Is the Lattice Fermionic Casimir effect Universal?*

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Introduction

Our Results: Universal ?

Summary

*Work done with Yash V. Mandlecha, arXiv:2207.00889, Phys. Lett B835,137558 (2022).

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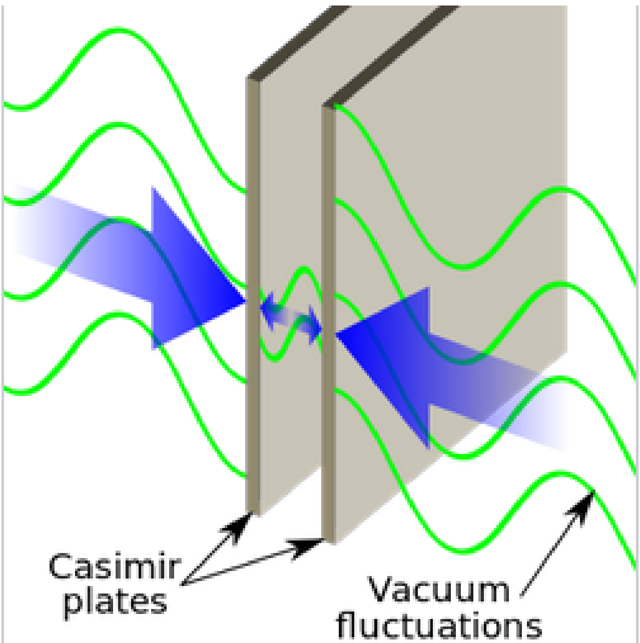

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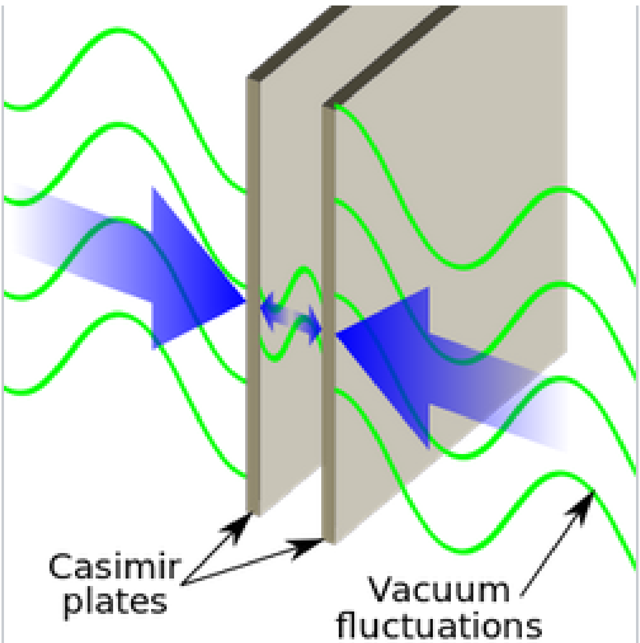

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- It has now been measured experimentally (Bressi et al PRL 2002; Lamoreaux PRL 1997). Both the variation d^4 and the magnitude $[K_{th} = 1.30 \times 10^{-27}$ N m^2 and $K_{exp} = 1.22(18) \times 10^{-27}$ N m$^2]$ of the Casimir force agrees with the theory.

♠ As depicted in the figure, the zero point energy based computation proceeds simply by taking the the difference in the its spectrum with and without the plates:

$$E_{cas}^{D+1} = E(d) - E(d \to \infty) \ ,$$

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♣ Interestingly for parallel plates, the Casimir force is attractive for vectors (i.e., photons considered originally), scalars, and fermions.

♡ It is widely recognised that the QCD vacuum is rather nontrivial, and has interesting properties of quark confinement and chiral symmetry breaking, related to various condensates. This makes a study of Casimir effect interesting in QCD.

♣ Indeed, the MIT Bag Model boundary conditions prevent the fermion current from crossing the boundary in order to ensure quark confinement.

♢ QCD on space-time lattice has proved to be the most reliable and productive tool to study its strong coupling domain of hadron spectrum, and the QCD Vacuum.

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♠ Employing both periodic and anti-periodic boundary conditions, when one has $ap_1^P(n) = 2\pi n/N$ and $ap_1^{AP}(n) = (2n+1)\pi/N$ respectively, the Casimir energy is

$$\begin{aligned} aE_{cas}^{D+1} &= aE_0(N) - aE_0(N \to \infty) \\ &= c_{\rm deg} \int \frac{d^{D-1}ap_\perp}{(2\pi)^2} \left[-\sum_n aE(ap_\perp, ap_1(n)) + N \int \frac{dap_1}{2\pi} aE(ap) \right] \quad (1) \end{aligned}$$

$c_{\rm deg}$ denotes degeneracy factor due to spin, fermion doubling etc.

♠ Nielsen-Ninomiya theorem implies lattice fermions have doubling(naıve,staggered) and/or broken chiral symmetry (Wilson) or non-locality (overlap, domain wall fermions).

♡ The detailed form of $D(p)$ in aE above is governed by this choice which for naıve fermions $(aE)^2 = \sum_{k=1}^{D} \sin^2 ap_k + (am)^2$.

