Quantum state recovery and black holes

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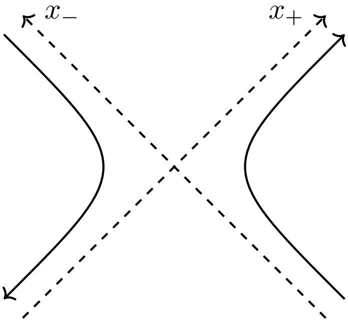
Special Relativity

The quintessential picture of Special Relativity ($x^{\pm} = x \pm t$):



Boost:

$$t(\tau) = \cosh(a\tau)t + \sinh(a\tau)x$$
$$x(\tau) = \sinh(a\tau)t + \cosh(a\tau)x.$$

Unruh effect

Let

$$\beta = \frac{2\pi}{a}$$

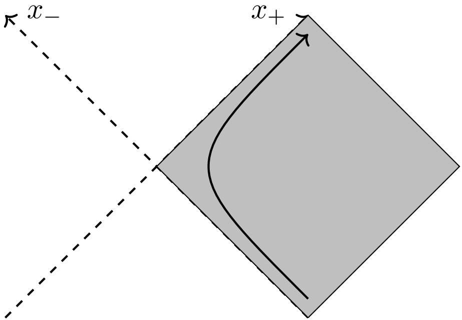
By the addition theorem for $\sinh$ and $\cosh$:

$$\begin{aligned} t(\tau + i\beta) =& t(\tau) \\ x(\tau + i\beta) =& x(\tau). \end{aligned}$$

This suggests that vacuum state correlation functions of a quantum field ϕ should be periodic when expressed in terms of τ provided that their analytic continuation in τ makes sense. This is the case if we restrict to the right wedge $\pm x^{\pm} > 0$:

Unruh effect

Right wedge:

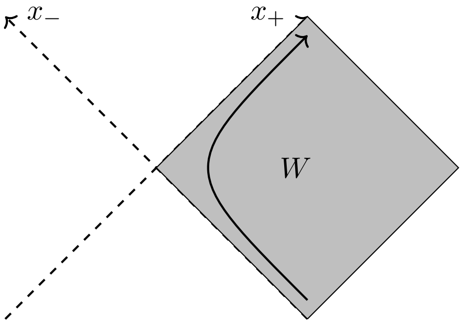


Imaginary boost parameter time periodicity ($\xi = \sqrt{x^2 + t^2}, y, z$):

$$\langle \Omega | \phi(\tau + i\beta, \xi) \phi(\tau', \xi') | \Omega \rangle = \langle \Omega | \phi(\tau', \xi') \phi(\tau, \xi) | \Omega \rangle .$$

The order of the operators has changed because we approach the real τ-axis from above / below on the left / right sides of this equation.

Unruh effect

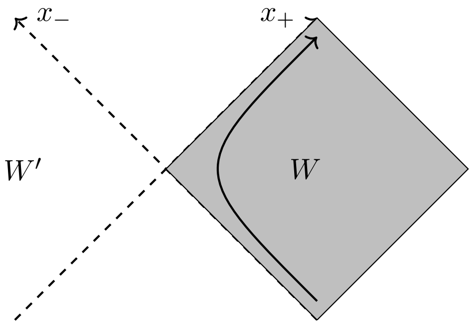


We would get the same periodicity if we had

$$_{W'W}\langle\Omega|\phi(\tau,\xi)\phi(\tau',\xi')|\Omega\rangle_{W'W} \propto \mathrm{Tr}_W\, \phi(\tau,\xi)\phi(\tau',\xi')e^{-\beta K}.$$

where K is the generator of boosts in the right wedge W.

Unruh effect

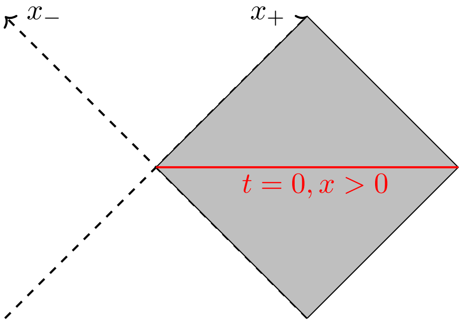


Formally,

$$\mathscr{H} = \mathscr{H}_{W'} \otimes \mathscr{H}_{W},$$

because the wedge W and the opposite wedge W' are causally disjoint.

