Nonlinear Constraints Abelianization of a Prototypical Second-Class System using BFFT method

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Abstract

We apply the BFFT formalism to a prototypical second-class system, aiming to convert its constraints from second to first-class. The proposed system admits a consistent initial set of second-class constraints and an open potential function providing room for applications to mechanical models as well as field theory such as the non-linear sigma model. The constraints can be arbitrarily non-linear, broadly generalizing previously known cases. We obtain a sufficient condition for which a simple closed expression for the Abelian converted constraints and modified involutive Hamiltonian can be achieved. As an explicit example, we discuss a particle on a torus model, obtaining the full first-class abelianized constraints in closed form and the corresponding involutive Hamiltonian.

1. Introduction

The quantization of constrained dynamical systems has been extensively studied in the theoretical physics literature from many different perspectives [1]. One of the successful path to quantizing systems with second-class constraints through their conversion to first-class, using auxiliary variables is Batalin-Fradkin-Fradkina-Tyutin method [2].We propose a prototypical second-class dynamical system which includes many of the physical models as particular cases [3, 4]. In the present work, we considerably extend that model in order to properly use the BFFT formalism [3, 4].

2. Prototypical Non-linear Second Class Constraint System

Given a symmetric invertible square matrix $f_{ij}(q^k)$ depending on the variables q^k, with $i,j,k=1,\dots,N$, and M thrice differentiable functions $T_\alpha(q^k)$, with $\alpha=1,\dots,M$, we define our prototypical system by the Lagrangian [3, 4],

$$L(l^\alpha,q^k,\dot q^k)=\frac{1}{2}f_{ij}(q^k)\dot q^i\dot q^j-V(q^k)-l^\alpha T_\alpha(q^k)\,. \tag{1}$$

Although the variables l^α in a certain sense enter in (1) as Lagrange multipliers, at this point they are considered as configuration space variables in very much the same level as q^k.
We obtain the canonical Hamiltonian corresponding to Lagrangian in eqn.(1) as,

$$H=\frac{1}{2}f^{ij}(q^k)p_ip_j+V(q^k)+l^\alpha T_\alpha(q^k) \tag{2}$$

and a first set of M trivial primary constraints

$$\chi_{(1)\alpha}=\pi_\alpha\,,\quad \alpha=1,\dots,M\,. \tag{3}$$

The upper index functions $f^{ij}(q^k)$ introduced in the Hamiltonian (2) denote the inverse of the previous lower index ones $f_{ij}(q^k)$. We shall often need the partial derivatives of the functions $f^{ij}(q^k)$, $V(q^k)$ and $T_\alpha(q^k)$, for that matter we introduce the condensed brief notations

$$f^{ij}{}_{,k}\equiv\frac{\partial f^{ij}}{\partial q^k}\,,\qquad V_i\equiv\frac{\partial V}{\partial q^i}\qquad T_{\alpha i}\equiv\frac{\partial T_\alpha}{\partial q^i} \tag{4}$$

Following further the DB algorithm [1], we impose the stability of the constraints under time evolution. As a result, three more constraint families are generated,

$$\chi_{(2)\alpha}=T_\alpha\,,\qquad \chi_{(3)\alpha}=f^{ij}p_iT_{\alpha j}\,, \tag{5}$$

and

$$\chi_{(4)\alpha}=\frac{1}{2}Q^{ij}_\alpha p_ip_j-v_\alpha-l^\beta w_{\alpha\beta}\,, \tag{6}$$

constituting a total of $4M$ constraints in phase space. For notation convenience, in the RHS of equation (6) we have introduced the q^k-dependent quantities Q^{ij}_α, v_α and $w_{\alpha\beta}$ defined explicitly by

$$Q^{ij}_\alpha\equiv\left(f^{il}T_{\alpha l}\right)_{,k}f^{kj}+\left(f^{jl}T_{\alpha l}\right)_{,k}f^{ki}-f^{kl}T_{\alpha k}f^{ij}{}_{,l}\,, \tag{7}$$

$$v_\alpha\equiv f^{ij}T_{\alpha i}V_j\,,\quad\text{and}\quad w_{\alpha\beta}\equiv f^{ij}T_{\alpha i}T_{\beta j}\,. \tag{8}$$