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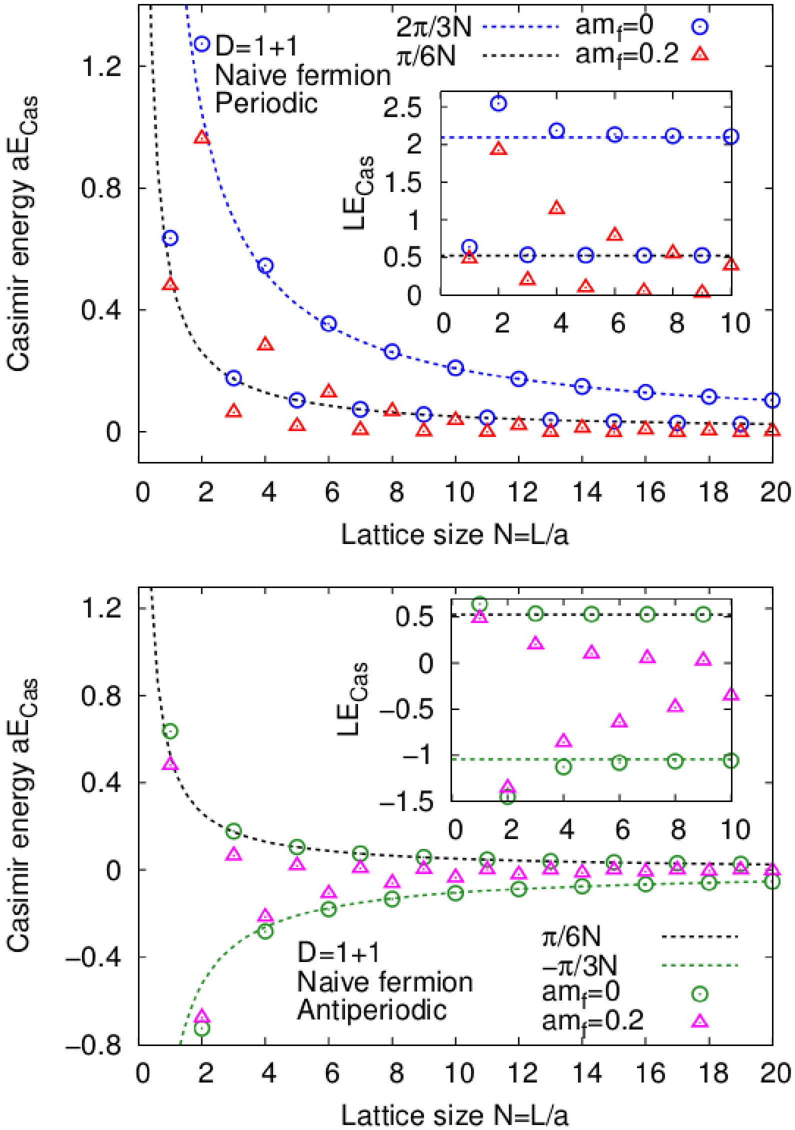
♡ For naıve fermions in 1+1 dimensions, Ishikawa et al obtained,

$$aE^P = \frac{2N}{\pi} - \cot\frac{\pi}{2N} \quad (\text{odd}) \qquad aE^P = \frac{2N}{\pi} - 2\cot\frac{\pi}{N} \quad (\text{even})$$
$$aE^{AP} = \frac{2N}{\pi} - \cot\frac{\pi}{2N} \quad (\text{odd}) \qquad aE^{AP} = \frac{2N}{\pi} - 2\csc\frac{\pi}{N} \quad (\text{even}) \qquad (2)$$



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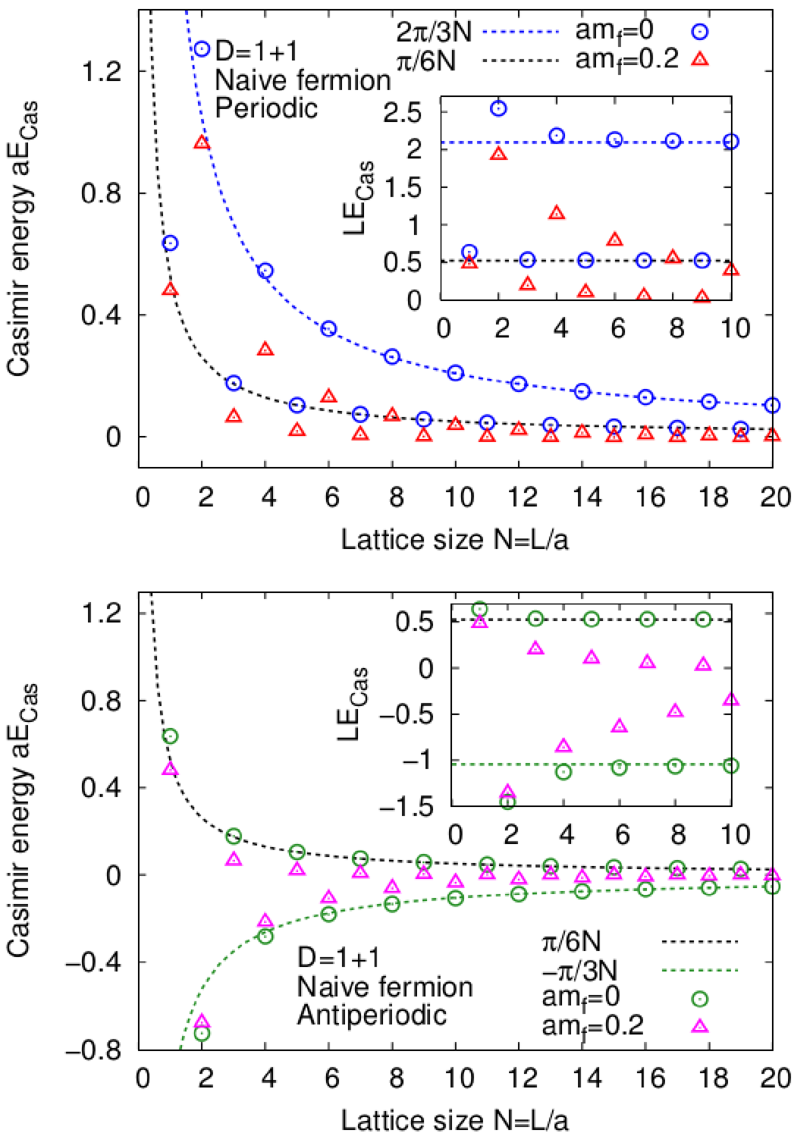