Unruh effect



Also, formally the generator of boosts in the right wedge ($x > 0$) is:

$$K = \int_{t=0,x>0} x^+ T_{t+} - x^- T_{t-}$$

Unruh effect

Interpretation

To an observer moving with constant acceleration a in the x-direction, the vacuum state looks thermal with Unruh-Hawking temperature

$$T = \frac{a}{2\pi}$$

(natural units $\hbar = c = k_B = G = 1$).

Bisognano-Wichmann theorem

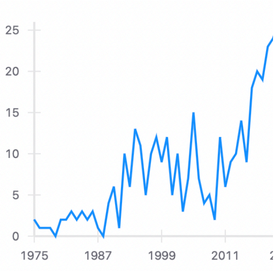


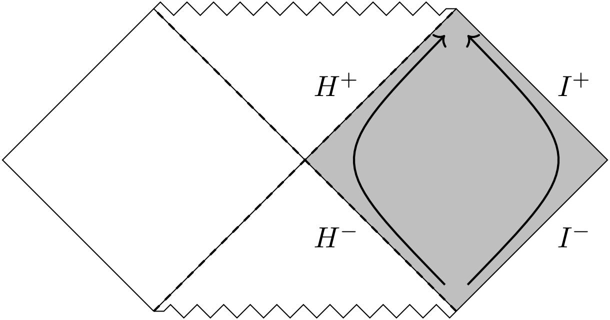
Figure: Citation statistics for Bisognano & Wichmann.

The above argument (with an important caveat regarding the status of the "density matrix" $e^{-\beta K}$) as presented is essentially due to Bisognano & Wichmann, who however did not themselves make the connection with the Unruh effect. Instead, their motivation was to establish a property of the observable algebras $\mathcal{A}_W$ associated with the right wedge.

Black Holes

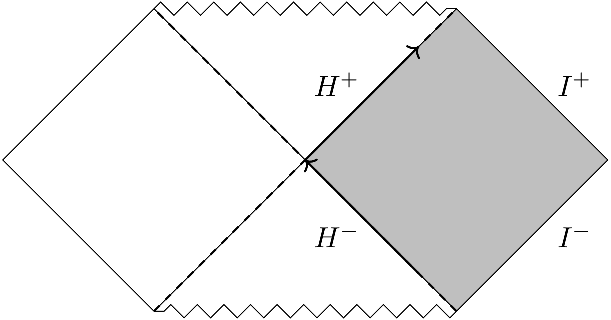
Almost literally the same reasoning applies to a Schwarzschild black hole:

$$ds^2 = -f d\tau^2 + f^{-1} dr^2 + r^2 d\Omega^2, \quad f = 1 - \frac{2M}{r}.$$



Now the right wedge is the exterior of the black hole $r > r_0 = 2M$ (outside the Schwarzschild radius r_0). The boosts correspond to shifts in τ. The full picture is best constructed using Kruskal-Szekeres coordinates which are analogous to $x^{\pm}$.

Black Holes

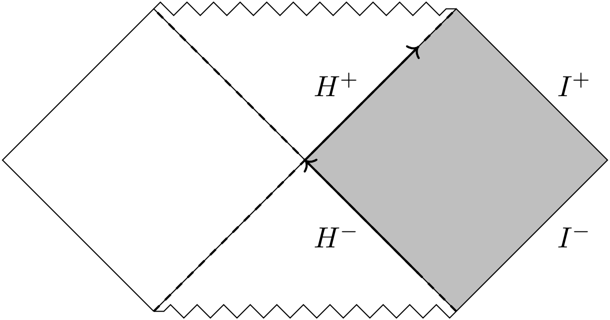


An observer just hovering along the event horizon H^+ (or H^-) given by $r = r_0$ must be accelerated with

$$a = \frac{M}{r_0^2}$$

(Newton's law).

Black Holes



By contrast to the Unruh effect, a (called the "surface gravity") is fixed by requiring that the τ-coordinate be normalized so that

$$ds^2 \sim -d\tau^2 + dr^2 + r^2 d\Omega^2 \quad \text{as } r \to \infty,$$

so τ becomes the usual inertial time t near infinity (redshift effect).

