Realizing late-time cosmology in the context of Dynamical Stability Approach

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Theme of this talk

Publication & Pre-print:

- **"Dynamical stability of k-essence field interacting non-minimally with a perfect fluid,"**

  A. Chatterjee, Saddam Hussain, Kaushik Bhattacharya

  Phys. Rev. D **104**, 103505 (2021)

- **"Ghost condensates and pure kinetic k-essence condensates in presence of field-fluid non-minimal coupling in the dark sector,"**

  A. Chatterjee, Saddam Hussain, Kaushik Bhattacharya

  arXiv: 2203.10607

Plan for today's talk

Theme of the presentation

- **Motivation & Constituents** of Non-minimal coupling of field-fluid sectors.
- **Theoretical Framework** of this coupled model.
- Essence of **non-canonical** scalar field.
- Evolution of coupled system in **FLRW background**.
- Techniques of **Dynamical Stability Analysis**.
- Comparative study on **two types of scalar field potential**.
- Results & Discussion.
- Conclusion.

Late time cosmology in Dynamical Stability Approach: Coupled field-fluid scenario (Phys. Rev. D **104**, 103505 (2021)) & (arXiv: 2203.10607)

Motivation & Constituents:

- To solve **cosmological coincidence problem** & Alleviate **Hubble Tension**.
- Explore interacting field-fluid model which can be derived from a **variational approach**.
- Field $\rightarrow$ ***k*-essence dark energy**; Fluid $\rightarrow$ **Relativistic fluid**.

Theoretical Framework: Total action of the coupled system

$$
\begin{aligned}
S \quad = \quad & \int_{\Omega} d^4x \Big[\sqrt{-g}\frac{R}{2\kappa^2} - \sqrt{-g}\rho(n,s) + J^{\mu}(\varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha^A_{,\mu}) \\
& - \sqrt{-g}\mathcal{L}(\phi, X) \ + S_{\rm int} \Big]
\end{aligned}
$$

- 1st term $\rightarrow$ Gravitational part of the action.
- 2nd & 3rd term $\rightarrow$ Action for a perfect fluid.
- 4th term $\rightarrow$ Action for the k-essence scalar field.
- $S_{\rm int} : -\sqrt{-g}\ f(n, s, \phi, X)$ $\rightarrow$ Action for Non-minimal coupling (depends on energy density & entropy for fluid; scalar field & kinetic term of k-essence sector for field).

Theoretical Framework

Fluid Sector:

- **Current Density** $(J^\mu) \Rightarrow \sqrt{-g}nu^\mu$.
- **Velocity four vector** $(u^\mu) \Rightarrow u^\mu u_\mu = -1$.
- **Energy-Momentum Tensor of fluid** $(T^{(M)}_{\mu\nu}) \Rightarrow \rho u_\mu u_\nu + \left(n\frac{\partial\rho}{\partial n} - \rho\right)(u_\mu u_\nu + g_{\mu\nu})$
- **Pressure & energy density (Fluid)** $\Rightarrow P_M = \left(n\frac{\partial\rho}{\partial n} - \rho\right)$

Field Sector:

- **Modified field equation** $\Rightarrow \mathcal{L}_{,\phi} + \nabla_\mu(\mathcal{L}_{,X}\nabla^\mu\phi) \; + f_{,\phi} + \nabla_\mu(f_{,X}\nabla^\mu\phi) = 0$
- **Energy-Momentum Tensor of field** $(T^{(\phi)}_{\mu\nu}) \Rightarrow -\mathcal{L}_{,X}(\partial_\mu\phi)(\partial_\nu\phi) - g_{\mu\nu}\mathcal{L}$
- **Pressure & energy density (field)** $\Rightarrow \rho_\phi = \mathcal{L} - 2X\mathcal{L}_{,X}$ **and** $P_\phi = -\mathcal{L}$

Interacting Sector (Field & Fluid):

