Automated Evaluation of Feynman integrals using GKZ hypergeometric systems

Based on: arxiv.2211.01285

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Need for precision calculations

- SM is not sufficient to explain the entire spectrum of observable physical phenomena - DM, dark energy, gravity ...
- Ongoing searches for BSM physics: higher luminosities and energy frontiers!
- Technological and financial challenges
- Extract more insights from whatever data we already have
- Fixed order precision calculations - involve multiloop multiscale Feynman diagrams. Complementary to the EFT approach[1].
- Calculate the corresponding integrals - usually very complicated.

[1] Jonas Klappert. "Precision calculations for Higgs physics in the standard model and beyond". PhD thesis. RWTH Aachen U., 2020. DOI: 10.18154/RWTH-2020-05782.

Our contribution

A proof-of-concept implementation demonstrating the utility of recent mathematical developments in evaluating such Feynman integrals in the form of a *Mathematica* package `FeynGKZ`[2].

[2] B. Ananthanarayan et al. “FeynGKZ: a Mathematica package for solving Feynman integrals using GKZ hypergeometric systems”. In: (Nov. 2022). arXiv: *2211.01285 [hep-th]*.

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Feynman integrals

The momentum representation

- Typically involve tensor structures in numerator - do tensor reduction
- Calculate the scalar integrals
- Momentum representation:

$$I_\Gamma(\nu, D) = \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{1}{\prod_{j=1}^{n}(-q_j^2 + m_j^2)^{\nu_j}} \tag{1}$$

l: number of loops
D: the space-time dimension
$\nu = (\nu_1, ..., \nu_n)$: propagator powers
k_r-s and q_j-s are the loop-momenta and internal-momenta for the Feynman graph Γ.

The Lee-Pomeransky representation

- An alternate form[3]:

$$
\begin{aligned}
I_\Gamma(\nu, D) &= \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D}{2}-\omega)}\Big(\prod_{i=1}^{n}\int_{\alpha_i=0}^{\infty}\frac{d\alpha_i\,\alpha_i^{\nu_i-1}}{\Gamma(\nu_i)}\Big)G(\alpha)^{-\frac{d}{2}} \\
&= \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D}{2}-\omega)\Gamma(\nu)}\int_{\mathbb{R}_+^n} d\alpha\,\alpha^{\nu-1}G(\alpha)^{-\frac{d}{2}}
\end{aligned}
\tag{2}
$$

- Lee-Pomeransky polynomial: $G(\alpha) = U(\alpha) + F(\alpha)$.

[3] Roman N. Lee and Andrei A. Pomeransky. “Critical points and number of master integrals”. In: *JHEP* 11 (2013), p. 165. DOI: 10.1007/JHEP11(2013)165. arXiv: 1308.6676 [hep-ph].

The Lee-Pomeransky representation (contd.)

- Generalized G-polynomial:

$$G_z(\alpha) = \sum_{a_j \in A} z_j \alpha^{a_j} = \sum_{j=1}^{N} z_j \prod_{i=1}^{n} \alpha_i^{a_{ij}} \tag{3}$$

  $z_j \to$ generic/indeterminate

- Generalized Feynman integral:

$$I_{G_z}(\nu, \nu_0) = \Gamma(\nu_0) \int_{\mathbb{R}_+^n} d\alpha\, \alpha^{\nu-1} G_z(\alpha)^{-\nu_0} \tag{4}$$

  where, $\nu_0 = \frac{D}{2}$

The associated GKZ system and its solutions

The associated GKZ system

$I_{G_z}(\nu, \nu_0)$ satisfies a holonomic system of PDEs called a *GKZ hypergeometric system*[4].

Ideals
Let $P = \mathbb{F}[x_1, ..., x_n]$ be some polynomial ring in $x_1, ..., x_n$ over $\mathbb{F}$.
$\mathcal{I} \subset P$ is said to be an ideal if

- $0 \in \mathcal{I}$
- $f + g \in \mathcal{I} \quad \forall f, g \in \mathcal{I}$
- $f \cdot g \in \mathcal{I} \quad \forall f \in P, g \in \mathcal{I}$

Thus, $\langle \mathcal{S} \rangle = \sum_i f_i g_i;\ f \in P, g \in S$ is the ideal spanned by $S \subset P$.

[4] I.M. Gelfand, Mikhail M. Kapranov, and Andrey V. Zelevinsky. "Discriminants, Resultants, and Multidimensional Determinants". In: 1994; I.M. Gelfand, Mikhail M. Kapranov, and Andrei Zelevinsky. "Generalized Euler integrals and A-hypergeometric functions". In: *Advances in Mathematics* 84 (1990), pp. 255–271.

The associated GKZ system (contd.)