The reduced density matrix

Either in the black hole case or the wedge, we can formally summarize the situation by saying that the reduced density matrix for the right wedge W is

$$\rho_W = \mathrm{Tr}_{W'} |\Omega\rangle\langle\Omega| \propto e^{-\beta K},$$

with $\beta = 2\pi/a$ the inverse Hawking-Unruh temperature and with trace taken over the opposite wedge W'.

Entropy of Black Holes

Bekenstein, Hawking and others argued that the entropy of a black hole should be $1/4$ of the area of the event horizon.

One might naively think that this is equal to its

Entanglement entropy

$$S(\rho_W) = -\operatorname{Tr} \rho_W \ln \rho_W .$$

because the EE quantifies our lack of knowledge about the complementary region, in this case related to the black hole.

However, instead Sorkin et al. found in the early 80's that $S(\rho_W) = \infty$.

Operator Algebras: Types

The root cause of $S(\rho_W) = \infty$ found by Sorkin et al. is that the algebra of observables associated with the right wedge $\mathcal{A}_W$ is mathematically of an unexpected nature. In the 1940s, Murray and von Neumann classified the possible algebras of bounded operators on a Hilbert space $\mathscr{H}$ into the following types, later refined by Connes and others in the 1970s:

- Type I: The algebra of all bounded operators on a Hilbert space $\mathscr{H}$.
- Type II_1 (has a normalized trace) and II_∞ (has no pure states),
- Type III_λ, $\lambda \in [0, 1]$ (have no pure states and no trace),
- Direct sums of types I,II,III.

It was first understood by Araki in a special case and later By Buchholz et al. in general that:

Type III

The algebras in QFT such as $\mathcal{A}_W$ are of type III_1 and so do not have a trace and no pure states.

Quantum gravity: type II [Witten 2022]

Operator Algebras: Modular theory

For algebras of type III, the reduced density matrix ρ_W does not exist mathematically and $S(\rho_W)$ simply is not defined. In practical approaches cutoffs are introduced turning the algebras into type I but then $S(\rho_W)$ diverges with the cutoff. This is more precisely what was found by Sorkin et al. However, there is a related object which does exist in type III, namely the "modular operator", Δ.
Again, formally,

$$\mathscr{H} = \mathscr{H}_{W'} \otimes \mathscr{H}_W, \quad \Delta = \rho_{W'}^{-1} \otimes \rho_W.$$

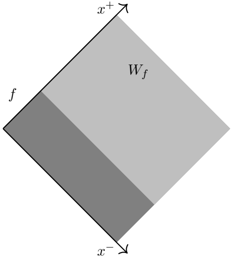
The tensor product and ρ_W is a formal illusion, but Δ can be defined rigorously via Tomita-Takesaki theory of von Neumann algebras, developed throughout the 1970s and crucial in the finer classificaction into types.

Borchers' observation

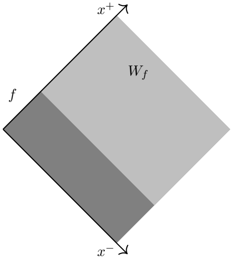
We can define $\mathcal{A}_W$ to be the algebra of all operators in the right wedge, then we have for example

$$\Delta^{it}\phi(\tau,\xi)\Delta^{-it} = \phi(\tau+t,\xi), \quad \phi(\tau,\xi) \in \mathcal{A}_W,$$

i.e. the Heisenberg evolution of Δ corresponds to boosts (Schwarzschild time-translations).
An important observation by Borchers from the 1990s is that we can get more interesting operators from two nested wedges:



Borchers' observation



The second wedge is ($y =$ angles)

$$W_f = \{x^+ > f(y) > 0, x^- > 0\}.$$

It has a modular operator Δ_f. So we have <u>two</u> modular operators.

