Diffraction and Low-x 2022

High Energy 2 -> 2 QCD Scattering Amplitudes

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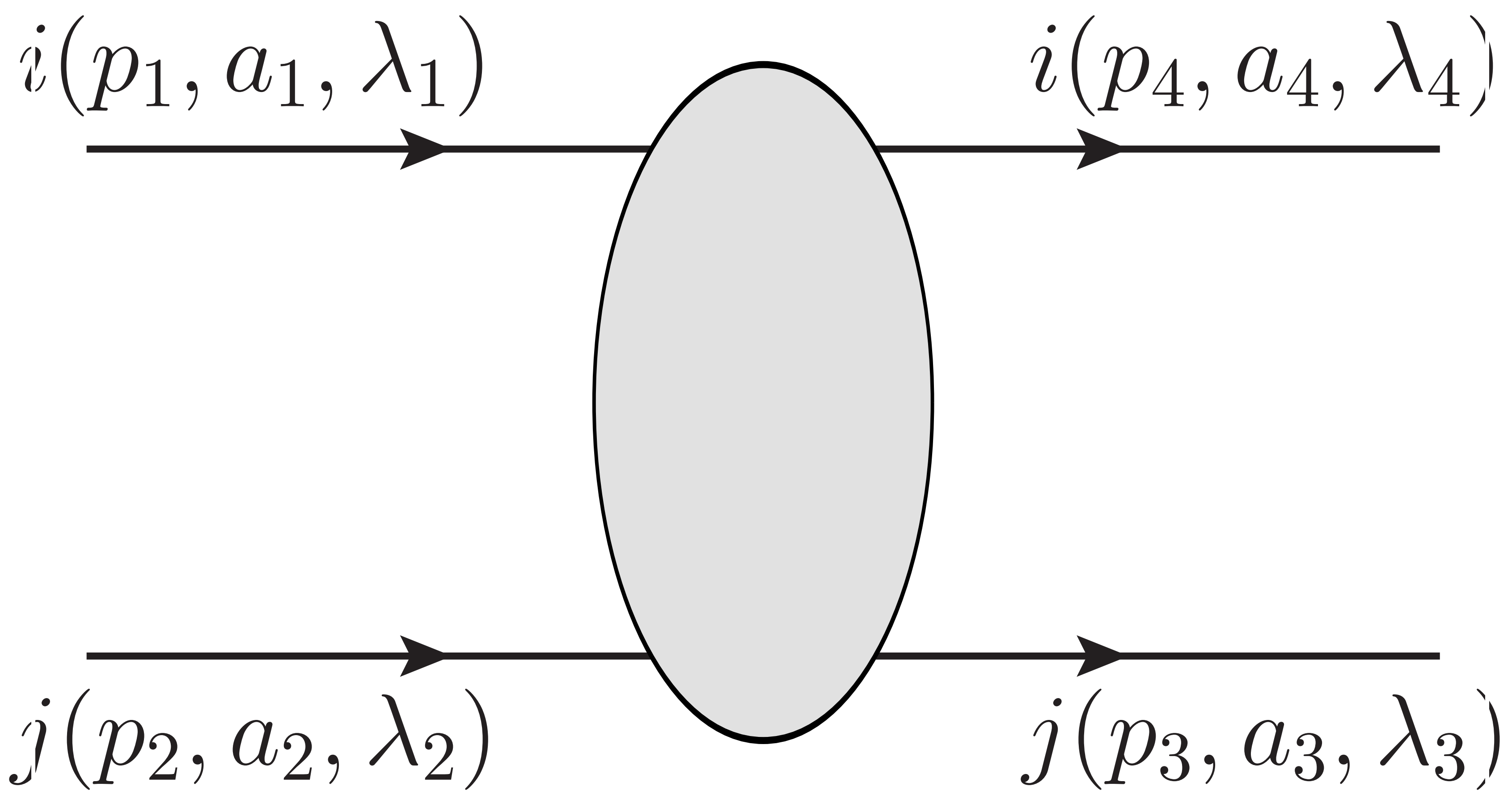
with thanks to collaborators Gulio Falcioni, Einan Gardi, Niamh Maher and Leonardo Vernazza

Introduction to 2->2 Amplitudes

- We consider 2 -> 2 partonic (colourful) scattering amplitudes in a massless gauge theory



- Described in terms of Mandelstam invariants

$$s = (p_1 + p_2)^2 > 0 \qquad t = (p_1 - p_4)^2 < 0 \qquad u = (p_1 - p_3)^2 < 0$$