- We describe the GKZ system as follows:

$$H_{\mathcal{A}}(\underline{\nu}) = I_{\mathcal{A}} \cup \langle \mathcal{A} \cdot \theta + \underline{\nu} \rangle \tag{5}$$

$$\begin{aligned} \mathcal{A} &= \{a_{ij}; i \in \{1, ..., n+1\}, j \in \{1, ..., N\}\} | a_{ij} = 1; i = 1\} \\ \underline{\nu} &= (\nu_0, \nu_1, ..., \nu_n)^T \end{aligned} \tag{6}$$

- $\mathcal{A} \rightarrow (n+1) \times N$ matrix; $n + 1 \leq N$
- $\theta = (\theta_1, \ldots, \theta_N)^T$; $\theta_i = z_i \partial_i \rightarrow$ Euler operators
- **Assume**: (1,...,1) lies in $\mathbb{Q}$-row span of $\mathcal{A}$

The associated GKZ system (contd.)

- $H_{\mathcal{A}}(\underline{\nu}) I_{G_z}(\nu, \nu_0) = 0$
- $I_{G_z}(\nu, \nu_0) \to$ *GKZ hypergeometric function*![5]

[5] René Pascal Klausen. "Hypergeometric Series Representations of Feynman Integrals by GKZ Hypergeometric Systems". In: (2020). eprint: *1910.08651*; Leonardo de la Cruz. "Feynman integrals as A-hypergeometric functions". In: (2019). eprint: *1907.00507*.

Solving the GKZ system

- **Algebraically**: the SST algorithm[6] $\to$ the Gröbner deformation method
- **Geometrically**: the triangulation method
- Both are equivalent!
- In this talk, focus on the **geometric** picture.

[6] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama. *Gröbner deformations of hypergeometric differential equations*. Vol. 6. Springer Science & Business Media, 2013.

Solving the GKZ system (contd.)

- We saw:

$$\mathcal{A} = \begin{pmatrix} 1 \\ A \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_N \end{pmatrix} \in \mathbb{Z}_{\geq 0}^{(n+1)\times N} \tag{7}$$

- A defines an assembly of N points (a point configuration) in $\mathbb{Z}^n$

$$\text{Conv}(A) := \left\{ \sum_{j=1}^{N} k_j a_j \middle| k \in \mathbb{R}_{\geq 0}^{N}, \sum_{j=1}^{N} k_j = 1 \right\} \tag{8}$$

- Newton polytope of $G_z(\alpha)$:

$$\Delta_{G_z} := Conv(A) \tag{9}$$

Solving the GKZ system (contd.)

- Triangulate Δ_{G_z}!
- Triangulation structure: $T = \{\sigma_1, ..., \sigma_r\}$.
- $\sigma_i \subset \{1, ..., N\}$ is some index set.

Solving the GKZ system (contd.)

Can always obtain a regular triangulation![7]

[7] Israel M. Gelfand, Mikhail M. Kapranov, and Andrei Zelevinsky. "HYPERGEOMETRIC FUNCTIONS, TORIC VARIETIES AND NEWTON POLYHEDRA". In: 1991.

Solving the GKZ system (contd.)

Can always obtain a unimodular regular triangulation ($\mathrm{vol}_0(\sigma_i) = 1$)![8]

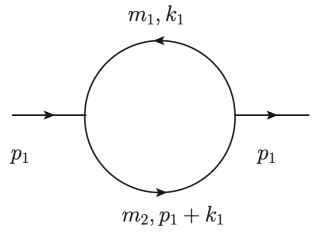
[8] W. Bruns and J. Gubeladze. *Polytopes, Rings, and K-Theory*. ISBN: 9780387763569; Finn F. Knudsen. "Construction of nice polyhedral subdivisions". In: 1973.

Solving the GKZ system (contd.)

- Regular triangulations can be used to construct a basis for the finite-dimensional solution space of $H_{\mathcal{A}}(\underline{\nu})$
- Each element: Γ-series
- Whole solution: linear combination of the Γ-series elements
- Unimodularity: one $\sigma_i \to$ one Γ-series
- Might as well use just the unimodular regular triangulations to construct a basis!

Demonstration

Bubble diagram with two unequal masses



The corresponding integral in momentum-representation:

$$I_\Gamma(\nu_1, \nu_2, D; p_1^2) = \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \frac{1}{(-k_1^2 + m_1^2)^{\nu_1}(-(p_1 + k_1)^2 + m_2^2)^{\nu_2}} \tag{10}$$

with two unequal masses m_1 and m_2, and external momentum p_1.

Bubble diagram with two unequal masses (contd.)