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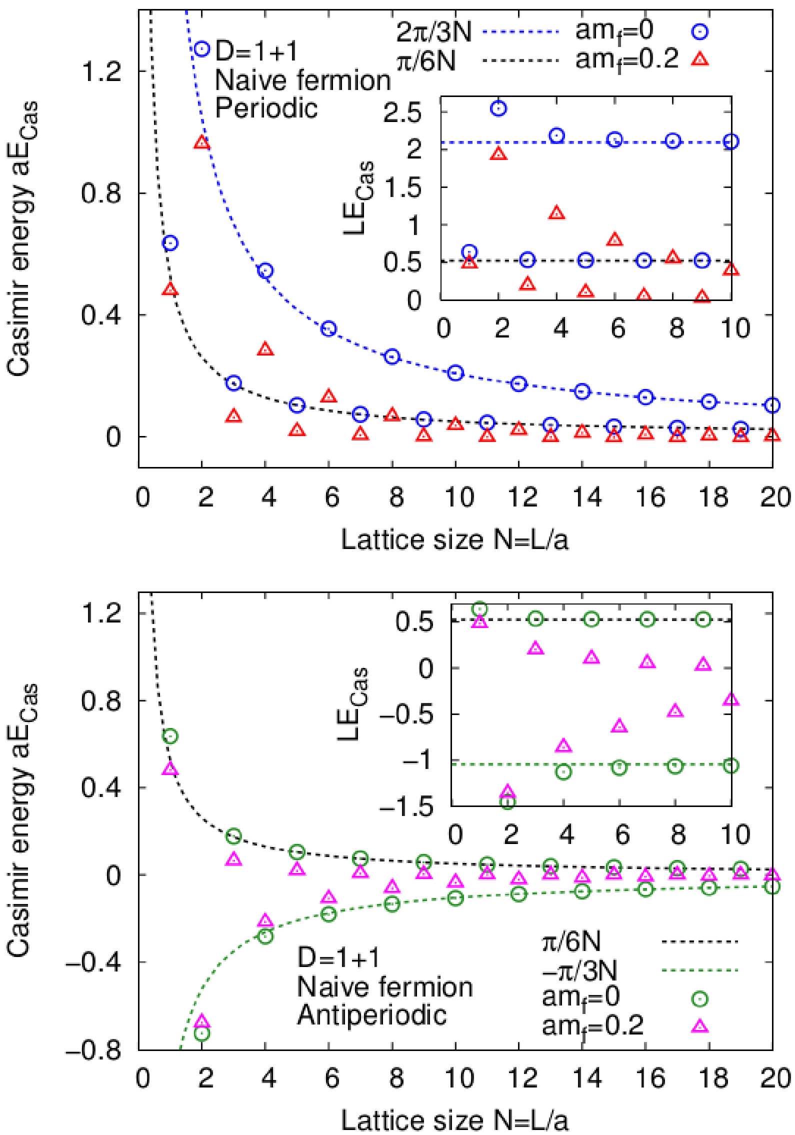
- One sees oscillations as N grows for both periodic(P)/anti-periodic(AP) case.
- Analytically, the $N \to \infty \equiv a/L \to 0$ limit leads to **three different** answers, namely $\pi/6L$ for odd N for both P/AP, and $2\pi/3L(-\pi/3L)$ for P (AP) for even N. Visible in the figures !



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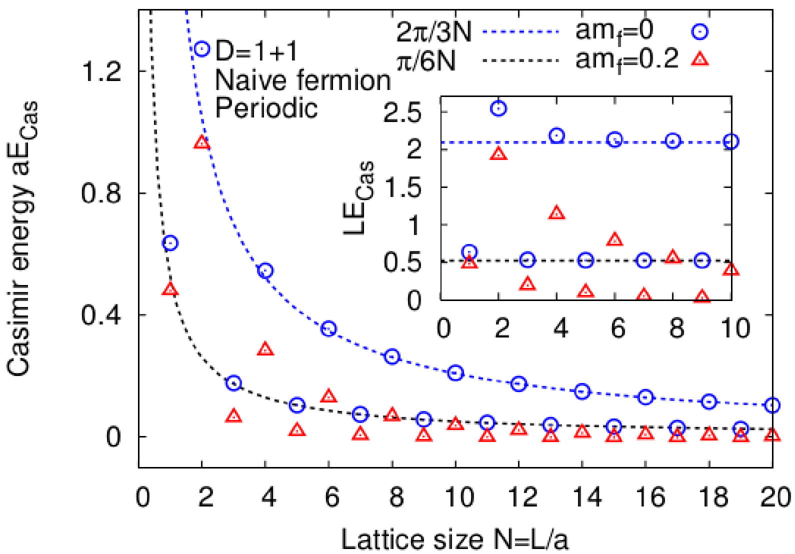

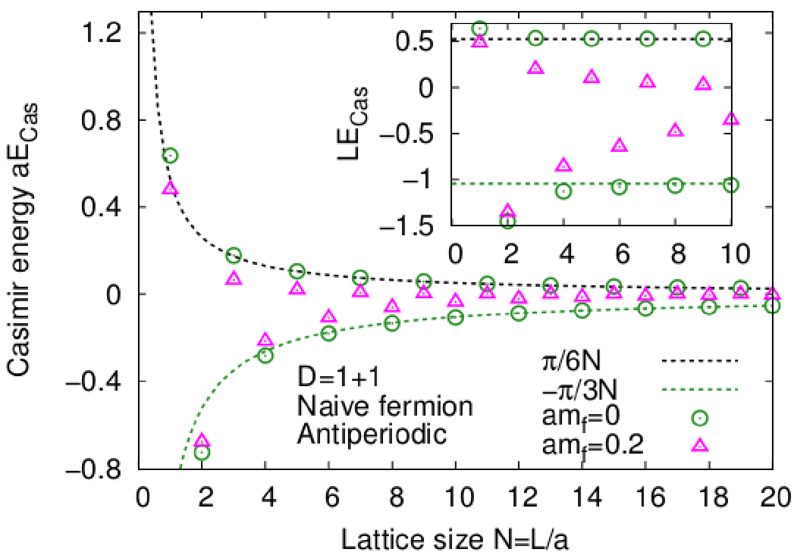

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- Violation of **Universality** !

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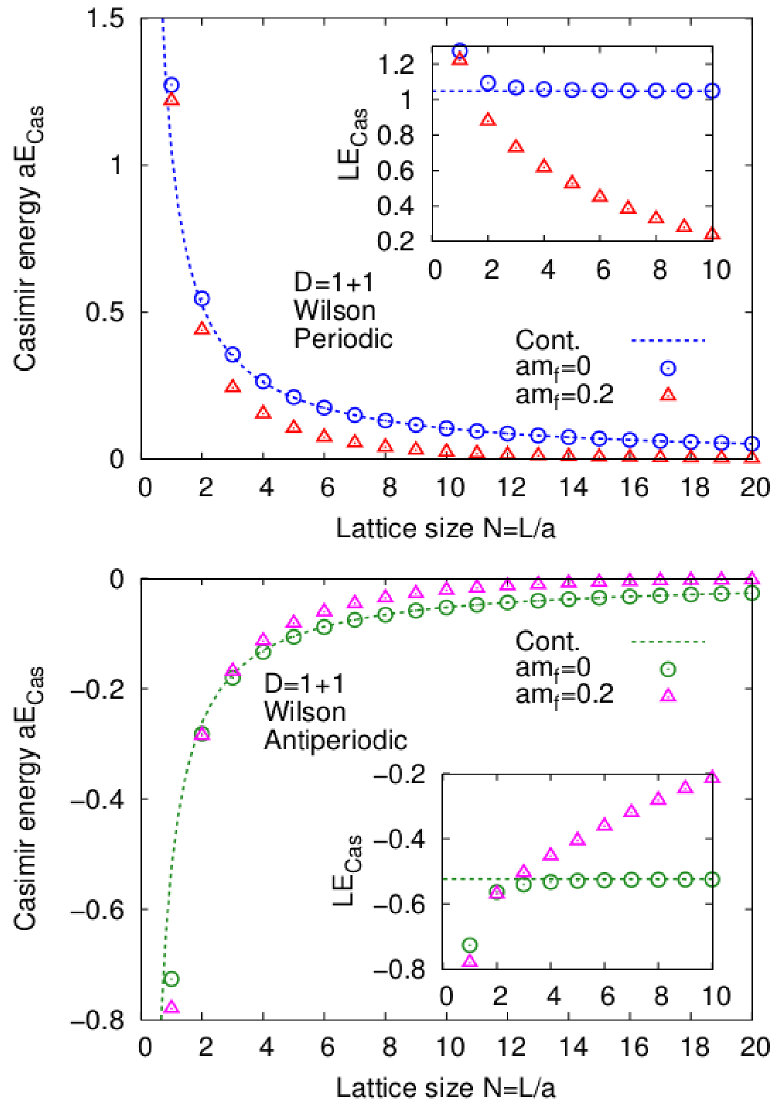