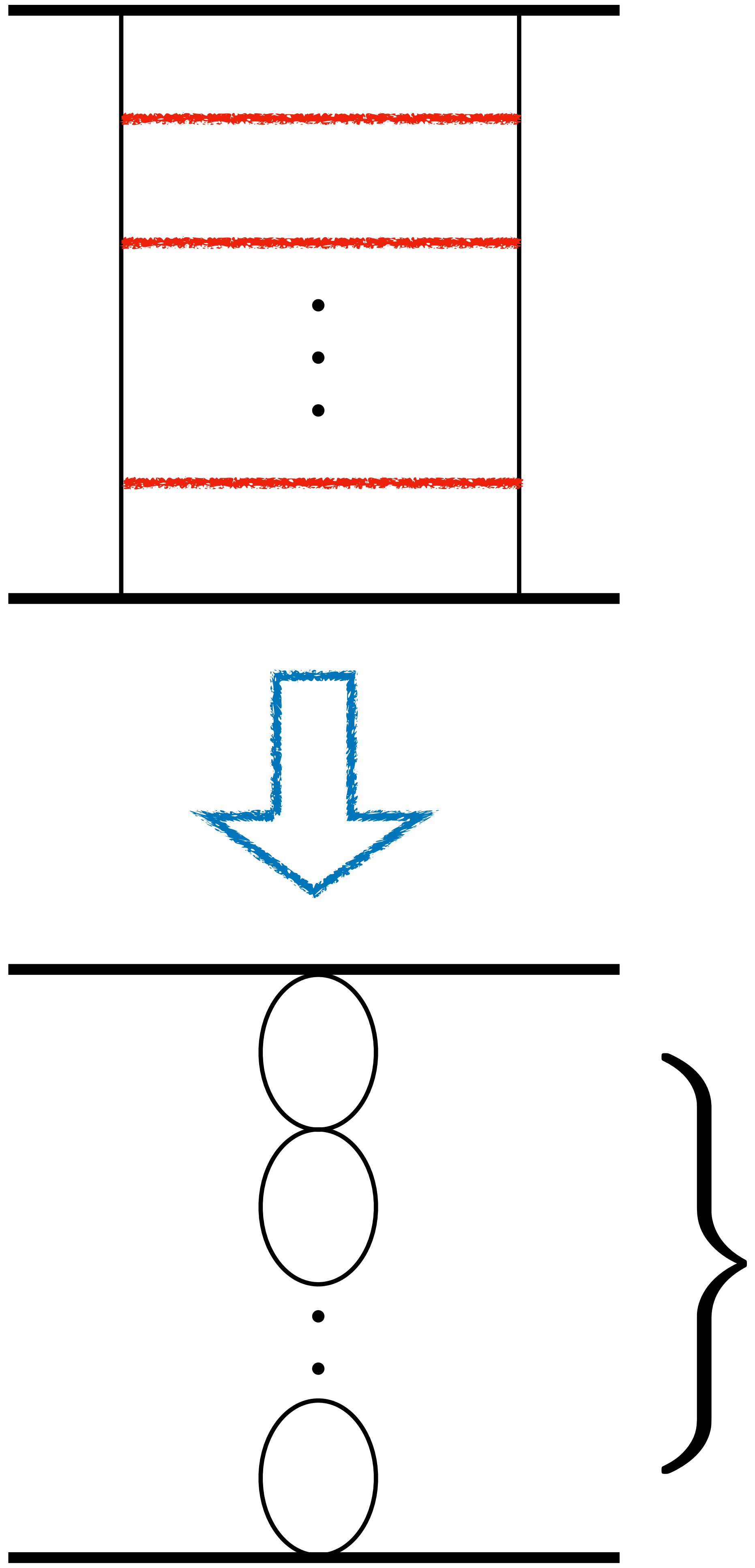
$$s + t + u = 0$$

- At fixed order in α_s
  - Planar $\mathcal{N} = 4$ sYM known to all orders, courtesy of the BDS ansatz [Bern, Dixon, Smirnov 05]
  - Three-loop full colour $\mathcal{N} = 4$ sYM and recently in QCD for all channels
    [Henn, Mistlberger 16] [Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi 21, 21, 22]
- The high-energy limit is defined to be where the centre of mass energy is much greater than the momentum transfer

$$s \gg -t$$

Why care about high-energy amplitudes?

- Formal - simplification gives opportunity to study high-loop orders and to resum perturbative amplitudes
- Boundary information for fixed order calculations via differential equations
- Constraints on infrared structure of amplitudes
- Perturbative meaning to Regge poles and Regge cuts



Complex Angular Momentum Plane

Let us travel back to the 1960s, prior to QCD

[Regge '59, '60; Eden, Landshoff, Olive, Polkinghorne '66; Collins '77]

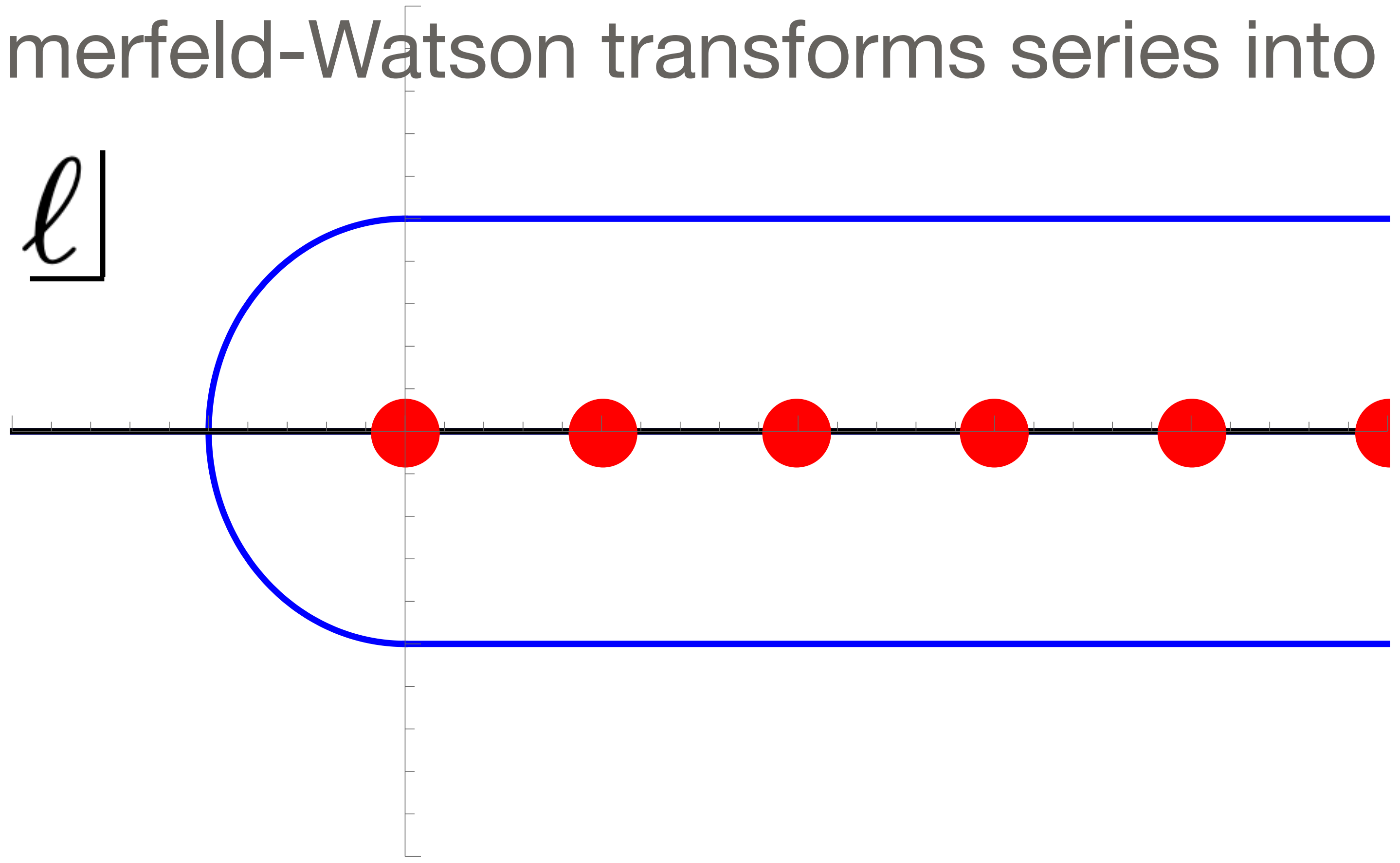
Start with partial wave expansion of the scattering amplitude

