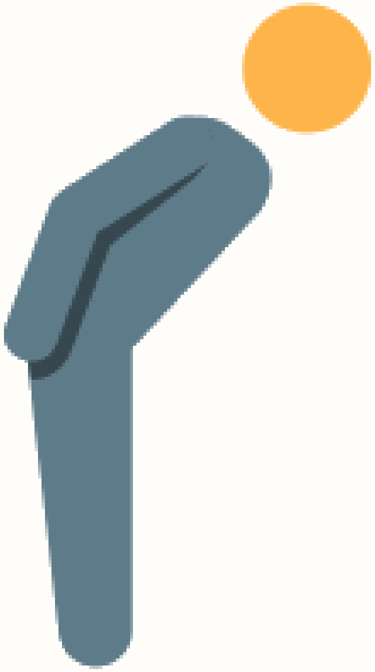
Values of LECs and the dependent variables

T_{Lat} (MeV)	$\frac{M}{\pi T_0}$	T_0 (MeV)	$\chi^2_{best-fit}$	d^3	d^4	$\tilde{d}^6$	λ
177	2	650	5.80×10^{-11}	$0.1940^{+0.0103}_{-0.0169}$	$1.2537^{+0.0778}_{-0.0732}$	$-0.0062^{+0.0768}_{-0.0449}$	$465.35^{+95.55}_{-80.68}$

In the Chiral Limit,

$T_c(0)$ = Critical temperature, κ_2 = curvature of critical line, κ_4 = higher order curvature

$T_c(0)$ (MeV)	κ_2	κ_4
$147.19^{+4.64}_{-4.47}$	$0.0169^{+0.0004}_{-0.0004}$	$0.00014^{+0.00001}_{-0.00001}$



Thank You.

Back-up Slides

Free Energy related to MFT

- The Free energy density of this MFT is calculated to be,

$$\Omega(\Sigma, m, T) = -\mathcal{N}\left[\frac{T_0^2}{4\lambda}\Sigma^2 + I^{\text{tot}}(\Sigma, m, T)\right]$$

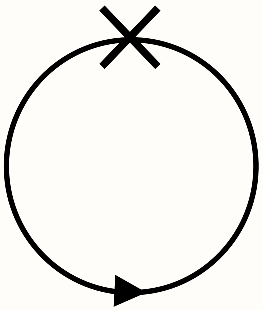
where, $I^{\text{tot}}(\Sigma, m, T) = I(\Sigma, m, T) + I^g(\Sigma, m, T)$

$$I^g(\Sigma, m, T) = -\frac{\tilde{d}^6}{(d^4)^3 T_0^2}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{(d^4)^4 \mathbf{p}^4}{p^2 + m^2}$$

$$= \frac{\tilde{d}^6}{(d^4)^3 T_0^2}\frac{m^6\left(23 - 30\log\left(\frac{m^2}{(d^4)^2 M^2}\right)\right)}{768\pi^2 (d^4)^3}$$

$$+ \frac{\tilde{d}^6}{(d^4)^3 T_0^2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3 (d^4)^3}\frac{p^4}{E_p\left(e^{E_p/T} + 1\right)}$$

From gradient term



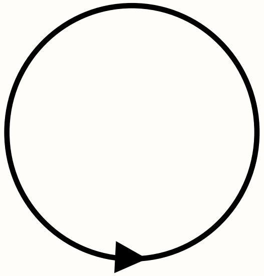
After performing dimensional regularization in 3 dimension

$$I(\Sigma, m, T) = \frac{1}{2}\int \frac{\mathrm{d}^4 p}{(2\pi^4)}\log(p^2 + m^2)$$

$$= \frac{m^4}{64\pi^2 (d^4)^3}\left(\log\left(\frac{m^2}{(d^4)^2 M^2}\right) - \frac{3}{2}\right)$$

$$+ \frac{T}{(2\pi)^3 (d^4)^3}\int \mathrm{d}^3 p\ \log\left(e^{-\frac{E_p}{T}} + 1\right)$$

From rest of the Fermionic Lagrangian



The Gap Equation and Condensate

- The value of the condensate can be obtained as a solution of the Gap equation.
- The Gap Equation correspond to the equation, $\frac{\partial \Omega}{\partial \Sigma} = 0$
- Hence, written implicitly the gap equation becomes,

$$-\mathcal{N}\left[\frac{T_0^2}{2\lambda}\Sigma + I_1(m,T) + I_1^g(m,T)\right] = 0$$

- Where the subscript 1 in I terms, signifies the first derivative with respect to condensate has been taken.
- Hence the condensate turn out to be,

$$\frac{T_0^2}{2\lambda}\Sigma = -\left[I_1(m,T) + I_1^g(m,T)\right]$$

- Explicit expressions of derivatives of `I' s are quite cumbersome and are not provided here for simplicity.