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♣ Since MIT bag boundary conditions are physically more appropriate for eventual QCD Casimir study, we proposed to check whether universality is restored by employing them. In our case, these amount to,

$$(1 + i\gamma^1)\psi|_{x^1=0,d} = 0 \ .$$

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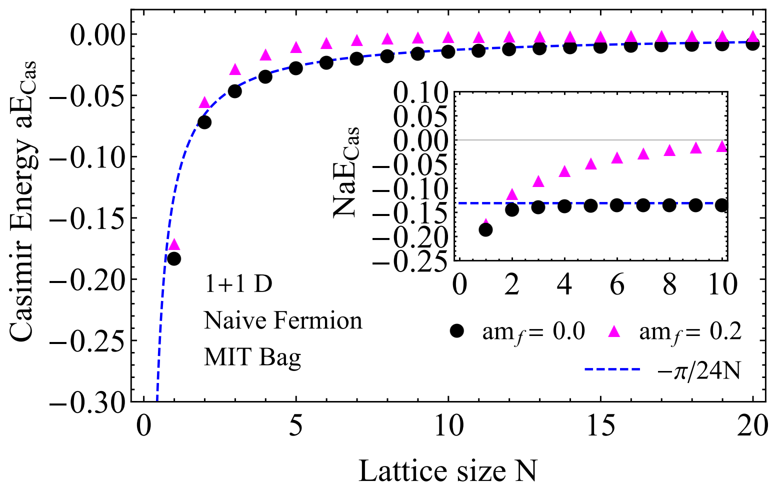
- In the $a \to 0$ limit **both** naıve and Wilson fermions lead to the $E_{cas} = -\pi/24d$ (modulo doubling for former) which in turn is the result for the continuum case with MIT bag boundary conditions.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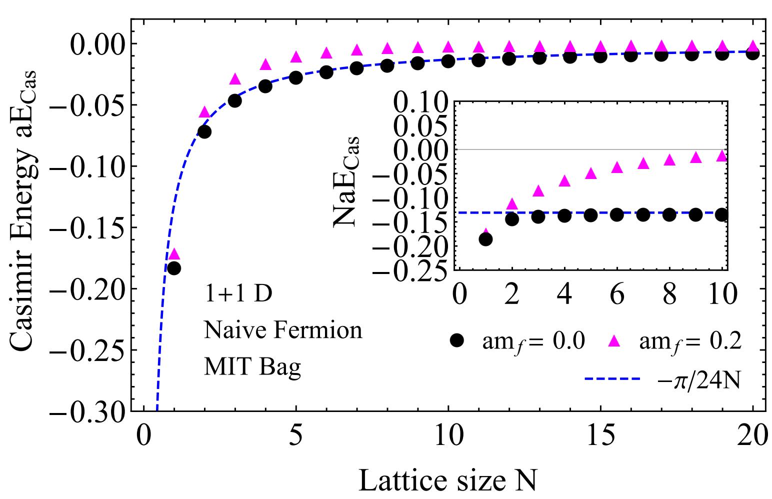
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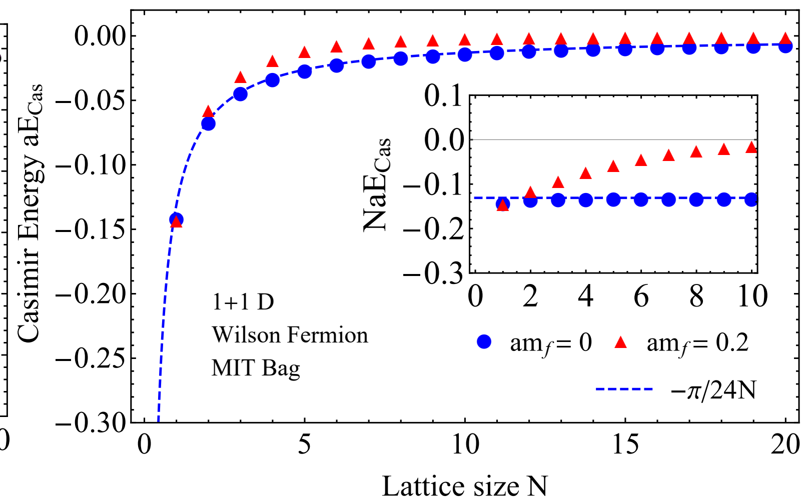
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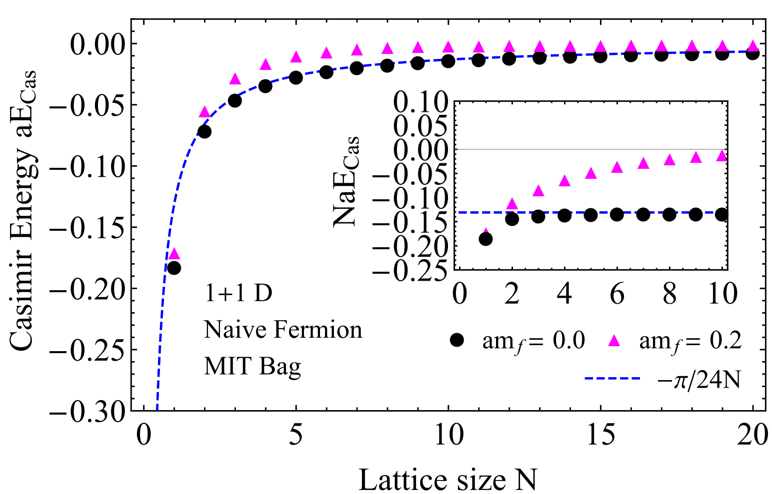