angular momentum in t-channel

dependence of a state of angular momentum ℓ are the Legendre polynomials

$$\mathcal{M} \sim \sum_{\ell=0}^{\infty} (2\ell+1) A_\ell(t) P_\ell(s)$$

Sommerfeld-Watson transforms series into a counter integral, picking up poles in the complex angular momentum plane



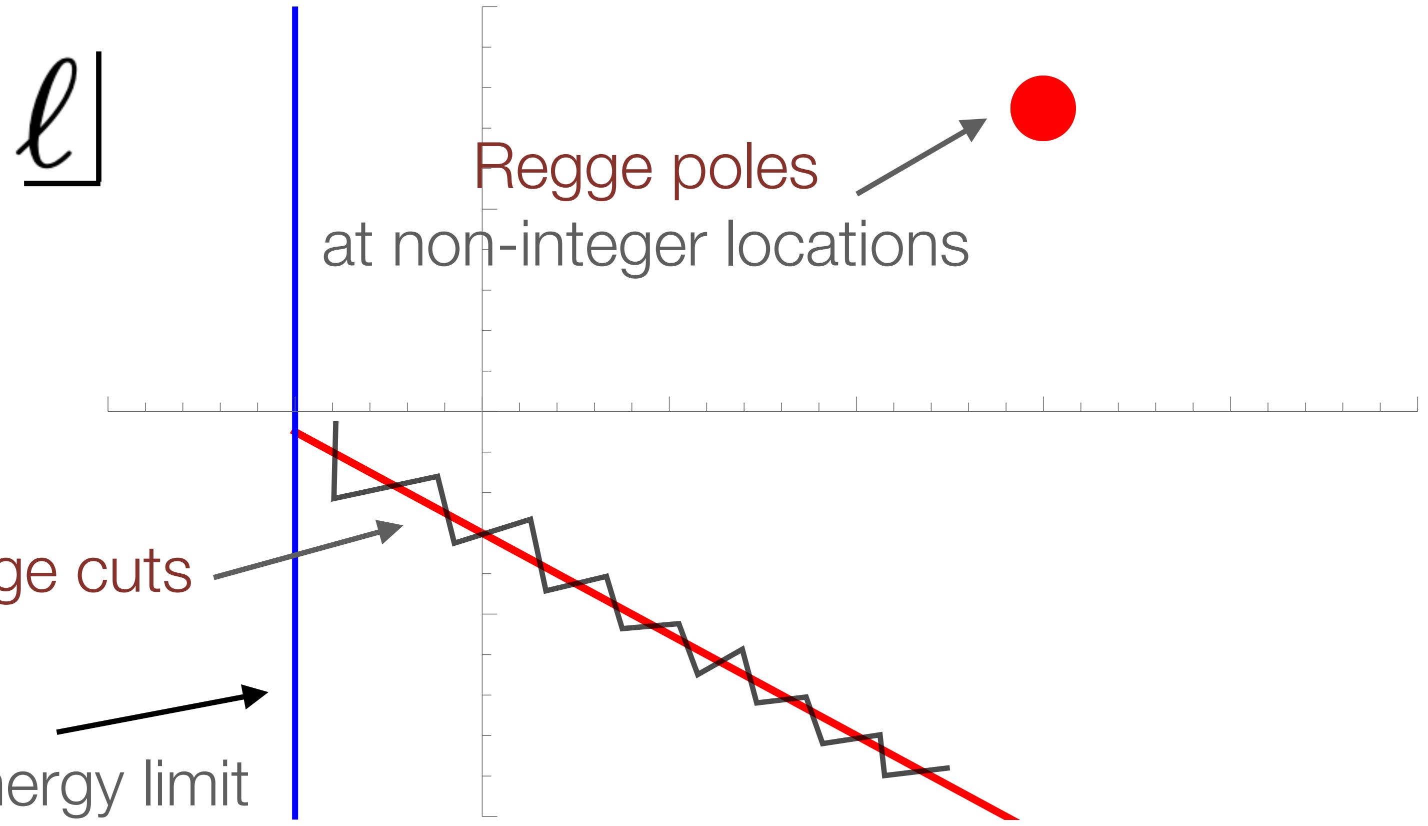
$$\mathcal{M} \sim \oint d\ell\, (2\ell+1) A_\ell(t) P_\ell(s) \frac{\pi}{\sin \pi\ell}$$

Legendre polynomials have the asymptotic behaviour

$$\lim_{s\to\infty} P_\ell(s) \sim s^\ell$$

Now we open up the contour

Doing so we will pick up other analytic behaviour



Regge poles at non-integer locations

Regge cuts

This contour is subleading in high energy limit

Regge cuts arise from only nonplanar diagrams [Mandelstam '63]

Only from Feynman integral analysis, there is no colour (yet)