- **Energy-Momentum Tensor of Int. sector** $(T^{(\mathrm{int})}_{\mu\nu}) \Rightarrow n\frac{\partial f}{\partial n}u_\mu u_\nu + \left(n\frac{\partial f}{\partial n} - f\right)g_{\mu\nu} - f_{,X}(\partial_\mu\phi)(\partial_\nu\phi)$
- **Pressure & energy density (Int.)** $\Rightarrow \rho_{\mathrm{int}} = f - 2Xf_{,X} \quad P_{\mathrm{int}} = \left(n\frac{\partial f}{\partial n} - f\right)$
- **Total energy momentum tensors** $\Rightarrow T^{\mathrm{tot.}}_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T^{(M)}_{\mu\nu} + T^{(\mathrm{int})}_{\mu\nu}$
- **Conservation of Total energy momentum tensors** $\Rightarrow \boxed{\nabla^\mu T^{\mathrm{tot.}}_{\mu\nu} = 0.}$

k-essence Dark energy model

Details of k-essence Model:

- A Lagrangian with **non-canonical kinetic terms** expressed as $L = V(\phi)F(X)$ with **Kinetic term** $X = \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$, $g^{\mu\nu}$ is the metric, $V(\phi)$ and $F(X)$ are functions of ϕ and X respectively.
- In the background of **FLRW space-time**, k-essence scalar field $\phi(t,\vec{x}) = \phi(t)$. **Kinetic term** $\rightarrow$ $X = \frac{1}{2}\dot{\phi}^2$.
- Stress-energy tensor is equivalent to that of an ideal fluid with **Energy density** $\rho = V(\phi)(2XF_{,X} - F)$ and **Pressure** $p = V(\phi)F(X)$.
- **EOM for k-essence sector** $\rightarrow$ $(F_{,X} + 2XF_{,XX})\ddot{\phi} + 3HF_{,X}\dot{\phi} + (2XF_{,X} - F)\frac{V_\phi}{V} = 0$
- For **constant potential** and **homogeneous scalar field** in **FLRW background** ensure the **scaling relation** $\rightarrow$ $XF_{,X}^2 = Ca^{-6}$, where C is a constant & $F_{,X} = \frac{dF}{dX}$.

M. Born and L. Infeld, Proc.Roy.Soc.Lond A144(1934)

C. Armendariz-Picon & V.F. Mukhanov Phys. Rev. Lett. 85, 4438–4441 (2000)

Coupled system (Background of FLRW metric)

Modified Friedmann's equation (Background of FLRW metric):

- **Energy density relation:** $3H^2 = \kappa^2 \left(\rho + \rho_\phi + \rho_{\rm int}\right)$
- **Pressure relation:** $2\dot{H} + 3H^2 = -\kappa^2 \left(P + P_\phi + P_{\rm int}\right)$

Modified field equation (Background of FLRW metric):

$$[\mathcal{L}_{,\phi} + f_{,\phi}] - 3H\dot{\phi}\,[\mathcal{L}_{,X} + f_{,X}] + \frac{\partial}{\partial X}(P_{\rm int} + f)(3H\dot{\phi})$$
$$\ddot{\phi}\,[(\mathcal{L}_{,X} + f_{,X}) + 2X(\mathcal{L}_{,XX} + f_{,XX})] - \dot{\phi}^2(\mathcal{L}_{,\phi X} + f_{,\phi X}) = 0$$

Conserving Quantities:

- Conservation in **particle number density** $\Rightarrow$ $\nabla_\mu(nu^\mu) = 0 \Rightarrow \dot{n} + 3Hn = 0$
- Conservation in **entropy** $\Rightarrow$ $\nabla_\mu(nsu^\mu) = 0 \Rightarrow \dot{s} = 0$

Technique of Dynamical Stability Analysis

Motivation & Techniques:

