Geometric Quantization of a Constrained System

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Abstract

In mathematical physics, geometric quantization is a method of defining quantum theory corresponding to an existing classical theory. It has been successfully applied to many field theoretic models. Also, Constrained systems occur frequently in physics, since they typically arise in the Hamiltonian formulation of classical sys- tems with gauge symmetries. Here, we will try to understand the geometric quantization from the perspective of a constrained system.

1. Introduction

Constrained systems occur frequently in physics, since constraints typically arise in the Hamiltonian formulation of classical systems with (gauge) symmetries [1]. Therefore their quantisation deserves particular attention and numerous devices have been developed to handle such situ- ations. A quantisation scheme which seems particularly well adapted to this goal but nevertheless somewhat neglected is geometric quantisation [2], since it-like constrained systems [3]-admits a very 'symplectic' formulation. Thus classical concepts of these systems are most easily incorporated into the quantum theory. Previous investigations of the applicability of geometric quantisation to constrained systems have been performed by Gotay [3] Ashtekar and Stillerman [4] and Blau [5].

2. Geometry of Constrained Systems

Classical Hamiltonian mechanics finds its natural setting in the framework of symplectic geometry. A manifold is called symplectic if it carries a closed non-degenerate 2-form ω and cotangent bundles (i.e. $M = T^{\star}Q$ for some 'configuration' manifold Q; ω is the canonical 2-form). The symplectic form ω gives rise to an isomorphism between the tangent and cotangent bundles of M, and defines a Poisson bracket $\{,\}$ on $C^{\infty}(M)$. In local coordinates on $T^{\star}Q$, ω takes the familiar form $w = dq^k \wedge dp^k$. M represents the phase space of a dynamical system.

In many physically important examples, however, only a submanifold $\bar{M}$ of M is classically accessible for the system. This manifold can (locally) be described by the vanishing of certain 'constraint' functions [1] $\phi_\alpha, \alpha = 1, ..., m$, and if the Poisson bracket between any two constraints vanishes on M (i.e. $\{\phi_\alpha, \phi_\beta\} = C^{\gamma}_{\alpha\beta}\phi_\gamma$ for some functions $C^{\gamma}_{\alpha\beta}$), the constraints are said to be first class.

In this work we shall restrict our attention to first-class constraints for two reasons: they are the most frequent and important, generally arising as the consequence of a classical 'gauge' symmetry; and constraints which are not first class can be eliminated by the introduction of Dirac brackets or by regarding them as gauge fixing constraints of a bigger (first-class) theory. The first-class property can be invariantly expressed by saying that $\bar{M}$ is a co-isotropic submanifold of M, i.e. $T\bar{M}^{\perp} \subset T\bar{M}$, where '$()^{\perp}$' denotes the orthogonal complement in $T_{\bar{M}}M$ with respect to ω. Since ω is closed, $T\bar{M}^{\perp}$ is integrable, giving rise to the so-called null foliation of the constraint surface. We shall assume that this foliation is actually a fibration and denote the quotient of $\bar{M}$ by the integral manifolds of $T\bar{M}^{\perp}$ by $\hat{M}$. $\hat{M}$ is called the reduced phase space (or 'space of true degrees of freedom', since $T\bar{M}^{\perp}$ is spanned by the Hamiltonian vector fields of the constraints).

Finally two functions f and g on M are regarded as equivalent if they are equal on $\bar{M}$, and a function f on M is called weakly gauge invariant if $\{f, \phi_\alpha\}$ vanishes on M for all constraints, i.e. if $\{f, \phi_\alpha\} = f^{\beta}_{\alpha}\phi_\beta$ for some functions f^{β}_{α}. Observables are to be identified with equivalence classes of weakly gauge-invariant functions.

3. Geometric Qunatization

Geometric quantisation [2], developed independently by Kostant and Souriau is a mathematically well defined geometrical quantisation scheme (an assignment of unitary operators on a Hilbert space to classical observables), which generalises and makes rigorous the usual canonical quantisation procedure.