We will show how to disentangle cuts and poles in perturbative QCD

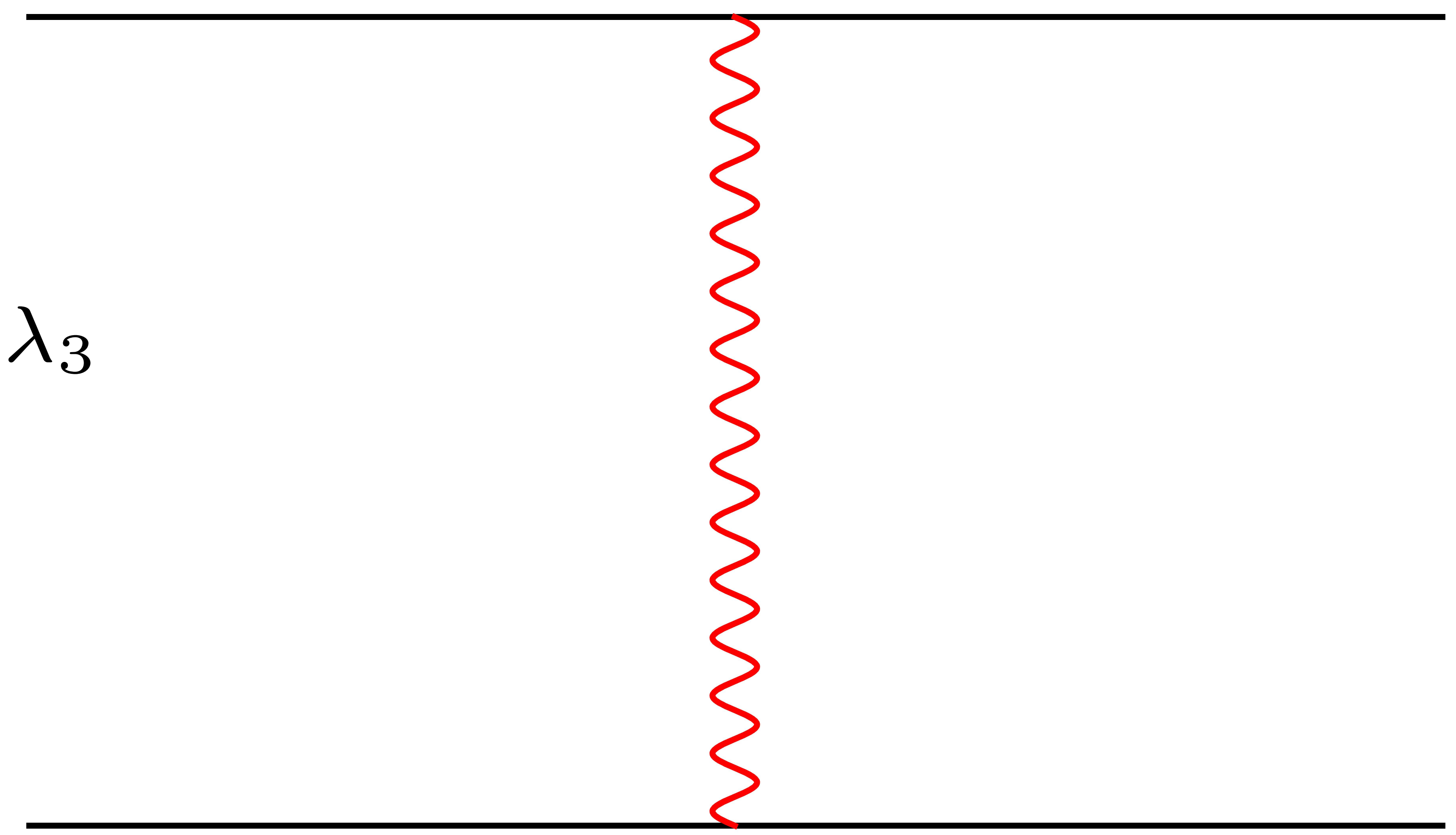
Signature Symmetry

- In the high-energy limit we have $s \approx -u$
- We define even and odd amplitudes under this signature symmetry $\mathcal{M}^{(\pm)}(s,t) = \frac{1}{2}\Big[\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big]$
- We also define the symmetric-even logarithm $L \equiv \log\left(\frac{s}{-t}\right) - \frac{i\pi}{2} = \frac{1}{2}\left[\log\left(\frac{-s-i0}{-t}\right) + \log\left(\frac{-u-i0}{-t}\right)\right]$
- Expanding in L $\quad \mathcal{M}^{(\pm)} = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{\infty} L^m \mathcal{M}^{(\pm,n,m)}$
    - The coefficients $\mathcal{M}^{(-,n,m)}$ are purely real and $\mathcal{M}^{(+,n,m)}$ are purely imaginary [Caron-Huot, Gardi, Vernazza '17]
- Bose symmetry means it occurs for colour quantum numbers of gluon-gluon
    - In the t-channel decomposition $\underline{\mathbf{8}} \otimes \underline{\mathbf{8}} = \underline{\mathbf{1}} \oplus \underline{\mathbf{8}_a} \oplus \underline{\mathbf{8}_s} \oplus (\underline{\mathbf{10}} \oplus \underline{\overline{\mathbf{10}}}) \oplus \underline{\mathbf{27}} \oplus \underline{\mathbf{0}}$
- Real, odd parts correspond to $\underline{\mathbf{8}_a}$ $\underline{\mathbf{10}} \oplus \underline{\overline{\mathbf{10}}}$ and imaginary, even parts to $\underline{\mathbf{1}}$ $\underline{\mathbf{8}_s}$ $\underline{\mathbf{27}}$ $\underline{\mathbf{0}}$

Tree Level

- Leading power in the large ratio $\left(\frac{s}{-t}\right)$

$$\mathcal{M}^{\text{tree}} = g_s^2 \frac{2s}{t} \mathbf{T}_i \cdot \mathbf{T}_j \delta_{\lambda_1 \lambda_4} \delta_{\lambda_2 \lambda_3}$$