Borchers' observation and ANEC

Borchers and others [Wiesbrock, Araki, Zsido, Longo 1990s, Casini 2020s] showed that:

- $\Delta_f^{it}\Delta^{-it} = e^{i(e^{2\pi t}-1)T_f}$,
- $T_f \geq 0$ as an operator,
- The T_f's and boosts generate the Bondi-Metzner-Sachs group,
- T_f is the angle-averaged ANEC operator ($y =$ angles)

$$T_f = r_0^2 \int \left(\int_{-\infty}^{\infty} T_{++}(x^+, x^- = 0, y) dx^+ \right) f(y)\, dy,$$

- Since $T_f \geq 0$ for all $f > 0$, the ANEC holds!

$$\mathrm{ANEC}(y) = \int_{-\infty}^{\infty} T_{++}(x^+, x^- = 0, y) dx^+ \geq 0$$

QNEC

The ANEC is a very strong non-perturbative constraint on matrix elements of the stress energy operator. An alternative proof using traditional high-energy methods was obtained only recently in a very well-known paper by [Hartman 2017].

Despite this success, the problem remains that in the type III case, the von Neumann entropy of a state is always infinite! We would like to give sense and prove the "QNEC" [Balakrisnan et al. 2019, Ceyhan et al. 2019, Bousso et al., Wall 2015, ...]:

$$2\pi \langle\psi|T_{++}(x^+, x^- = 0, y)|\psi\rangle \geq \frac{\delta^2}{\delta f(y)^2} S(\rho_f)|_{f=x^+}$$

where $\rho_f = \mathrm{Tr}_{W'_f} |\psi\rangle\langle\psi|$.

For this, we need "relative" entropy.

Relative Entropy: Classical

Relative entropy $=$ relative average surprise

$$S(p|q) := \langle \ln \frac{1}{q_i} \rangle - \langle \ln \frac{1}{p_i} \rangle = \sum p_i \ln \frac{p_i}{q_i}$$

- We believe probability distribution is $\{q_i\}$.
- It actually is $\{p_i\}$.
- Our surprise at seeing i-th event: $\ln \frac{1}{q_i}$.
- Our surprise should have been $\ln \frac{1}{p_i}$.

Relative Entropy: Quantum

Quantum case (type I)

$$S(\sigma|\rho) := \mathrm{Tr}\,\sigma \ln \sigma - \mathrm{Tr}\,\sigma \ln \rho.$$

- We believe density matrix is ρ.
- It actually is σ.
- $S =$ information gained if we assumed state is σ but now we learn ρ.
- Defined for arbitrary algebra types ("$\infty - \infty$")!

Relative Modular Theory [Araki, Connes, Haagerup, Takesaki, Tomita, ... 70's]

Given a decomposition $\mathscr{H} = \mathscr{H}_{W'} \otimes \mathscr{H}_W$ and a pure state $|\eta\rangle$ on $\mathscr{H}$, we get a mixed state on $\mathcal{A}_W$:

$$(\omega_\eta)_W = \mathrm{Tr}_{W'}(|\eta\rangle\langle\eta|)_{W'W}.$$

Conversely, any density matrix on $\mathscr{H}_W$ can be purified (“Schmidt-form”)

$$|\eta\rangle_{W'W} = \sum_k \lambda_k |\phi'_k\rangle_{W'} |\phi_k\rangle_W.$$

In the 70s, it was discovered that a “standard” version of the Schmidt-form can be found for algebras of arbitrary type (I-III) even though the tensor product factorization does not really exist [Connes, Araki, Haagerup, ... 70s]. The existence of such a “standard form” allows one to define a “relative modular operator”, formally:

$$\Delta_{\eta,\psi} = (\omega_\eta^{-1})_{W'} \otimes (\omega_\psi)_W.$$

Relative Entropy

Relative entropy [Umegaki 60s, Araki 70s]

$$S(\omega_\psi|\omega_\eta) := -\langle\psi|\ln(\Delta_{\eta,\psi})\psi\rangle.$$