Critical Temperature at Chiral Limit

- We can also proceed to calculate the Critical Temperature T_c in the chiral limit, i.e. at $d^3 = 0$.
- The expression of critical temperature can be obtained from solving the equation,

$$\left.\frac{\partial^2\Omega(\Sigma, T_c)}{\partial^2\Sigma}\right|_{d^3=0} = 0$$

Which leads to,

$$-\mathcal{N}\left[\frac{T_0^2}{2\lambda} + I_2(0, T_c) + I_2^g(0, T_c)\right] = 0$$

- And we get the relation,

$$\frac{1}{\lambda} = -\left[2I_2(0, T_c) + 2I_2^g(0, T_c)\right]$$

$$= \frac{T_c^2}{12(d^4)^3T_0^2} + \frac{7\pi^2\tilde{d}^6T_c^4}{24(d^4)^6T_0^4} = \frac{T_c^2}{12(d^4)^3T_0^2}\left[1 + \frac{7\pi^2\tilde{d}^6T_c^2}{2(d^4)^3T_0^2}\right]$$

Constraint on Gradient Term and Critical Temperature

- Considering only the case when $\lambda > 0$ we obtain,

$$\tilde{d}^6 > -\frac{2(d^4)^3 T_0^2}{7\pi^2 T_c^2}$$

- Looking carefully at the relation for λ with T_c we obtain two solutions for critical temperature,

$$T_c = T_0 (d^4)^{3/2} \frac{\sqrt{\sqrt{168\pi^2 \tilde{d}^6 + \lambda} + \sqrt{\lambda}}}{\sqrt{7}\pi\sqrt{-\tilde{d}^6}\lambda^{1/4}} , \quad T_c = T_0 (d^4)^{3/2} \frac{\sqrt{\sqrt{168\pi^2 \tilde{d}^6 + \lambda} - \sqrt{\lambda}}}{\sqrt{7}\pi\sqrt{\tilde{d}^6}\lambda^{1/4}}$$

- At, $\tilde{d}^6 > 0$ the solution at LHS is ruled out.
- At $\tilde{d}^6 < 0$ both the solutions can result in real temperature iff,

$$168\pi^2 \tilde{d}^6 > -\lambda, \implies |\tilde{d}^6| < \frac{\lambda}{168\pi^2}$$

- It can be shown that the solution at rhs is the consistent solution for T_c, and the relation at lhs gives rise to a new different solution for critical temperature.

Constraint on Gradient Term and Critical Temperature

- The second solution for critical temperature is found to be at temperature,

$$\sqrt{-T_c^2 - \frac{2(d^4)^3 T_0^2}{7\tilde{d}^6 \pi^2}}$$

- This solution tends to infinity as $\tilde{d}^6 \rightarrow 0^-$. As $\tilde{d}^6$ is decreased from 0^-,
- At, $\tilde{d}^6 = -\frac{(d^4)^3 T_0^2}{7\pi^2 T_c^2}$ the second solution coincides with T_c and,
- At, $\tilde{d}^6 = -\frac{2(d^4)^3 T_0^2}{7\pi^2 T_c^2}$ the second solution results $T_c = 0$.

We neglect, the Larger solution if

$$0 > \tilde{d}^6 > -\frac{(d^4)^3 T_0^2}{7\pi^2 T_c^2}$$

We neglect, the Smaller solution if

$$-\frac{(d^4)^3 T_0^2}{7\pi^2 T_c^2} > \tilde{d}^6 > -2\frac{(d^4)^3 T_0^2}{7\pi^2 T_c^2}$$

Curvature coefficients at Chiral limit

- At finite chemical potential we know the curvature coefficients are defined as, [A. Bazzavov *et al*. [HotQCD] Phys. Lett. B 795 (2019)],

$$T_c(\mu_B) = T_c(0) - \kappa_2 \frac{\mu_B^2}{[T_c(0)]} - \kappa_4 \frac{\mu_B^4}{[T_c(0)]^3} + \mathcal{O}(\mu^6)$$

- At, $\mu_B = 3\mu$, we get, $T_c(0) = T_c$
- We find at chiral limit the curvature coefficients at leading order at $\tilde{d}^6$ is given by following expressions,

$$\kappa_2 = \left[\frac{1}{6\pi^2} - \frac{\tilde{d}^6 T_c(0)^2}{3(d^4)^3 T_0^2}\right] = \frac{1}{6\pi^2}\left[1 - \frac{2\pi^2 \tilde{d}^6 T_c(0)^2}{(d^4)^3 T_0^2}\right]$$

and,

$$\kappa_4 = \left[\frac{1}{72\pi^4} - \frac{5\tilde{d}^6 T_c(0)^2}{54(d^4)^3 \pi^2 T_0^2}\right] = \frac{1}{72\pi^4}\left[1 - \frac{20\pi^2 \tilde{d}^6 T_c(0)^2}{3(d^4)^3 T_0^2}\right]$$