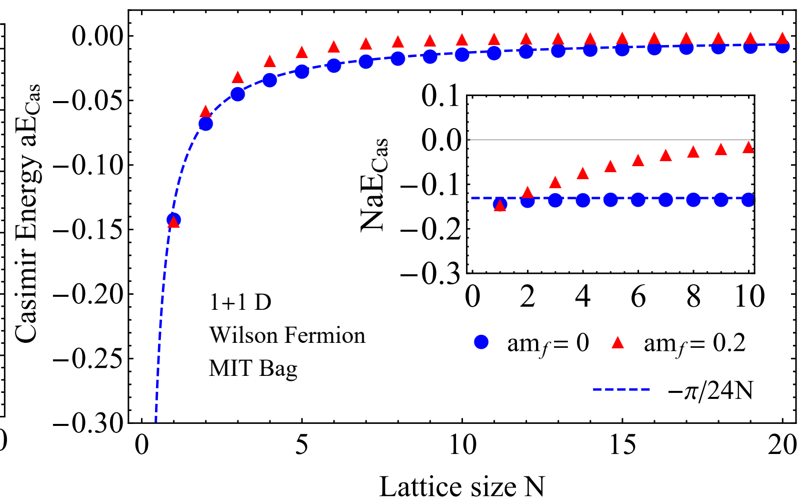

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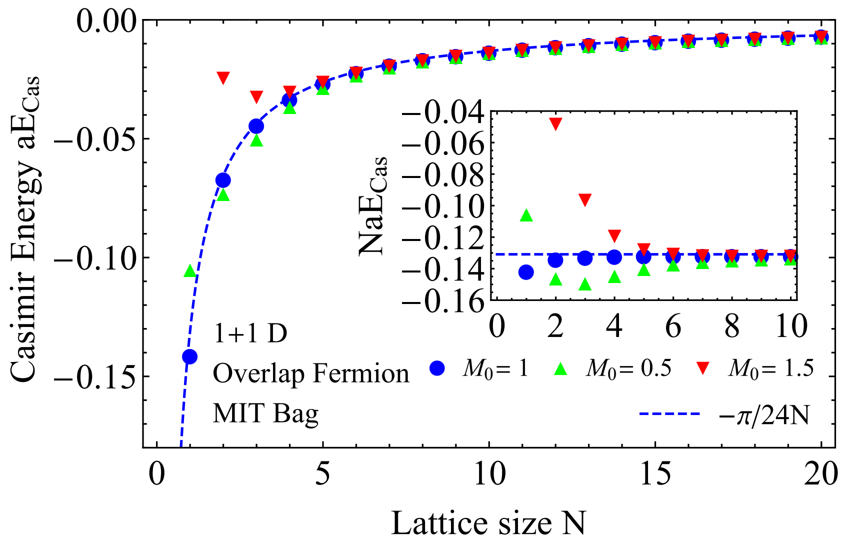

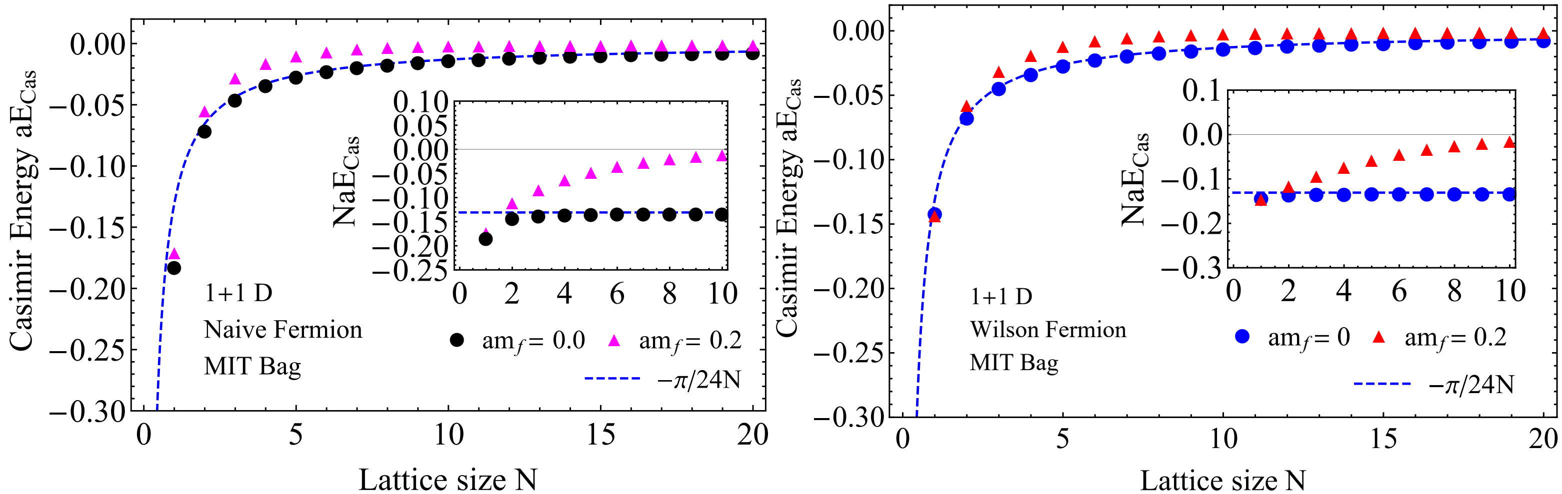

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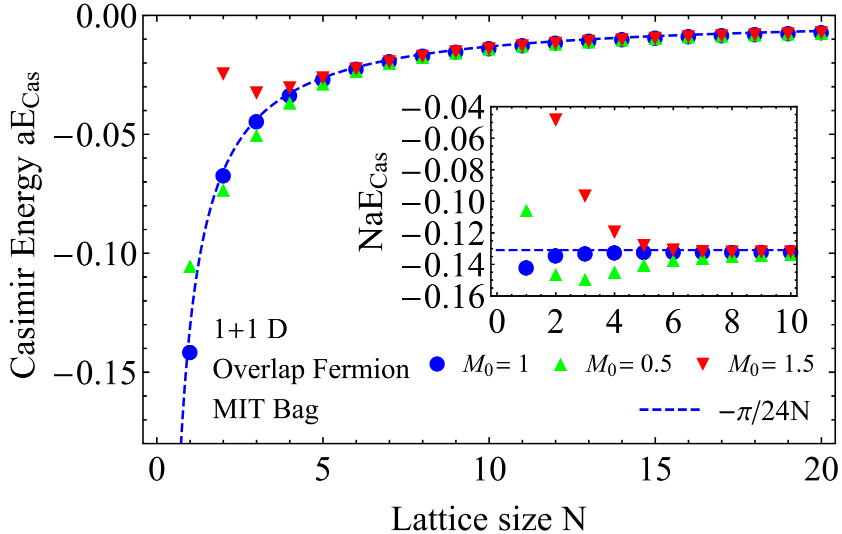