- Gluon exchange in the t-channel
- Helicity is conserved
- Colour matrices in representation of scattered partons
- For gluon-gluon the tree-level is only in the $\underline{\mathbf{8}_a}$ representation in the t-channel basis

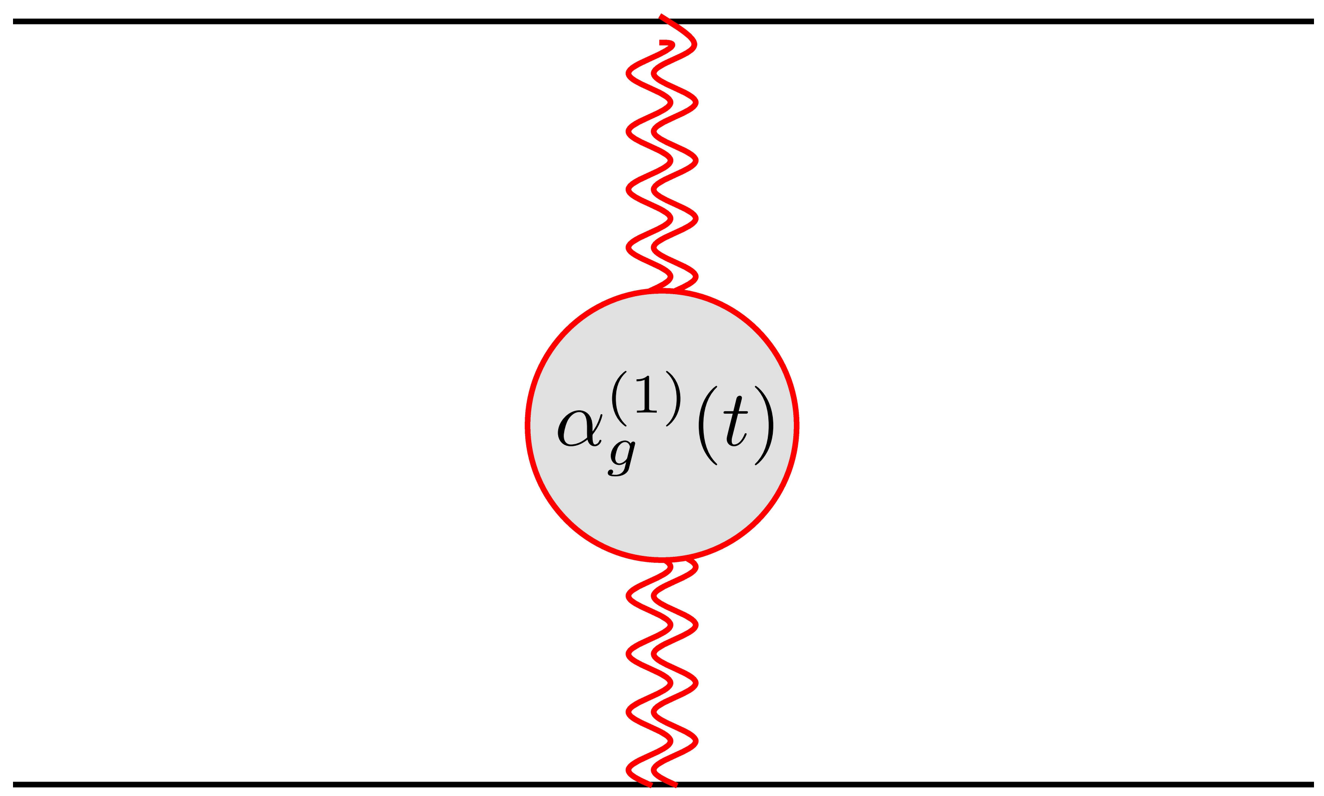
Leading Logarithms (LL)

- At each order in the perturbative expansion large logarithms develop
- The leading logarithms $(\alpha_s L)^n$ can be resummed by the famous Regge pole

[Kuraev, Fadin and Lipatov '76]

$$\mathcal{M}^{\rm LL} = e^{C_A \alpha_g(t,\mu^2) L} \mathcal{M}^{\rm tree} \qquad \alpha_g(t,\,\mu^2) = \frac{\alpha_s}{\pi}\frac{r_\Gamma}{2\epsilon}\left(\frac{\mu^2}{-t}\right)^{\epsilon} + \mathcal{O}(\alpha_s^2) \qquad r_\Gamma = 1 + \mathcal{O}(\epsilon^2)$$

- Infrared divergences regulated in dimensional regularisation $d = 4 - 2\epsilon$
- Purely real, proportional to tree-level
- Dresses the gluon propagator $\frac{1}{t} \to \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha_g(t,\mu^2)}$
  - The gluon *Reggeizes*