- Apply to any physical system **evolving with time**.
- To investigate the coupled system behavior from **early to late time phase** of the evolution.
- For continuous and finite system, x_i variables that define the dynamical system, expressed as $\frac{dx_i}{dt} = f_i(x_1, x_2, ..., x_i)$.
- Above equation is autonomous equations and fixed or critical points exist at $x_i = y_0$ for $f_i(y_0) = 0$.
- To check stability of the critical points $\rightarrow$ **Jacobian matrix** $\Rightarrow \mathcal{J}_{ij} = \frac{\partial f_i}{\partial x_j}$.
- Eigenvalue at the critical point of Jacobian matrix $\Rightarrow$ stability of the critical points.
- Sign. of eigenvalues (positive) $\Rightarrow$ **unstable / saddle critical points**.
  Sign. of eigenvalues (negative) $\Rightarrow$ **stable critical points**.
- For $n \times n$ Jacobian matrix, n eigenvalues exist.

CGB, NT and MW, Phys. Rev. D 91 (2015)

AC, SH and KB, Phys. Rev. D 104 (2021)

Comparative study: 3-D & 2-D Autonomous System

3-D System (PHYS. REV. D 104, 103505 (2021))

- Dimensionless variables:
  $x = \dot{\phi}$, $y = \frac{\kappa\sqrt{V(\phi)}}{\sqrt{3}H}$, $z = \frac{\kappa^2 f}{3H^2}$, $\sigma = \frac{\kappa\sqrt{\rho}}{\sqrt{3}H}$,
  $B = \frac{f_{,\phi} k^2}{H^3}$, $C = \frac{\kappa^2 P_{\rm int}}{3H^2}$, $D = \frac{\kappa^2}{3H^2} f_{,X}$,
  $E = \frac{\kappa^2}{H^3}\frac{\partial^2 f}{\partial\phi\partial X}$, $\lambda = -\frac{V_{,\phi}}{\kappa V^{3/2}}$.
- Constraint Eqn:
  $\sigma^2 = 1 - y^2\left(\frac{3}{4}x^4 - \frac{1}{2}x^2\right) - z + x^2 D$.
- Friedmann's Eqn:
  $\frac{\dot{H}}{H^2} = -\frac{3}{2}[\omega\sigma^2 + y^2 F + C + 1]$.
- Other variables:
  $\Omega_\phi = y^2(x^2 F_{,X} - F)$, $\Omega_{\rm int} = z - x^2 D$
- Critical Points:
  $x' = y' = z' = 0$. Prime denotes the derivative of the dynamical variables x, y, z with respect to Hdt.
- Chosen Forms:
  $F(X) = X^2 - X$ & $V(\phi) = \frac{\delta^2}{\kappa^2\phi^2}$
  ($\phi \rightarrow k$-essence scalar field, $\delta \rightarrow$ model parameter).
- Form of Interaction:
  $f = \alpha\rho^\epsilon(\frac{\phi}{\kappa})X$ & $f = \alpha\rho(\frac{\phi}{\kappa})^m X^n$.
  (ϵ, m, n $\rightarrow$ Model parameters).
- Study in **matter dominated ($\omega = 0$) background**.

2-D System [arXiv: 2203.10607]

- Dimensionless variables:
  $x = \dot{\phi}$, $\sigma = \frac{\kappa\sqrt{\rho}}{\sqrt{3}H}$, $y = \frac{\kappa^2 f}{3H^2}$, $z = \frac{H_0}{H}$,
  $C = \frac{\kappa^2 P_{\rm int}}{3H^2}$, $D = \frac{\kappa^2 f_{,X}}{3H^2}$, $\alpha = \frac{\kappa^2 V_0}{H_0^2}$.
- Constraint Eqn:
  $\sigma^2 = 1 - \frac{\alpha z^2}{3}(x^2 F_{,X} - F) - y + x^2 D$.
- Friedmann's Eqn:
  $\frac{\dot{H}}{H^2} = -\frac{3}{2}\left(\omega\sigma^2 + \frac{\alpha z^2}{3}F + C + 1\right)$.
- Other variables:
  $\Omega_\phi \equiv \frac{\alpha z^2}{3}(x^2 F_{,X} - F)$, $\Omega_{\rm int} \equiv y - x^2 D$
- Critical Points:
  $x' = z' = 0$. Prime denotes the derivative of the dynamical variables x, z with respect to Hdt.
- Chosen Forms:
  $F(X) = AX^2 + BX$ & $V(\phi) = V_0$ (Const.).
- Form of Interaction:
  $f = g\rho X^\beta$ & $f = gV_0\rho^q X^\beta M^{-4q}$
  (g, β, q, V_0 $\rightarrow$ Model parameters).
- Study in the context of **matter dominated ($\omega = 0$) background**.