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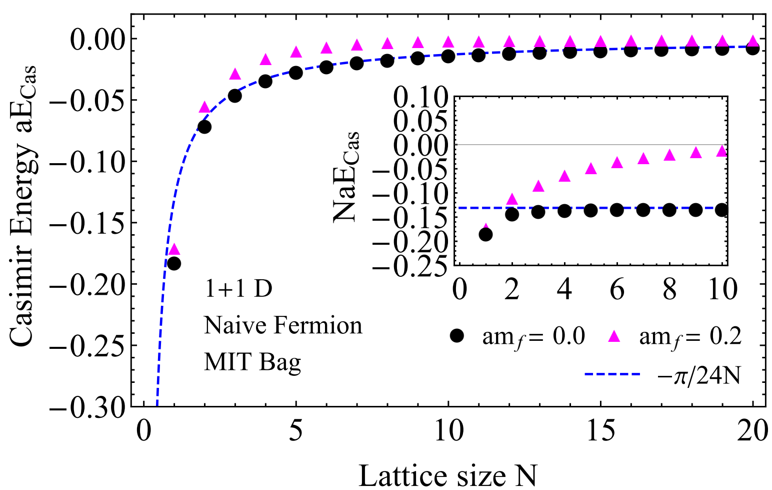

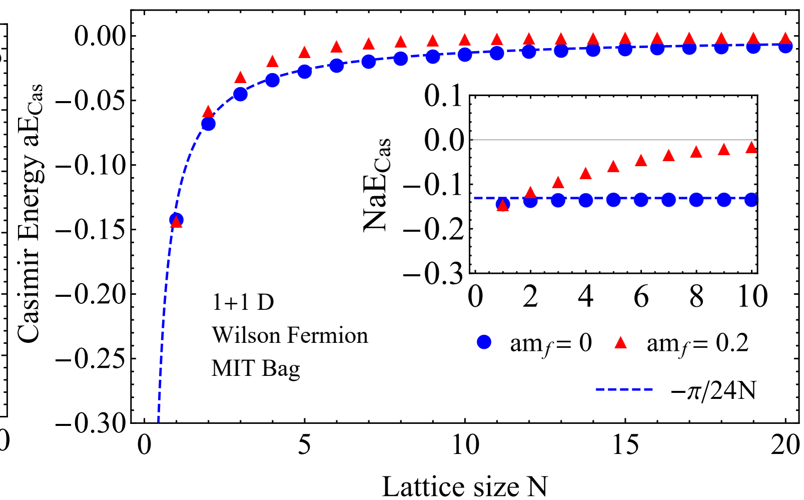

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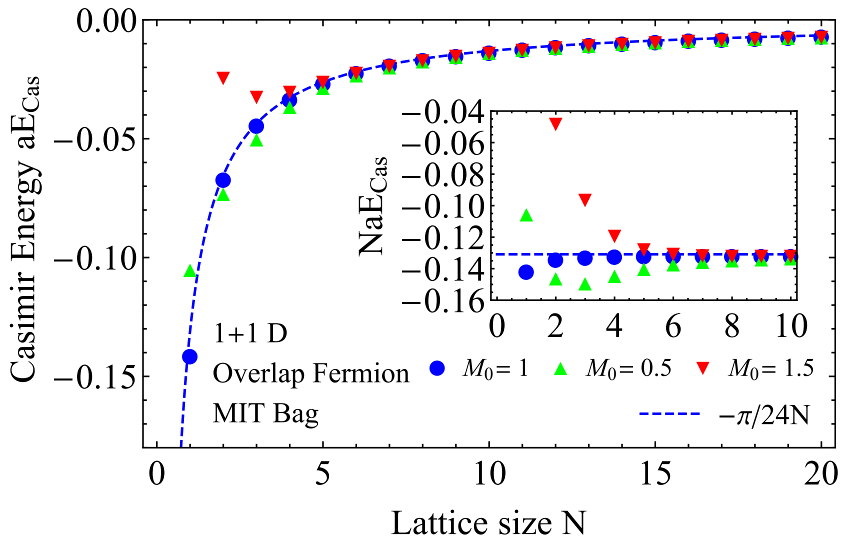

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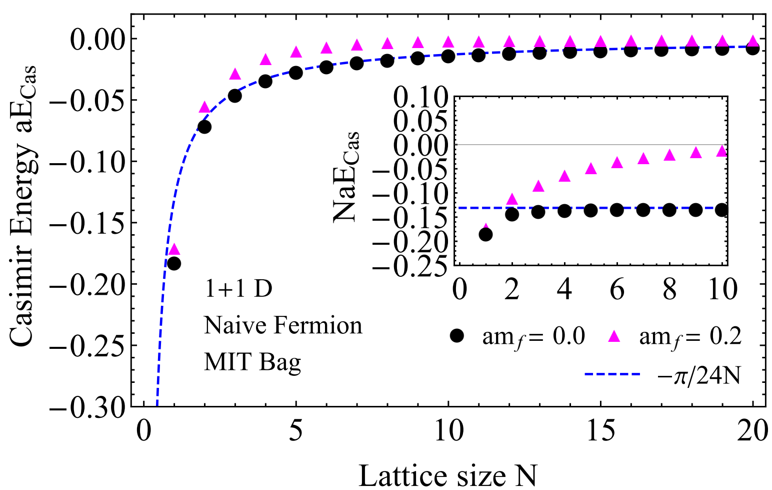