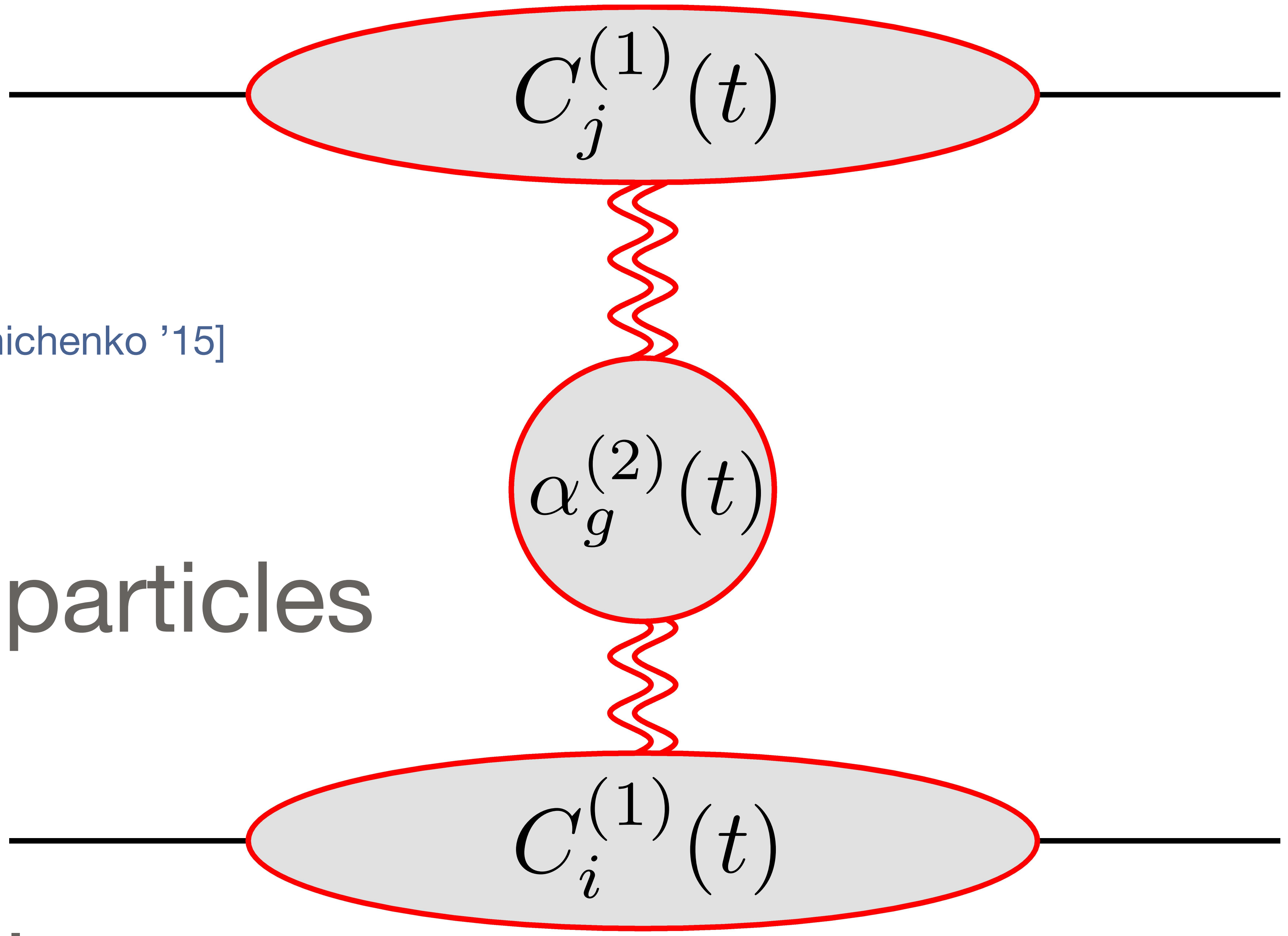
Next-to-Leading-Logarithms (NLL) Real

- Terms of order $\alpha_s^n L^{n-1}$
- The amplitude still factorises, *Regge factorisation*

[Fadin, Fiore, Kozlov, Reznichenko '06; Ioffe, Fadin, Lipatov '10; Fadin, Kozlov, Reznichenko '15]

$$\mathcal{M}^{\mathrm{NLL},(-)} = C_i(t) C_j(t) e^{C_A \alpha_g(t,\mu^2) L} \mathcal{M}^{\mathrm{tree}}$$

- Need impact factors that depend on the scattered particles
- Regge trajectory at two loops
- Proportional to the colour structure of the tree-level



Next-to-Leading-Logarithms (NLL) Imaginary

- In gluon-gluon the $\underline{\mathbf{1}}$ $\underline{\mathbf{8}_s}$ $\underline{\mathbf{27}}$ $\underline{\mathbf{0}}$ colour channels open up

  first appearance of a **cut**

- In the soft approximation, BFKL kernel simplifies

- Allows for an iterative solution

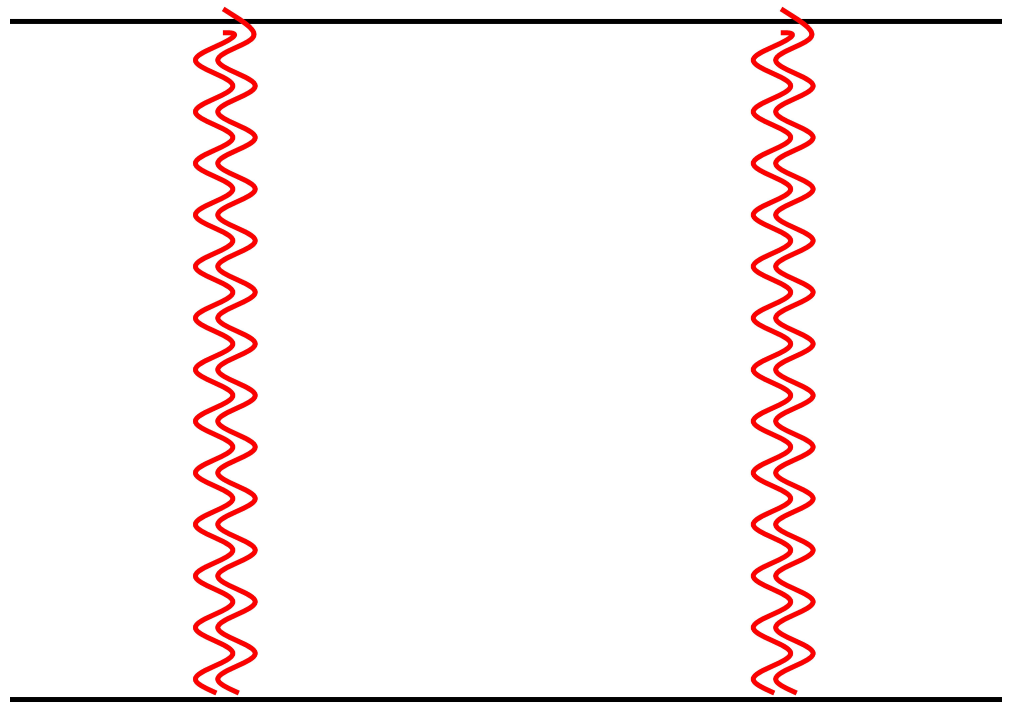
- Infrared divergences to all orders and resummed