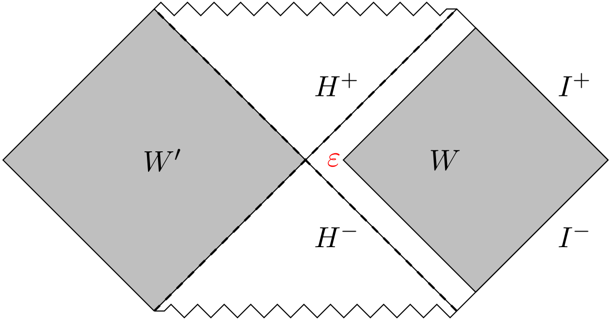
1. $\mathcal{A} =$ type I $\Longrightarrow S(\omega_\psi|\omega_\eta) = \mathrm{Tr}(\omega_\psi \ln \omega_\psi) - \mathrm{Tr}(\omega_\psi \ln \omega_\eta)$.
2. Individual terms not defined for general von Neumann algebra.
3. $\omega_\eta =$ tracial state $\Longrightarrow$ von Neumann entropy.
4. Measure of distinguishability between ω_η, ω_ψ.
5. Average information gained if we thought the state was ω_η but now we learn it is $\omega_\psi \Longrightarrow$ Operational meaning!
6. ω_η is thought of as reference (e.g. vacuum in QFT).

Relative Entropy and Black Holes

As an application of the relative entropy, consider again formally the equality

$$-\operatorname{Tr}\rho_W \ln \rho_W = \frac{1}{2} S(|\Omega\rangle\langle\Omega|_{WW'} \,|\, \rho_{W'} \otimes \rho_W) \quad (= \infty).$$

This is still infinite, but now we can give a natural regularization:



We leave a safety belt of size ε between W and W', then one can see that

$$S(|\Omega\rangle\langle\Omega|_{WW'} \,|\, \rho_{W'} \otimes \rho_W) \sim \frac{A}{\varepsilon^2},$$

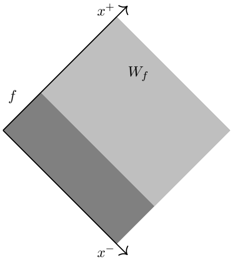
similar to Hawking's prediction for S but with a renormalized Newton constant $G \sim \varepsilon^2$. The belt means that all quantities are now well-defined, by the split property investigated in quantum field theory by Buchholz and others.

Relative entropy and QNEC

Another application is a rigorous formulation of the QNEC inequality:

$$2\pi\langle\psi|ANEC(y)|\psi\rangle \geq -\frac{\delta}{\delta f(y)} S_{W_f}(\omega_\psi|\omega_\Omega)|_{f=0},$$

where W_f is the shifted wedge and $|\Omega\rangle =$vac. This can be used to give a rigorous proof of QNEC [Faulkner et al. 2019]!



QNEC, channels, state recovery

The analysis of QNEC suggests that one should study

$$S_{W_f}(\omega_\psi|\omega_\Omega) - S_W(\omega_\psi|\omega_\Omega)$$

The restriction of a density matrix from W to W_f can be seen as a "channel". The QNEC is closely related to the ‘invertability" of this channel, which provides a link to "state recovery".

Channels

- **Observables:** elements a of some operator algebra $\mathcal{A}$.
- **States:** positive functionals $\omega : \mathcal{A} \to \mathbb{C}$ such that $\omega(1) = 1$. For type I: $\omega(a) = \mathrm{Tr}\, a\omega$.

On states/observables we have standard operations ("channels"):

- **Time evolution:** $a \to U^* a U$ ($U = e^{iHt}$).
- **von Neumann measurement:** $a \to PaP$ ($P =$ eigenprojection).
- **Ancillary systems:** $a \to a \otimes 1_{\mathbb{C}^n}$.

These three combine most generally into a **completely positive map** T (defined for arbitrary type).

Local operations

Typically, we have bipartite system $\mathscr{H}_W \otimes \mathscr{H}_{W'}$.

Local operations

Channels of form

$$T = \sum T_W \otimes T_{W'}$$

Example:

Teleportation of a state $|\beta\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ from W' to W. [Bennett, Brassard, Crepeau, Jozsa, Perez, Wootters 1993], can be seen as a local operation on a system $W'CW$.

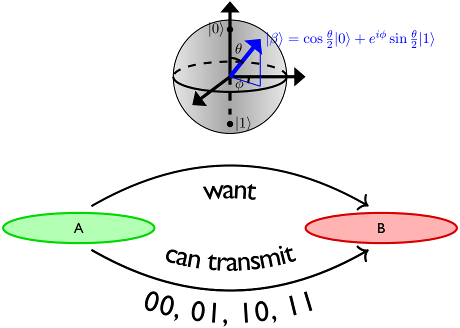


Figure: Teleportation of one q-bit.