As it turns out, if we assume

$$w\equiv\det w_{\alpha\beta}\neq0\,, \tag{9}$$

the whole set of constraints $\chi_{(r)\alpha}$ with $r=1,\dots,4$ is second-class. This can be seen by computing the Poisson brackets among all constraints and writing the resulting constraint matrix as

$$\Delta_{(rs)\alpha\beta}\equiv\{\chi_{(r)\alpha},\chi_{(s)\beta}\}=\begin{bmatrix}0&0&0&w_{\alpha\beta}\\0&0&w_{\alpha\beta}&D_{\alpha\beta}\\0&-w_{\alpha\beta}&M_{\alpha\beta}&R_{\alpha\beta}\\-w_{\alpha\beta}&-D_{\beta\alpha}&-R_{\beta\alpha}&N_{\alpha\beta}\end{bmatrix}, \tag{10}$$

for $r,s=1,\dots,4$, with the short-hand conventions

$$D_{\alpha\beta}\equiv\{\chi_{(2)\alpha},\chi_{(4)\beta}\}\,,\quad M_{\alpha\beta}\equiv\{\chi_{(3)\alpha},\chi_{(3)\beta}\}\,, \tag{11}$$

and

$$N_{\alpha\beta}\equiv\{\chi_{(4)\alpha},\chi_{(4)\beta}\}\,,\quad R_{\alpha\beta}\equiv\{\chi_{(3)\alpha},\chi_{(4)\beta}\}\,. \tag{12}$$

In fact, the determinant of the constraint matrix (10) depends only on its secondary diagonal and is given by

$$\det\Delta_{(rs)\alpha\beta}=w^4 \tag{13}$$

which clearly shows that, under the assumption (9), the prototypical system (1) is indeed second-class.
For completeness and future reference, we compute the remaining further entries below the main diagonal in the constraint matrix (10) and write them explicitly as

$$\begin{aligned}M_{\alpha\beta}&=f^{ij}p_k\left[\left(f^{kl}T_{\alpha l}\right)_{,i}T_{\beta j}-\left(f^{kl}T_{\beta l}\right)_{,i}T_{\alpha j}\right]=p_kM^k_{\alpha\beta}\,,\\D_{\alpha\beta}&=T_{\alpha i}Q^{ij}_\beta p_j\end{aligned} \tag{14}$$

$$\begin{aligned}N_{\alpha\beta}&=p_i\left[\chi_{(4)\alpha,j}Q^{ij}_\beta-\chi_{(4)\beta,j}Q^{ij}_\alpha\right]\\&=p_ip_jp_kQ^{ijk}_{\alpha\beta}+p_iV^i_{\alpha\beta}+p_il^\gamma N^i_{\alpha\beta\gamma}\end{aligned} \tag{15}$$

and

$$R_{\alpha\beta}=p_ip_jR^{ij}_{\alpha\beta}+f^{ij}T_{\alpha i}v_{\beta j}+l^\gamma f^{ij}T_{\alpha i}w_{\beta\gamma j} \tag{16}$$

3. BFFT Constraints Abelianization

The Batalin-Fradkin-Fradkina-Tyutin quantization approach [2] aims to convert the second-class constraints, in our case all $\chi_{(r)\alpha}$, to corresponding first-class ones by extending the initial phase space $(q^k,l^\alpha,p_k,\pi_\alpha)$ including new BFFT variables $\eta^{(r)\alpha}$. For the present prototypical system, the indexes run through $r=1,\dots,4$, $\alpha=1,\dots,M$, and we have a total of $4M$ $\eta^{(r)\alpha}$ BFFT variables. The sought converted constraints $\tilde\chi_{(r)\alpha}$ can be expanded in a power series in the auxiliary BFFT variables as [3, 4]

$$\tilde\chi_{(r)\alpha}=\sum_{n=0}^{\infty}X_{(rt_{(n)})\alpha\gamma_{(n)}}\eta^{(t_{(n)})\gamma_{(n)}}\,. \tag{17}$$