After successfully loading the package and installing its dependencies, specify the integral in its momentum representation as:

```
In[3]:=    MomentumRep = {{k1, m1, a1}, {p1 + k1, m2, a2}};
           LoopMomenta = {k1};
           InvariantList = {p1^2 -> -s};
           Dim = 4 - 2ϵ;
           Prefactor = 1;
```

Bubble diagram with two unequal masses (contd.)

Now derive the $\mathcal{A}$-matrix:

In[4]:=	FindAMatrixOut = FindAMatrix[{MomentumRep, LoopMomenta, InvariantList, Dim, Prefactor}, UseMB → False];
Prints ⇒	The Symanzik polynomials → $U = x_1 + x_2$, $F = m_1^2 x_1^2 + s x_1 x_2 + m_1^2 x_1 x_2 + m_2^2 x_1 x_2 + m_2^2 x_2^2$ The Lee-Pomeransky polynomial → $G = x_1 + m_1^2 x_1^2 + x_2 + s x_1 x_2 + m_1^2 x_1 x_2 + m_2^2 x_1 x_2 + m_2^2 x_2^2$ The associated $\mathcal{A}$−matrix → $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \end{pmatrix}$, which has codim = 2. Normalized Volume of the associated Newton Polytope → 3 Time Taken 1.50005 seconds

Bubble diagram with two unequal masses (contd.)

Compute the unimodular regular triangulations[9]:

```
In[5]:=    Triangulations = FindTriangulations[FindAMatrixOut];

Prints ⇒   Finding all regular triangulations ...
           Found 5 Regular Triangulations, out of which 3 are Unimodular
           The 3 Unimodular Regular Triangulations →
           1 ::  {{1,2,3},{2,3,4},{3,4,5}}
           2 ::  {{1,2,3},{2,4,5},{2,3,5}}
           3 ::  {{2,4,5},{1,3,5},{1,2,5}}
           Time Taken 0.126965 seconds
```

[9] Jörg Rambau. “TOPCOM: Triangulations of Point Configurations and Oriented Matroids”. In: *Proceedings of the International Congress of Mathematical Software.* 2002. URL: *http://www.zib.de/PaperWeb/abstracts/ZR-02-17*.

Bubble diagram with two unequal masses (contd.)

Calculate the Γ-series:

```
In[7]:=   SeriesSolution = SeriesRepresentation[Triangulations,2];
```

Prints ⇒

```
Unimodular Triangulation → 2
Number of summation variables → 2
```

Non-generic limit → $\{z_1 \to m_1^2,\ z_2 \to s + m_1^2 + m_2^2,\ z_3 \to 1,\ z_4 \to m_2^2,\ z_5 \to 1\}$

The series solution is the sum of following 3 terms.

Term 1 ::

$$\Big(\Big((-1)^{-n_1-n_2}\,\text{Gamma}[-2+\epsilon+a_1-n_1-n_2]\,\text{Gamma}[4-2\epsilon-a_1-a_2+n_2]\,\text{Gamma}[a_2+2n_1+n_2]\,(m_1^2)^{2-\epsilon-a_1}\left(\frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}\right)^{n_1}\left(\frac{m_1^2}{s+m_1^2+m_2^2}\right)^{n_2}(s+m_1^2+m_2^2)^{-a_2}\Big)\Big/\big(\text{Gamma}[a_1]\,\text{Gamma}[4-2\epsilon-a_1-a_2]\,\text{Gamma}[a_2]\,\text{Gamma}[1+n_1]\,\text{Gamma}[1+n_2]\big)\Big)$$

Term 2 ::

$$\Big(\Big((-1)^{-n_1-n_2}\,\text{Gamma}[-2+\epsilon+a_2-n_1-n_2]\,\text{Gamma}[4-2\epsilon-a_1-a_2+n_2]\,\text{Gamma}[a_1+2n_1+n_2]\,(m_2^2)^{2-\epsilon-a_2}\left(\frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}\right)^{n_1}\left(\frac{m_2^2}{s+m_1^2+m_2^2}\right)^{n_2}(s+m_1^2+m_2^2)^{-a_1}\Big)\Big/\big(\text{Gamma}[a_1]\,\text{Gamma}[4-2\epsilon-a_1-a_2]\,\text{Gamma}[a_2]\,\text{Gamma}[1+n_1]\,\text{Gamma}[1+n_2]\big)\Big)$$

Term 3 ::

$$\Big(\Big((-1)^{-n_1-n_2}\text{Gamma}[2-\epsilon-a_2+n_1-n_2]\,\text{Gamma}[2-\epsilon-a_1-n_1+n_2]\,\text{Gamma}[-2+\epsilon+a_1+a_2+n_1+n_2]\left(\frac{m_1^2}{s+m_1^2+m_2^2}\right)^{n_1}\left(\frac{m_2^2}{s+m_1^2+m_2^2}\right)^{n_2}(s+m_1^2+m_2^2)^{2-\epsilon-a_1-a_2}\Big)\Big/\big(\text{Gamma}[a_1]\,\text{Gamma}[4-2\epsilon-a_1-a_2]\,\text{Gamma}[a_2]\,\text{Gamma}[1+n_1]\,\text{Gamma}[1+n_2]\big)\Big)$$

```
Time Taken 0.066558 seconds
```

Bubble diagram with two unequal masses (contd.)