Data Processing Inequality [Lieb & Ruskai, Lindblad, Araki, Uhlmann, ... 70's]

DPI

If $T : \mathcal{B} \to \mathcal{A}$ is a channel, then

$$S(\omega_\psi|\omega_\eta) \geq S(\omega_\psi \circ T|\omega_\eta \circ T).$$

1. Decisive property of distinguishability measures, key for entanglement quantification [e.g. Plenio & Virmani 07, ...].
2. Shared by many other "divergences" of entropy type [e.g. Hiai 21].
3. Implies e.g. SSA for von Neuman entropy ($\mathcal{A} =$ type I), convexity, ...
4. Equality case: [Petz 80s]
5. Proofs: Subharmonic analysis [Lieb, Araki, Epstein, ... 70s], variational principles [Uhlmann 70s, Kosaki 80s], ...

Recovery channels

Question

Under what conditions can a channel $T : \mathcal{B} \to \mathcal{A}$ be approximately inverted in the sense that there exists $R : \mathcal{A} \to \mathcal{B}$ such that ω_ψ and $\omega_\psi \circ T \circ R$ are "close" for "a class of" states ω_ψ (depending on T)?

Ideas:

1. We should not lose too much information by applying T.
2. Precisely: For some reference state $|\eta\rangle$, the information loss $S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T)$ should be small.
3. We should have perfect recoverablity if this information loss is zero:

$$S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T) = 0 \quad \Longrightarrow \quad \omega_\psi = \omega_\psi \circ T \circ R.$$

Petz recovery channel

Suppose $\mathcal{A}$ acts on $\mathscr{H}$ and $\mathcal{B}$ on $\mathscr{K}$. Given: channel $T : \mathcal{B} \to \mathcal{A}$, reference state $|\eta_{\mathcal{A}}\rangle$ in natural cone $\Longrightarrow$ **"Petz-adjoint"** [Petz 80's, Longo 17 $\to$ bimodules]:

- **KMS inner product** on $\mathcal{A}$:

$$\langle a_1, a_2\rangle_{\mathcal{A}} = \langle \eta_{\mathcal{A}} | a_1^* \Delta_{\eta_{\mathcal{A}}}^{1/2} a_2 \eta_{\mathcal{A}}\rangle \quad (= \mathrm{Tr}(a_1^* \eta_{\mathcal{A}} a_2 \eta_{\mathcal{A}}) \quad \text{when } \mathcal{A} = \text{type I.})$$

- $|\eta_{\mathcal{A}}\rangle$ induces state $|\eta_{\mathcal{B}}\rangle$ in $\mathscr{K}$ via $T \Longrightarrow$ KMS inner product on $\mathcal{B}$.
- $T^+ : \mathcal{A} \to \mathcal{B} =$ **"Petz" adjoint** of T relative to KMS inner products.
- **This is a channel.**

Rotated Petz map [Junge et al. 18]

$$R^t = Ad(\Delta_{\eta_{\mathcal{B}}}^{-it}) \circ T^+ \circ Ad(\Delta_{\eta_{\mathcal{A}}}^{it})$$

Improved DPIs: History

- **Petz et al. 80s and 90s equality case**: $S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T) = 0. \implies T^+$ is a perfect recovery channel: $\omega_\psi = \omega_\psi \circ T \circ T^+$.
- **Fawzi and Renner 15**: first results on approximate recovery, when $S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T) \approx 0$.
- Subsequent **improvements/variations**: e.g. [Berta, Lemm, Wilde 15, Carlen, Vershynina 17, Gao, Wilde 20, Junge, Renner, Sutter, Wilde, Winter 18, Sutter, Tomamichel, Harrow 16, Sutter, Berta, Tomamichel 17,...]
- Restricted to **type I** von Neumann algebras
- Recent generalizations to **general von Neumann algebras** by [Faulkner, SH, Swingle, Wang 20, Faulkner, SH 20, SH 21, Junge, LaRacuente 20] using tools from non-commutative L_p spaces [Araki, Masuda 82, Terp 81, Haagerup 79, Yamagami 92, Izumi 00, Pisier, Q Xu 03, Vershynina 17, Berta, Scholz, Tomamichel 18, ...]