The converted first-class constraints $\tilde\chi_{(r)\alpha}$ will satisfy following conditions,

$$X_{(r)\alpha}=\chi_{(r)\alpha}\,. \tag{18}$$

and

$$\{\tilde\chi_{(r)\alpha},\tilde\chi_{(s)\beta}\}=0 \tag{19}$$

Substituting (17) into (19) and picking up the resulting $\eta^{(r)\alpha}$ zero-order term, it is straightforward to obtain the well-known BFFT first-step-condition

$$\Delta_{(rs)\alpha\beta}=-X_{(rt_1)\alpha\gamma_1}\omega^{(t_1t_2)\gamma_1\gamma_2}X_{(st_2)\beta\gamma_2}\,, \tag{20}$$

where $\omega^{(t_1t_2)\gamma_1\gamma_2}$, defined as

$$\omega^{(rs)\alpha\beta}\equiv\{\eta^{(r)\alpha},\eta^{(s)\beta}\}\,, \tag{21}$$

characterizes the symplectic structure among the BFFT variables. In practical terms, one chooses a convenient $\omega^{(rs)\alpha\beta}$ and solves (20) for $X_{(rs)\alpha\beta}$. For our current prototypical second-class system, aiming to preserve the initial constraints original structure as much as possible, we choose the symplectic BFFT algebra [3, 4]

$$\omega^{(rs)\alpha\beta}=\begin{bmatrix}0&0&\delta^{\alpha\beta}&0\\0&0&0&\delta^{\alpha\beta}\\-\delta^{\alpha\beta}&0&0&0\\0&-\delta^{\alpha\beta}&0&0\end{bmatrix} \tag{22}$$

and consider a deformation in phase space induced by the BFFT variables corresponding to the ansatz

$$\begin{aligned}\tilde\chi_{(1)\alpha}&=\pi_\alpha-\eta^{(3)}_\alpha\,,\qquad\tilde\chi_{(2)\alpha}=T_\alpha+\eta^{(2)}_\alpha\,,\\\tilde\chi_{(3)\alpha}&=\bar f^{ij}\left(p_i-T_{\beta i}\eta^{(4)\beta}\right)\bar T_{\alpha j}\,,\end{aligned} \tag{23}$$

and

$$\begin{aligned}\tilde\chi_{(4)\alpha}&=\frac{1}{2}\bar Q^{ij}_\alpha\left(p_i-T_{\beta i}\eta^{(4)\beta}\right)\left(p_j-T_{\gamma j}\eta^{(4)\gamma}\right)\\&\quad-\bar v_\alpha-l^\beta\bar w_{\alpha\beta}-\bar w_{\alpha\beta}\eta^{(1)\beta}\end{aligned} \tag{24}$$

with all barred quantities depending only on the variables q^k and $\eta^{(2)}_\alpha$. Now it is straightforward to check that the converted constraints (23) and (24) will generate the Abelian algebra (19) if the barred quantities, generically represented by $\bar F(q^k,\eta^{(2)}_\alpha)$, satisfy the condition [3, 4]

$$\bar F_{,i}=T_{\alpha i}\{\bar F,\eta^{(4)\alpha}\}\,, \tag{25}$$

The corresponding modified Hamiltonian in strong involution with the converted constraints can be directly obtained from (2), considering the same deformation in phase space introduced in equations (23) and (24), as

$$\begin{aligned}\tilde H&=\frac{1}{2}\bar f^{ij}(q^k,\eta^{(2)}_\alpha)\left(p_i-T_{\beta i}\eta^{(4)\beta}\right)\left(p_j-T_{\gamma j}\eta^{(4)\gamma}\right)\\&\quad+\bar V(q^k,\eta^{(2)}_\alpha)+(l^\alpha+\eta^{(1)\alpha})\left(T_\alpha(q^k)+\eta^{(2)}_\alpha\right)\end{aligned} \tag{26}$$