Check for an expression in terms of known hypergeometric functions[10]:

```
In[8]:=   GetClosedForm[SeriesSolution];
```

Prints ⇒ `Closed form found with Olsson!`

Term 1 ::

$$\frac{1}{\mathrm{Gamma}[a_1]}\,\mathrm{Gamma}[-2+\epsilon+a_1]\;\mathrm{H3}\left[a_2, 4-2\epsilon-a_1-a_2, 3-\epsilon-a_1, \frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}, \frac{m_1^2}{s+m_1^2+m_2^2}\right]\; m_1^4\,(m_1^2)^{-\epsilon-a_1}(s+m_1^2+m_2^2)^{-a_2}$$

Term 2 ::

$$\frac{1}{\mathrm{Gamma}[a_2]}\,\mathrm{Gamma}[-2+\epsilon+a_2]\;\mathrm{H3}\left[a_1, 4-2\epsilon-a_1-a_2, 3-\epsilon-a_2, \frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}, \frac{m_2^2}{s+m_1^2+m_2^2}\right]\; m_2^4\,(m_2^2)^{-\epsilon-a_2}(s+m_1^2+m_2^2)^{-a_1}$$

Term 3 ::

$$\left(\left(\mathrm{G1}\left[-2+\epsilon+a_1+a_2, 2-\epsilon-a_1, 2-\epsilon-a_2, -\frac{m_2^2}{s+m_1^2+m_2^2}, -\frac{m_1^2}{s+m_1^2+m_2^2}\right]\,\mathrm{Gamma}[2-\epsilon-a_1]\,\mathrm{Gamma}[2-\epsilon-a_2]\;\mathrm{Gamma}[-2+\epsilon+a_1+a_2]\,(s+m_1^2+m_2^2)^{2-\epsilon-a_1-a_2}\right)\Big/\left(\mathrm{Gamma}[a_1]\;\mathrm{Gamma}[4-2\epsilon-a_1-a_2]\,\mathrm{Gamma}[a_2]\right)\right)$$

`Time Taken 0.05827 seconds`

[10] B. Ananthanarayan et al. “Olsson.wl : a *Mathematica* package for the computation of linear transformations of multivariable hypergeometric functions”. In: (Dec. 2021). arXiv: `2201.01189 [cs.MS]`.

Bubble diagram with two unequal masses (contd.)

Evaluate the sum of the Γ-series terms numerically:

```
In[9]:=   SumLim = 30;
          ParameterSub = {ϵ → 0.001, a1 → 1, a2 → 1, s → 10, m1 → 0.4, m2 → 0.3};
          NumericalSum[SeriesSolution, ParameterSub, SumLim];
```

```
Prints ⇒  Numerical result = 997.382
          Time Taken 0.222572 seconds
```

Summary and Future Works

Summary

- Very rich mathematical structures appear in context of computing multiloop multiscale Feynman integrals
- In-depth analysis of such structures might furnish insights for developing novel computational frameworks and algorithms for evaluating these integrals.
- Laurent expansion of hypergeometric functions in the dim-reg parameter ϵ - critical bottleneck in this approach. Known automated implementations: *HypExp*, *XSummer*, also private in-house implementations.

Future Works

- Tackle the issue of ϵ-expansion of hypergeometric functions: a new algorithm has recently been proposed in arxiv.2208.01000[11].
- Convert the standing proof-of-concept implementation into a performance-driven one. Current performance bottlenecks stem from an excess of dependence on *Mathematica*, particularly in the numerical summation step.
- Explore the scope of ***FeynGKZ*** in evaluating stringy canonical forms, inspired by arxiv.2005.07395[12]. To be taken up soon.

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[11] Souvik Bera. "ϵ-Expansion of Multivariable Hypergeometric Functions Appearing in Feynman Integral Calculus". In: (Aug. 2022). arXiv: *2208.01000 [math-ph]*.

[12] Song He et al. "Stringy canonical forms and binary geometries from associahedra, cyclohedra and generalized permutohedra". In: *JHEP* 10 (2020), p. 054. DOI: *10.1007/JHEP10(2020)054*. arXiv: *2005.07395 [hep-th]*.

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