Main theorems

Thm. 1 [Faulkner, SH, Swingle, Wang 20, Faulkner, SH 20]

If $T : \mathcal{B} \to \mathcal{A}$ is a channel, then

$$S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T) \geq -\int_{\mathbb{R}} dt\, \beta_0(t) \ln F(\omega_\psi|\omega_\psi \circ T \circ R^t),$$

where $R^t : \mathcal{A} \to \mathcal{B}$ is rotated Petz recovery channel, $F =$ fidelity,
$\beta_0(t) = \frac{\pi}{\cosh(2\pi t)+1} =$ probability density.

Remarks:

- Small information loss in DPI implies proximity of ω_ψ and recovered $\omega_\psi \circ T \circ R^t$ to high fidelity.
- Generalizes [Junge, Renner, Sutter, Wilde, Winter 18] to von Neumann algebras
- In type I setting $F(\omega_\psi|\omega_\eta) = \mathrm{Tr}\sqrt{\sqrt{\omega_\eta}\,\omega_\psi\sqrt{\omega_\eta}}$.
- Integral can be pulled inside F (Jensen)

Main theorems

Thm. 2 [SH 21]

If $T : \mathcal{B} \to \mathcal{A}$ is a channel, then

$$S(\omega_\psi|\omega_\eta) - S(\omega_\psi \circ T|\omega_\eta \circ T) \geq S_{\text{meas}}\,(\omega_\psi|\omega_\psi \circ T \circ R),$$

where S_{meas} = measured relative entropy,

$$R = \int_{\mathbb{R}} dt\,\beta_0(t) R^t$$

integrated rotated Petz recovery channel.

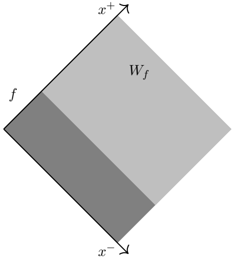
Remarks:

- Small information loss in DPI implies proximity of ω_ψ and recovered $\omega_\psi \circ T \circ R$ w.r.t. measured relative entropy.
- Generalizes [Sutter, Berta, Tomamichel 17] to von Neumann algebras.
- $S_{\text{meas}} = \sup\{S|_{\mathcal{C}} : \mathcal{C} \subset \mathcal{A}$ commutative$\}$.
- Integral cannot be pulled outside, but sharp in “classical” case.

Improved DPI and QNEC

Special case $\mathcal{A}_{W_f} \subset \mathcal{A}_W$ (inclusion channel), $|\Omega\rangle$ = vac.

$$S_W(\omega_\psi|\omega_\Omega) - S_{W_f}(\omega_\psi|\omega_\Omega) \geq -\int_1^\infty \ln F_W(\omega_\psi \,|\, e^{iyT_f}\omega_\psi e^{-iyT_f})^2 \, \frac{dy}{y^2}.$$



This gives a relation with the QNEC because $T_f =$ ANEC operator with angle average against f.

Conclusions

- Black hole horizons combined with quantum field theory have interesting connections to QIT
- Highly non-trivial results even for Lorentzian QFT in flat space
- Relationship with singularity theorems (quantum focussing conjecture, Bousso bounds, ...)
- Operator algebraic methods are very useful (and elegant!)

Tools: Non-commutative L_p-theory

For $\mathcal{A}$ = type I: $\mathscr{H}$ = Hilbert-Schmidt operators. Ordinary L_p-norms:

$$\|\zeta\|_p = [\mathrm{Tr}(\zeta\zeta^*)^{p/2}]^{1/p}$$

"Non-commutative" generalization: $|\psi\rangle \in \mathscr{H}$ faithful:

$$\|\zeta\|_{p,\psi} = [\mathrm{Tr}(\zeta\omega_\psi^{2/p-1}\zeta^*)^{p/2}]^{1/p}$$

Non-commutative L_p-spaces $\mathcal{A}$ = von Neumann [Araki, Masuda 82]

$1 \leq p \leq 2$:

$$\|\zeta\|_{p,\psi} = \sup_{\|\phi\|=1} \|\Delta_{\phi,\psi}^{(1/2)-(1/p)}\zeta\|.$$

$2 \leq p \leq \infty$: Similar variational definition.