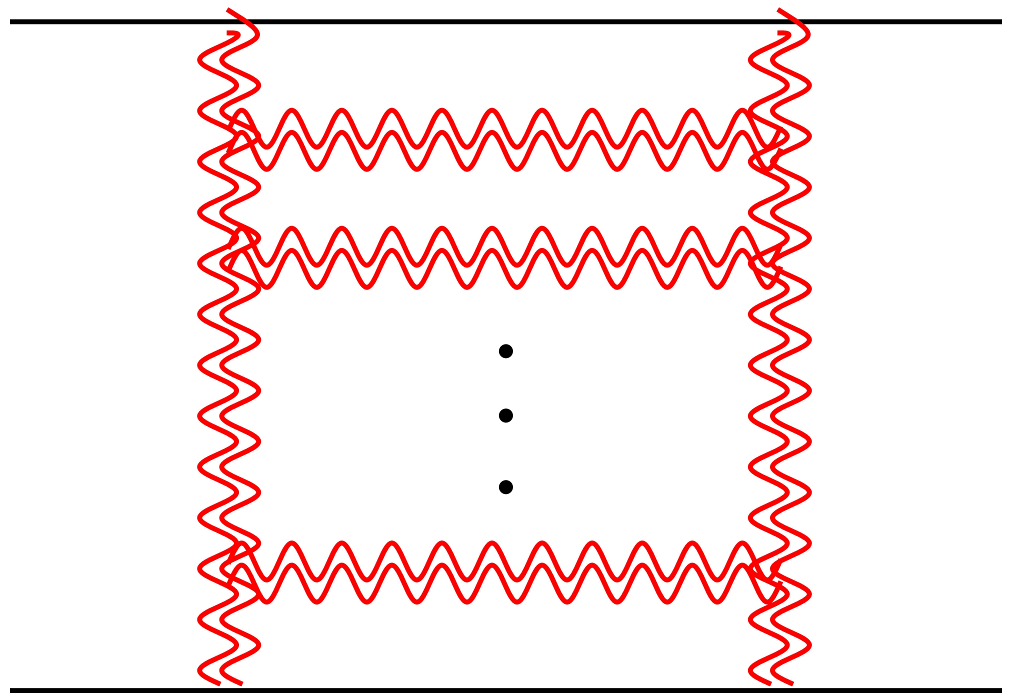
  [Caron-Huot, Gardi, Reichel, Vernazza '17]

$$\mathcal{M}_{[1]}^{\mathrm{NLL},(+)} = \frac{i\pi}{R(\epsilon)-1} \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n L^{n-1} \left(C_A \alpha_g^{(1)}\right)^n + \mathcal{O}(\epsilon^0)$$

- Finite terms known to arbitrary order (resummation unknown)

  [Caron-Huot, Gardi, Reichel, Vernazza '20]