Comparative study: Autonomous Equations

Autonomous Equations in 3-D system (PHYS. REV. D 104, 103505 (2021))

$$
\begin{aligned}
x' = \dot{x}/H &= \frac{(B/3 + \sqrt{3}\lambda y^3 F) + 3x\left(y^2 F_{,X} + C_{,X}\right) - x^2(E/3 + \sqrt{3}\lambda y^3 F_{,X})}{[(D - y^2 F_{,X}) + x^2(D_{,X} - y^2 F_{,XX})]} \\
y' = \dot{y}/H &= -\frac{\sqrt{3}\lambda y^2 x}{2} + \frac{3}{2} y\left[\omega\sigma^2 + y^2 F + C + 1\right] \\
z' = \dot{z}/H &= \left[-3(C+z) + \frac{B}{3}x + Dx\,x'\right] + 3z\left[\omega\sigma^2 + y^2 F + C + 1\right]
\end{aligned}
$$

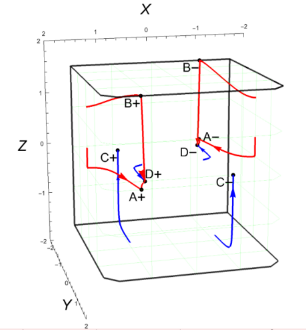
Autonomous Equations in 2-D system [arXiv: 2203.10607]

$$
\begin{aligned}
x' = \dot{x}/H &= \frac{3x\left(\frac{\alpha z^2}{3} F_{,X} + C_{,X}\right)}{\left[(D - \frac{\alpha z^2}{3} F_{,X}) + x^2\left(D_{,X} - \frac{\alpha z^2}{3} F_{,XX}\right)\right]} \\
z' = \dot{z}/H &= \frac{3}{2} z\left[\omega\,\sigma^2 + \frac{\alpha z^2}{3} F + C + 1\right].
\end{aligned}
$$

Results & Discussion

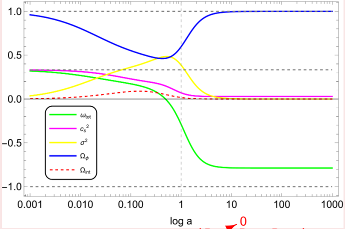
Phase Trajectory and Evolution Plot for Inverse Square Potential ($f = \alpha\rho(\frac{\phi}{\kappa})^m X^n$) [PHYS. REV. D 104, 103505 (2021)]

Phase Trajectory: $(n = 1,\, m = 3,\, \delta = 10)$



- 3-dimensional compact phase space of x, y, z.
- Total 8 critical points.
- Symmetric about x-y plane.
- Red curves $\rightarrow$ Repeller & Blue curves $\rightarrow$ Attractor.
- $A_\pm$ & $B_\pm$ $\Rightarrow$ Saddle points, and, $C_\pm$ & $D_\pm$ $\Rightarrow$ Stable fixed points.
- Critical point $D_\pm$ $\rightarrow$ Global Attractor.