- $p = 2$: $\|\zeta\|_{2,\psi} = \|\zeta\|$
- $p = 1$: $\|\zeta\|_{1,\psi} = F(\omega_\zeta|\omega_\psi)$
- $p = \infty$: $\|a\psi\|_{\psi,\infty} = \|a\|$.
- $\lim_{p\to 2-}(p/2-1)^{-1}\ln\|\eta\|_{p,\psi}^p = S(\omega_\psi|\omega_\eta)$.

Tools: Interpolation

Non-commutative interpolation [Faulkner, SH, Swingle, Wang 20]

$|G(z)\rangle$ $\mathscr{H}$-valued holomorphic on $\{0 < \mathrm{Re} z < 1/2\}$, bounded on closure, $0 < \theta < 1/2$, $p_0, p_1 \in [1,2]$ or $p_0, p_1 \in [2, \infty]$,

$$\frac{1}{p_\theta} = \frac{1-2\theta}{p_0} + \frac{2\theta}{p_1}.$$

Then

$$\begin{aligned} &\ln \|G(\theta)\|_{p_\theta,\psi} \\ \leq &\int_{-\infty}^{\infty} dt \left((1-2\theta)\alpha_\theta(t) \ln \|G(it)\|_{p_0,\psi} + (2\theta)\beta_\theta(t) \ln \|G(1/2+it)\|_{p_1,\psi} \right), \end{aligned}$$

where

$$\alpha_\theta(t) = \frac{\sin(2\pi\theta)}{(1-2\theta)(\cosh(2\pi t) - \cos(2\pi\theta))}, \quad \beta_\theta(t) = \frac{\sin(2\pi\theta)}{2\theta(\cosh(2\pi t) + \cos(2\pi\theta))}.$$

Tools: analytic vectors

To bring $T : \mathcal{B} \to \mathcal{A}$ into game, define [Petz 80s]

$$V_\psi(b|\psi_\mathcal{B}\rangle) := T(b)|\psi_\mathcal{A}\rangle \quad \Longrightarrow \quad V_\psi : \mathscr{K} \to \mathscr{H}$$

and analytic vector

$$|G(z)\rangle = \Delta^{z}_{\eta,\psi\mathcal{A}} V_\psi \Delta^{-z}_{\eta,\psi,\mathcal{B}} |\psi_\mathcal{B}\rangle.$$

- Properties of T + Tomita-Takesaki-Araki imply desired analyticity in $\{0 < \mathrm{Re} z < 1/2\}$ and boundedness ($V_\psi =$ contraction).
- Interpolation and $\theta \to 0$ (for $p_0 = 1, p_1 = 2$ in thm. 1 and $p_0 = \infty, p_1 = 2$ in thm. 2)
- Entropy difference in DPI pops out in thm. 1 (thm. 2 more difficult)

Tools: regularization, inequalities

Limit $\theta \to 1$ is very subtle for general von Neumann algebras!

Some further tools

- Regularized vectors needed $|\psi_P\rangle = f_P(\ln \Delta_{\eta,\psi})|\psi\rangle$ for f_P with suitable analyticity properties + $f_P(x) \to \delta(x)$.
- Works well with limit of entropy S as $P \to \infty$!

$$\limsup_P S(\psi_P|\eta) \leq S(\psi|\eta) + 2\ln(\|f\|_1/\|\tilde{f}\|_\infty)$$

  $\to$ optimize f (sharp Hausdorff-Young)
- Sandwiching inequalities (Harnack, ALT-inequality):

$$1 \geq \langle\zeta|\, \Delta_{\psi,\zeta}^{2/p-1}\, |\zeta\rangle \geq \left(\|\zeta\|_{p,\psi}\right)^2 \geq \left(\langle\zeta|\, \Delta_{\psi,\zeta}^{1-p/2}\, |\zeta\rangle\right)^{2/p}$$

- Thm. 2: $p_0 = n$ and $n \to \infty$ + Araki-Lie-Trotter + heavy use of Araki-Masuda results.

Conclusions

- Improved DPI
- Approximate recovery when loss in DPI is small
- Highly non-trivial inequalities for general von Neumann algebras
- Intimate connection (and improvement) to (of) QNEC