with

$$\bar V_{,i}=T_{\alpha i}\{\bar V,\eta^{(4)\alpha}\}\,. \tag{27}$$

4. Mechanical Example: Particle on torus

Particle on torus is a two dimensional surface embedded in three dimensional space. It has been studied by many physicists as a simple three dimensional model for constrained motion [4]. Lagrangian for a particle constrained to move on the surface of torus of radius r is

$$L=\frac{1}{2}m\dot r^2+\frac{1}{2}mr^2\dot\theta^2+\frac{1}{2}m(b+r\sin\theta)^2\dot\phi^2-l(r-a) \tag{28}$$

where (r,θ,ϕ) are toroidal co-ordinates. The canonical Hamiltonian corresponding to the Lagrangian in eqn(28)is then written as,

$$H=\frac{p_r^2}{2m}+\frac{p_\theta^2}{2mr^2}+\frac{p_\phi^2}{2m(b+r\sin\theta)^2}+l(r-a) \tag{29}$$

where p_r, p_θ,p_ϕ and p_λ are the canonical momenta conjugate to the coordinate r, θ ϕ and l respectively. All the constraints can be written as,

$$\begin{aligned}\chi_1&=\pi\approx0\\\chi_2&=T=(r-a)\approx0\\\chi_3&=f^{rr}p_rT_r=\frac{p_r}{m}\approx0\\\chi_4&=\frac{1}{2}\{Q^{\theta\theta}p_\theta^2+Q^{\phi\phi}p_\phi^2\}-l\omega\\&=\frac{1}{m}\{\frac{p_\theta^2}{mr^3}+\frac{p_\phi^2\sin\theta}{m(b+r\sin\theta)^3}-l\}\approx0\end{aligned} \tag{30}$$

Thus the matrix Δ_{rs} between the constraints has the form

$$\Delta_{rs}=\{\chi_r,\chi_s\}_P=\begin{pmatrix}0&0&0&\frac{1}{m}\\0&0&\frac{1}{m}&0\\0&-\frac{1}{m}&0&\frac{3}{m^3}\{\frac{p_\theta^2}{r^4}+\frac{p_\phi^2\sin^2\theta}{(b+r\sin\theta)^4}\}\\-\frac{1}{m}&0&-\frac{3}{m^3}\{\frac{p_\theta^2}{r^4}+\frac{p_\phi^2\sin^2\theta}{(b+r\sin\theta)^4}\}&0\end{pmatrix} \tag{31}$$

BFFT Converted constraints and Hamiltonian can be written as,

$$\begin{aligned}\tilde\chi_{(1)}&=\pi-\eta^{(3)}\\\tilde\chi_{(2)}&=(r-a)+\eta^{(2)}\\\tilde\chi_{(3)}&=\frac{1}{m}(p_r-\eta^{(4)})\bar T\\\tilde\chi_{(4)}&=\frac{1}{m}\{\frac{p_\theta^2}{m(r+\eta^{(2)})^3}+\frac{p_\phi^2\sin\theta}{m[b+(r+\eta^{(2)})\sin\theta)]^3}\\&\quad-l-\eta^{(1)}\}\end{aligned} \tag{32}$$

$$\begin{aligned}\tilde H&=\frac{(p_r-\eta^{(4)})^2}{2m}+\frac{p_\theta^2}{2m(r+\eta^{(2)})^2}+\frac{p_\phi^2}{2m[b+(r+\eta^{(2)})\sin\theta]^2}\\&\quad+(l+\eta^{(1)})(r+\eta^{(2)}-a)\end{aligned} \tag{33}$$

It can easily be verified that the modified constraints and Hamiltonian are involutive in nature.

5. Conclusion

In this work we have successfully studied the constraint structure and Abelianization procedure of a prototypical nonlinear second-class system. We have also studied particle on torus as an example of this kind of system.

References

[1] P. A. M. Dirac, Can. J. Math. 2, 129 (1950).
[2] I. A. Batalin and I. V. Tyutin, IJMPA 6 (1991) 3255.
[3] V. K. Pandey and R. Thibes, MPLA 37 (2022) 2250086.
[4] V. K. Pandey and R. Thibes, arXiv:2103.05626 [hep-th].