Evolution plot: $(n = 1,\, m = 3,\, \delta = 10)$



- Grand EOS $\Rightarrow \frac{P_{tot}}{\rho_{tot}} = \frac{(\cancelto{0}{P_M}+P_\phi+P_{int})}{(\rho_M+\rho_\phi+\rho_{int})}$

  Sound Speed $\Rightarrow c_s^2 = \frac{dP_{tot}/dX}{d\rho_{tot}/dX}$
- Accelerating Universe $\Rightarrow -\frac{1}{3} \leq \omega_{tot.} \leq -1.$

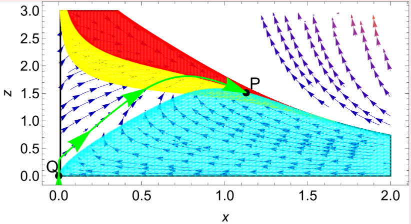
  Fluid density $\Rightarrow 0 \leq \sigma^2 \leq 1.$
  Field density $\Rightarrow 0 \leq \Omega_\phi \leq 1.$

  Sound speed $\Rightarrow 0 \leq c_s^2 \leq 1.$
  Int. energy density $\Rightarrow \Omega_\phi \neq 0.$
- Energy transfer from field to fluid and then fluid to field has also been observed.
- Coupled field-fluid system starting from radiation to matter and end up at accelerating phase with negligible sound speed.

Results & Discussion

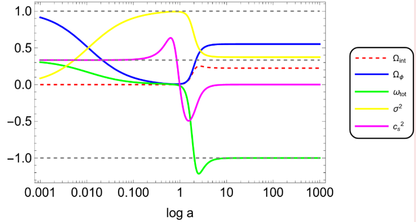
Phase Trajectory and Evolution Plot for Constant Potential ($f = gV_0\rho^q X^\beta M^{-4q}$) [arXiv: 2203.10607]

Phase Trajectory:
($A = \frac{1}{2}, B = \frac{1}{3}, q = -1, g = \frac{1}{2}, \alpha = 1, \beta = \frac{1}{3}, M = 1$)



- 2-dimensional compact phase space of x, z.
- 1 non-trivial stable fixed point (P).
- Constraint on the phase space from $0 \leq \Omega_\phi \leq 1$ & $0 \leq \sigma^2 \leq 1$.
- Red region $\rightarrow$ Phantom Behavior, Yellow region $\rightarrow$ Accelerating Universe & Blue region $\rightarrow$ sound speed is between 0 and 1. Green lines $\rightarrow$ lines of stability go towards stable fixed point (P).

Evolution plot:
($A = \frac{1}{2}, B = \frac{1}{3}, q = -1, g = \frac{1}{2}, \alpha = 1, \beta = \frac{1}{3}, M = 1$)



- Energy density of k-essence sector dominates over the early and late time.
- Fluid density (σ^2) is dominating when $\omega_{\text{tot.}}$ crosses zero line.
- Energy transfer from field to fluid and then fluid to field has also been observed.
- Total EOS starts from radiation dominated phase and ended up at accelerating phase with $\omega_{\text{tot}} = -1$.
- Interacting energy density ($\Omega_{\text{int.}}$) exist at late time.

Overall Conclusion

Conclusion

- Investigation of cosmological effects of **non-minimally coupled** k-essence scalar field and a pressure-less relativistic fluid using the **variational method**.
- A **non-minimal interaction** term $f(n, s, \phi, X)$ depends both on the fluid (n, s) and field sector's (ϕ, X) variables.
- Presence of **interaction** term, **Field and Friedmann equations** are modified in the background of **FLRW universe**.
- We develop the phase space using **dimensionless variables** and examine the dynamics of **power law and constant potential** in coupled k-essence sector.
- Evolutionary dynamics reveal **field-to-fluid-to-field** energy transfer.
- A **stable late-time cosmic accelerating scenario** has been observed through this non-minimally coupled field-fluid model.
- From **Early to late time phase** of the universe has been realized through evolutionary dynamics of the non-minimally coupled sectors.


Thank you!