The Hilbert space of states is constructed from the space of sections of a complex line bundle L over the symplectic manifold (M, ω) . A connection ∇ on L allows one to lift vector fields on M to L and thus in particular to associate linear differential operators (acting on sections of L) with functions on M (via their Hamiltonian vector fields). In order to enforce the basic Poisson bracket-commutator relation, the curvature of ∇ is required to be proportional to (the pull back of) ω. If ∇ is additionally compatible with a Hermitian structure on the fibres of L, the pair (L, ∇) is called a 'prequantisation' of M and exists iff ω defines an integral cohomology class on M (since it represents the first Chern class of L). The prequantum operator corresponding to an observable f (denoted by O_f) is defined by

$$O_f := (\frac{\hbar}{i})\nabla_{X_f} + f \tag{1}$$

where X_f, is the Hamiltonian vector field of f, and satisfies

$$[O_f, O_g] = i\hbar O_{\{f,g\}} \tag{2}$$

for all $f, g \in C^{\infty}(M)$ If $(M, \omega) = (T^{\star}Q, -d\theta)$ the cohomology class of ω is trivial and thus $L \simeq M \times \mathbb{C}$.Sections of L can then be identified with functions on M and the prequantum Hilbert space $\mathcal{H}$ is isomorphic to $L^2(M, \omega^n)$. In this case ∇ and O_f, explicitly are

$$\nabla_X = X - (i/\hbar)\theta_X \tag{3}$$

and

$$O_f = (\hbar/i)X_f + (f - \theta(X_f)) \tag{4}$$

In order to reduce the Hilbert space $\mathcal{H}$ of sections of L (square integrable with respect to the Liouville measure) to a physically more reasonable Hilbert space of sections depending on half the number of variables, one introduces what is called a polarisation P: an involutive $n(= dimM)$-dimensional Lagrangian $(P^{\perp} = P)$ distribution on M. Then one regards only those sections as representing physical states which are covariantly constant along P with respect to ∇. The resulting linear space of 'polarised sections' (yet to be equipped with a scalar product) will be denoted by $P\mathcal{H}$.

If the space of integral manifolds of P (denoted M_p) has the structure of a Hausdorff manifold, P is called admissible or reducible.

Locally a polarisation is spanned by n Hamiltonian vector fields X_{f^k}, and the requirement of being Lagrangian translates into the condition $\{f^i, f^k\} = 0$. Thus the notion of a polarisation generalises the traditional concept of 'a complete set of commuting observables'.

If $M = T^{\star}Q$ there is a natural polarisation called the vertical polarisation vp, whose integral manifolds are the fibres of $\pi : T^{\star}Q \to Q$. It is spanned by the vector fields d/dp^k and the corresponding complete set of observables is q^k. Polarised elements of the prequantum Hilbert space $\mathcal{H}$ can be identified with functions on $Mp = T^{\star}Q_{vp} = Q$ and $vp\mathcal{H} \simeq L^2(Q)$.

Iff P is invariant under the flow generated by f,i.e. iff $[X_f, u] \subset P, \forall u \in P$ then the prequantum operators O_f, are compatible with P in the sense that they map polarised states to polarised states. Observables f satisfying this condition are called quantisable with respect to P.

4. Geometric Qunatization of a Constrained System

Let $\bar{M}$ be a coisotropic submanifold of M and $\hat{M}$ the reduced phase space. Under reasonable conditions $\hat{M}$ is symplectomorphic to the cotangent bundle of a reduced configuration space $\hat{Q}$, perhaps equipped with a non-canonical symplectic structure [3, 4, 5].

In the quantisation of this system we are confronted with different physically reasonable alternatives. (i) ('Dirac' or 'prequantum operator constraint' method.) The prequantum Hilbert space $\mathcal{H}$ will consist of functions ψ on M, constraints will be imposed as prequantum operator constraints (i.e. $O_\alpha, \psi := O_{\phi_\alpha}\psi = 0$) on $\mathcal{H}$ and a polarisation P is introduced on M. Physical states will then be elements of $ker\ 0_\alpha \cap P\mathcal{H} =: H$. (ii) (Reduced phase space method.) The prequantum Hilbert space $\hat{\mathcal{H}}$ consists of functions $\hat{\psi}$ on $\hat{M}$ and a polarisation $\hat{P}$ defines the space of physical states $\hat{P}\hat{\mathcal{H}}$.