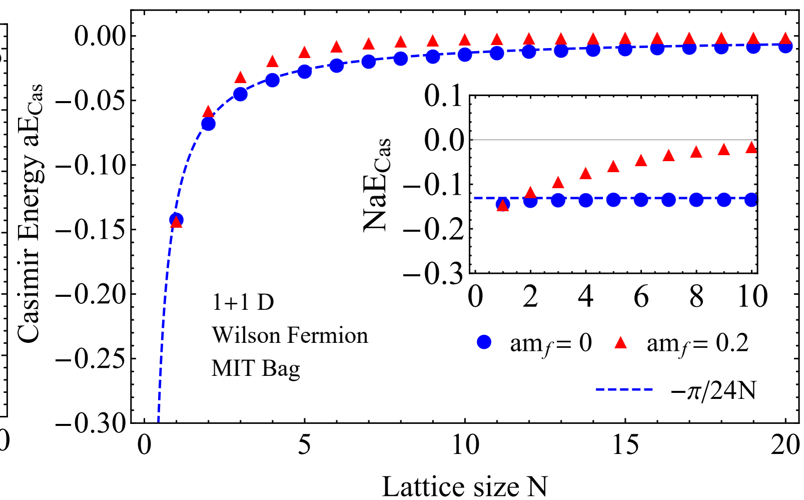

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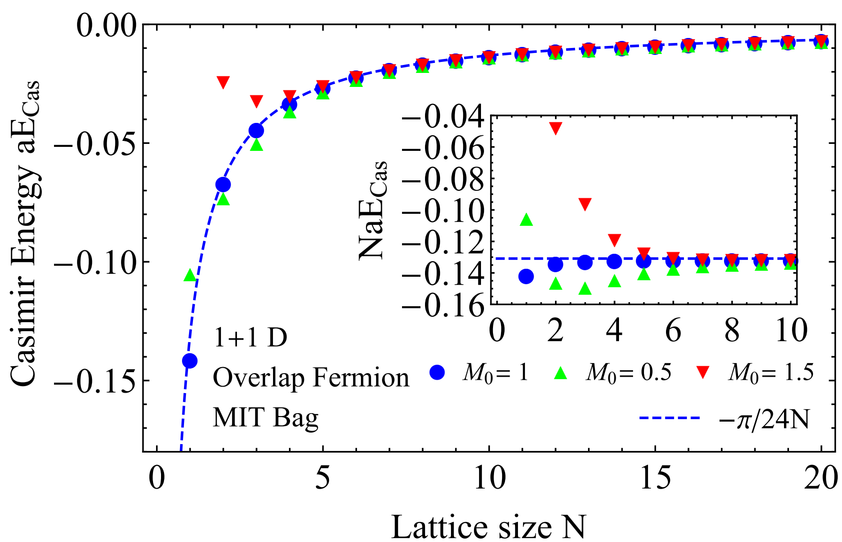

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♢ Checked to be true for all fermion types in three dimensions as well with $E_{cas} = -7\pi^2/2880d^3$.

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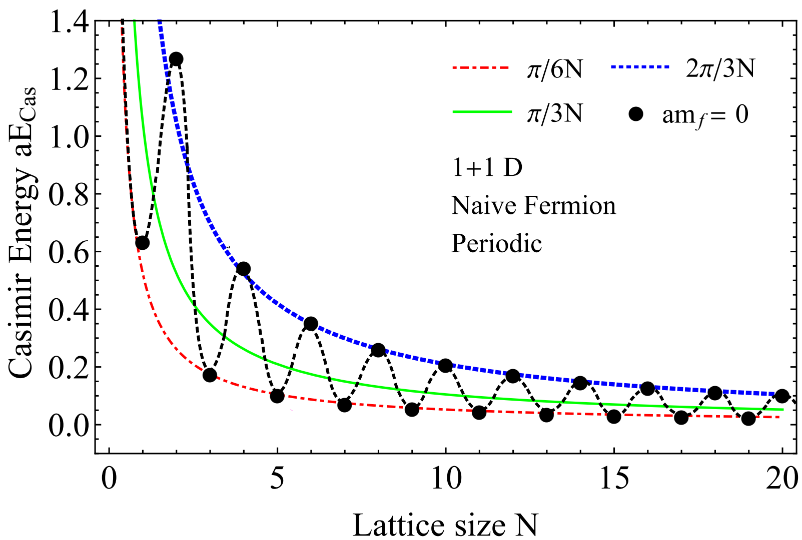

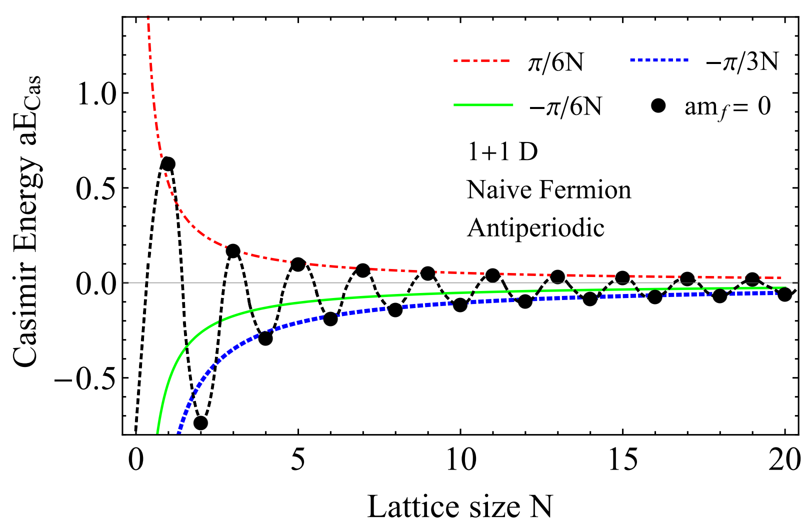

From Mandlecha+RVG PLB835 '22.

The oscillations average to expected continuum result (green line)!