4.1 Prequantization

Let $\{\phi_\alpha, \alpha = 1, ..., m\}$ be a set of constraint functions on M satisfying (1) and (2), and let $\mathcal{H}$ be the prequantum Hilbert space $L^2(T^{\star}Q, \omega^n)$. Then a tempting way of imposing the constraints ϕ_α, at the (pre)quantum level is to realise them as prequantum operator constraints on $\mathcal{H}$ $O_\alpha\psi = 0 \quad (\alpha = 1, ..., m)$ (where we have set $O_\alpha := O_{\phi_\alpha}$). By virtue of (2) we have

$$[O_\alpha, O_\beta] = i\hbar O_{\{\alpha,\beta\}} := i\hbar O_{\{\phi_\alpha,\phi_\beta\}} \tag{5}$$

and the conditions (5) are internally consistent if the structure functions $C^{\gamma}_{\alpha\beta}$ are constants. In the general case however these conditions imply non-trivial integra- bility conditions. This can be seen by noting that the prequantum operator correspond- ing to the product of two functions f and g is neither the product of the operators O_f and O_g, nor of the form $fO_g + gO_f$ but rather is given by the expression

$$Ofg = fO_g + gO_f - fg \tag{6}$$

4.2 Polarisation

Assuming for the moment that the $C^{\gamma}_{\alpha\beta}$ are indeed constant we can pursue the geometric quantisation programme outlined above by introduction of a polarisation P on M. In order that O_α maps polarised states to polarised states the constraints ϕ_α, must be quantisable with respect to P, i.e. P must satisfy

$$[X_\alpha, u] \subset P \tag{7}$$

$(X_\alpha := X_{\phi_\alpha})$. This incidentally shows that in the case $M = T^{\star}Q$ the usual vertical polarisation vp is inappropriate if the constraints are more than linear in the momenta. This observation is almost certainly relevant for the study of (toy models of) quantum gravity, where the Hamiltonian constraint is indeed quadratic in the momenta.

Following [4, 5] a polarisation P satisfying (7) will be called 'compatible with the constraints'. It is important to realise that this notion of compatibility depends not only on M and $\bar{M}$, but also on the particular choice of constraints ϕ_α, used to define the constraint surface. Thus reparametrising the surface will require a new polarisation.

Taking the intersection of a compatible P with $T\bar{M}$ and adjoining the 'gauge' directions X_α , not already contained in this intersection one obtains a sub-bundle $\bar{P}$ of $T\bar{M}$ with the following properties. (i) dim $\bar{p}_{\bar{m}} = n \quad \forall \bar{m} \in \bar{M}$. (ii) The pull-back of ω to $\bar{p}_{\bar{m}}$ is zero. (iii) $\bar{P}$ is integrable. (iv) $\bar{P}$ contains all gauge directions. (Integrability of $\bar{P}$ requires the compatibility of P.) Generally a distribution on $\bar{M}$ satisfying these four conditions (whether it was obtained from a polarisation on M or not) will be called 'constrained polarisation' and it is easy to see [4] that there is a 1 : 1 correspondence between these constrained polarisations and genuine polarisations on $\hat{M}$. Furthermore, there is a natural isomorphism between

$$\bar{P}\bar{\mathcal{H}} := \{\bar{\psi} \in \bar{\mathcal{H}} : \nabla_{\bar{u}}\bar{\psi} = 0, \forall \bar{u} \in \bar{P}\} \tag{8}$$

and $\hat{P}\hat{\mathcal{H}}$, where $\hat{P} = T\bar{\pi}\bar{P}$.

Furthermore, every element of P satisfying the prequantum operator constraints (5) yields-upon restriction to $\bar{M}$- an element of $\bar{P}\bar{\mathcal{H}}$.

5. Conclusion

In this work we have successfully tried to understand the geometric quantization of a Constrained system.

References

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