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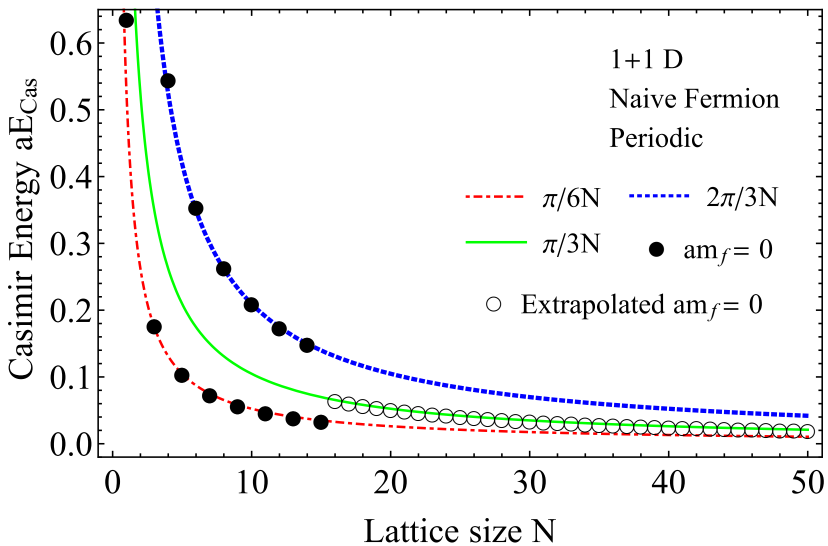
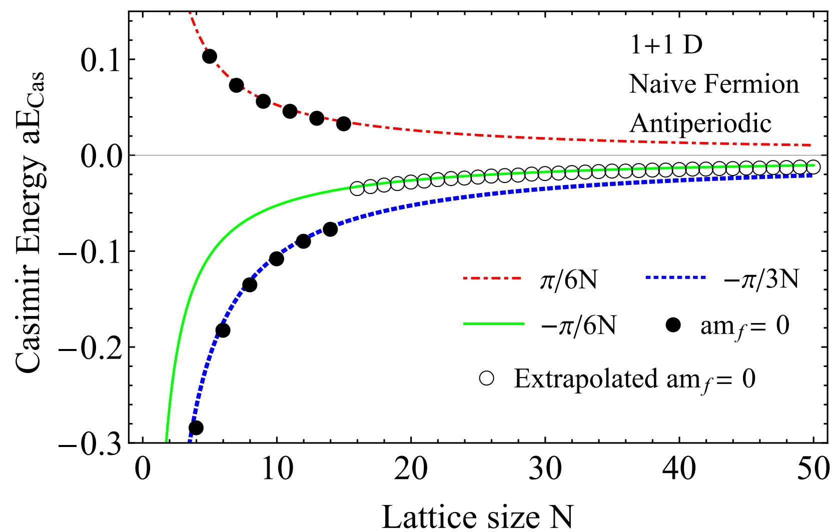
◇ The Euler-MacLaurin formula is applied when last term of a rapidly oscillating series tends to zero as $n \to \infty$, just as NF-difference above. It is implemented in *Mathematica* which we employed to obtain,

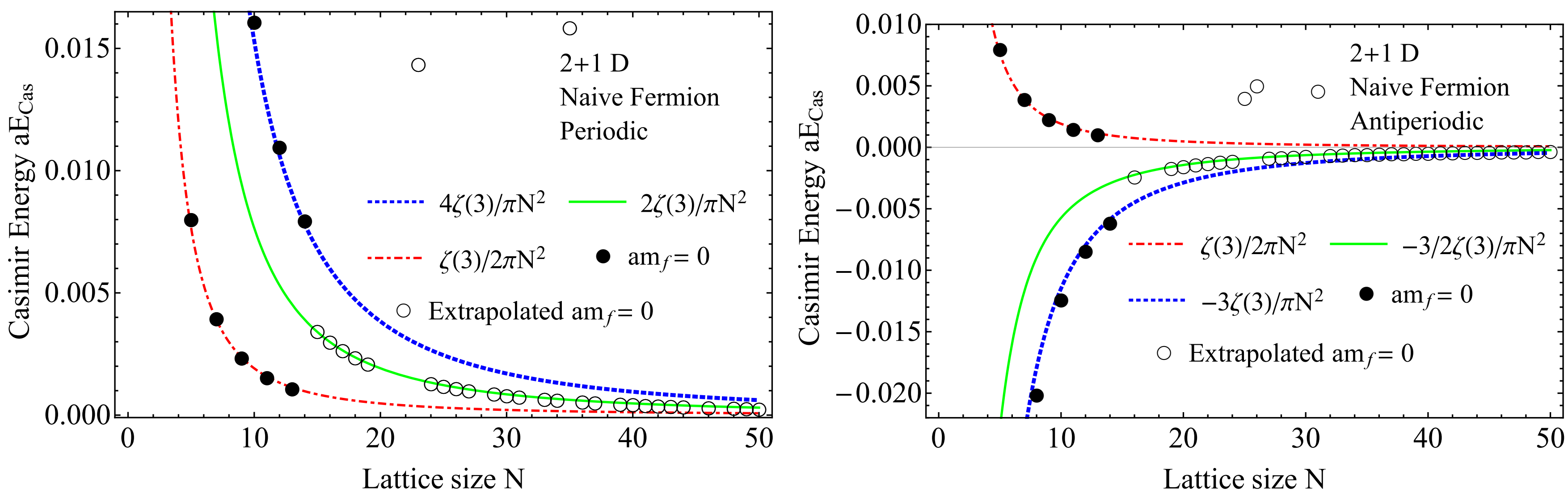




From Mandlecha+RVG PLB835 '22.

The full(open) circles are actual(extrapolated) result. Universal again!

♠ We checked that this solution is not limited to 1+1 dimensions but works for 2+1 dimensions (shown below) and 3+1 dimensions although the details vary. Indeed, it works even for other cases such as Wilson quarks with negative mass where such oscillations are observed



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♡ Earlier studies of free lattice fermion Casimir effects claimed violation of universality: naive fermions and other type of fermions differ in the continuum limit.

♣ Employing the MIT bag boundary conditions, we showed that **all** types of fermions lead to the same continuum result in the lattice spacing $a \to 0$ limit, as expected from universality. This was demonstrated analytically for naıve and Wilson fermions in 1+1 dimensions and numerically for other cases.

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♠ Casimir effect, known to arise due to quantum fluctuations of vacuum, should be investigated for theories with nontrivial vacuum such as QCD.

♡ Earlier studies of free lattice fermion Casimir effects claimed violation of universality: naive fermions and other type of fermions differ in the continuum limit.

♣ Employing the MIT bag boundary conditions, we showed that **all** types of fermions lead to the same continuum result in the lattice spacing $a \to 0$ limit, as expected from universality. This was demonstrated analytically for naıve and Wilson fermions in $1+1$ dimensions and numerically for other cases.

♢ Observing the odd N and even N series to differ by vanishing terms in the continuum limit but with rapid oscillations, we a) treated the two series as one and b)employed a suitable extrapolation method. It was shown to restore universality, leading to the same answer as with other fermion types in $a \to 0$ limit.