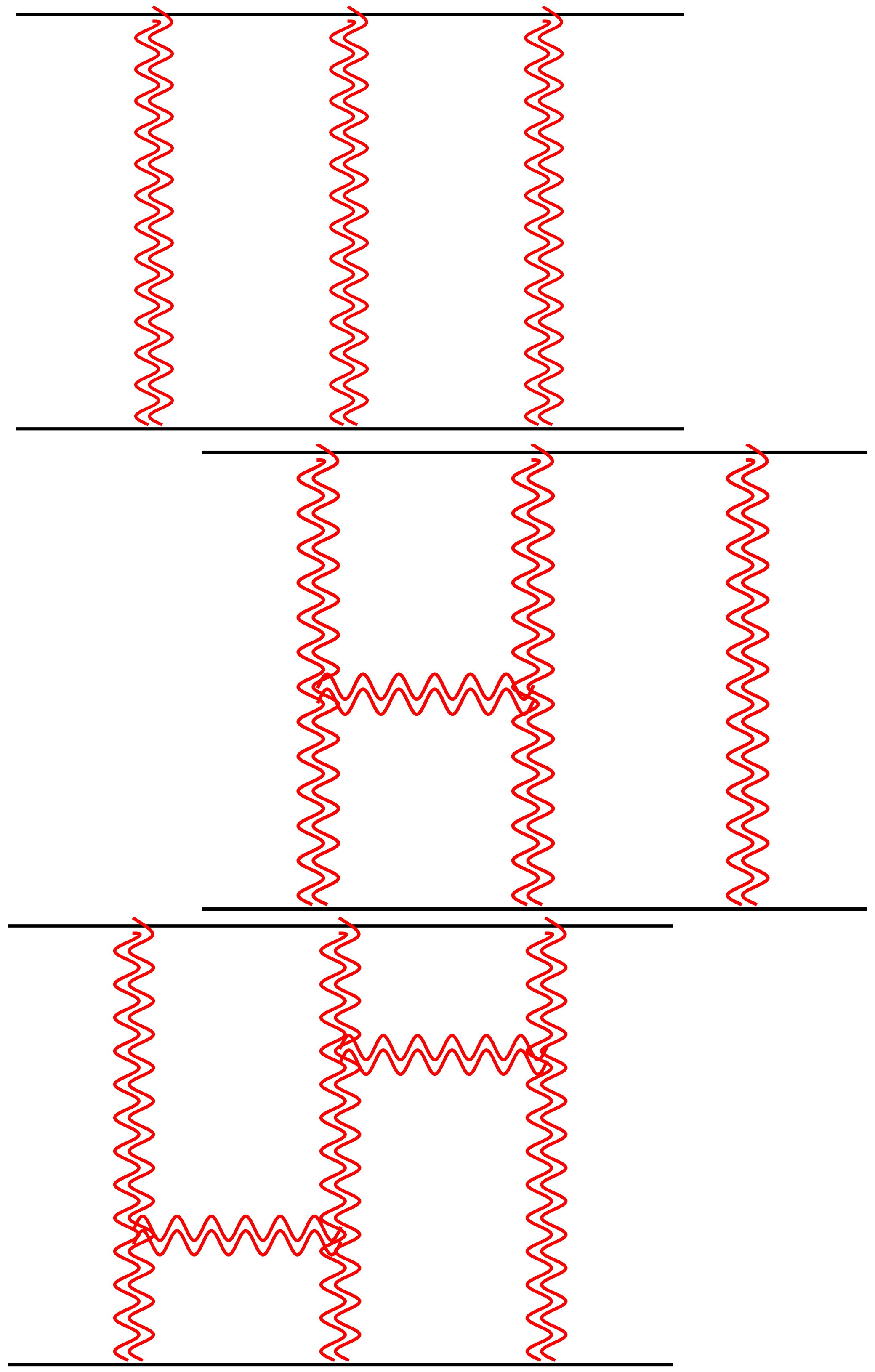


NNLL Real

- Regge factorisation breaks, need to add a term as a new channel opens up

$$\mathcal{M}^{\mathrm{NNLL},(-)} = C_i(t)C_j(t)e^{C_A\alpha_g(t,\mu^2)L}\mathcal{M}^{\mathrm{tree}} + \mathcal{M}^{\mathrm{NNLL},(-),\mathrm{NF}}$$

$\underline{\mathbf{8}_a}$ $\qquad$ $\underline{\mathbf{10} \oplus \overline{\mathbf{10}}}$



- The new term can be accessed by Balitsky-JIMWLK [Caron-Huot '13]

universal across gauge theories

- Computations up to four loops for any scattered partons

[Caron-Huot, Gardi, Vernazza '17; Falcioni, Gardi, **CM**, Vernazza '20]

- The four-loop is non-planar

- Is this a Regge pole and Regge cut separation?

Regge Cuts

[Falcioni, Gardi, Maher, **CM**, Vernazza '21, **PRL**]

- Let us see the NNLL amplitude at two loops

$$\mathcal{M}^{(-,2,0)} = \left(C_i^{(2)} + C_j^{(2)} + C_i^{(1)} C_j^{(1)}\right) \mathcal{M}^{\text{tree}} + \mathcal{M}^{(-,2,0),\text{NF}}$$

- Free to move terms that are in $\underline{\mathbf{8}_a}$ from the extra term into a new definition of the impact factors
- What criteria do we have? Regge cuts are non-planar [Mandelstam '63]
- Move all planar contributions into impact factors and Regge trajectory

$$\begin{aligned}\mathcal{M}^{\text{NNLL},(-)} &= \left(C_i(t)C_j(t)e^{C_A\alpha_g L}\mathcal{M}^{\text{tree}} + \mathcal{M}^{\text{NNLL},(-),\text{NF}}\big|_{\text{planar}}\right) + \mathcal{M}^{\text{NNLL},(-),\text{NF}}\big|_{\text{non-planar}} \\ &= \tilde{C}_i(t)\tilde{C}_j(t)e^{C_A\tilde{\alpha}_g L}\mathcal{M}^{\text{tree}} + \mathcal{M}^{\text{NNLL},(-),\text{NF}}\big|_{\text{non-planar}}\end{aligned}$$

- Gives a perturbative description of the Regge pole and Regge cut separation

Recent Progress

- Matching to fixed order computations we can find pole parameters at NNLL

  [Ahmed, Henn, Mistlberger ’19; Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi ’21]

- Three-loop Regge trajectory which has the expected infrared structure

  [Falcioni, Gardi, Maher, **CM**, Vernazza ’21, **PRL**; see also Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi ’21] [Korchemskaya, Korchemsky ’94, ’96]

- Two-loop impact factors at higher orders in ϵ

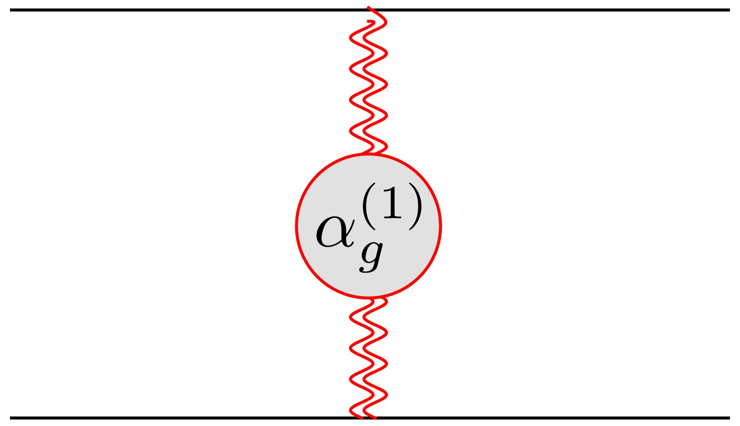
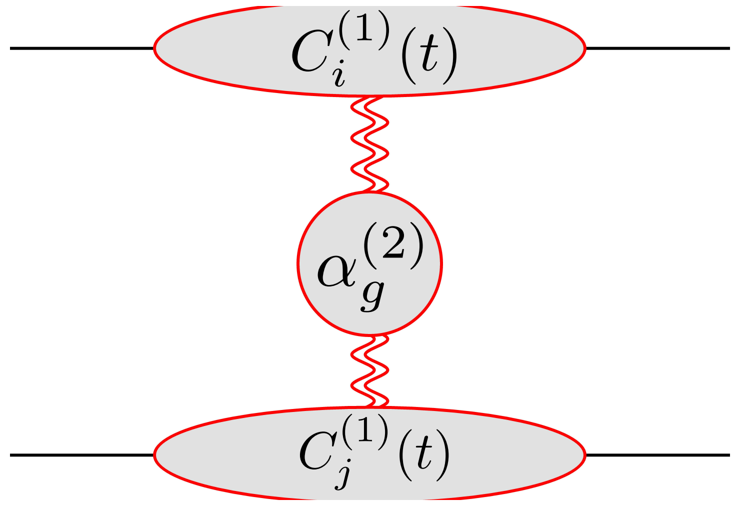
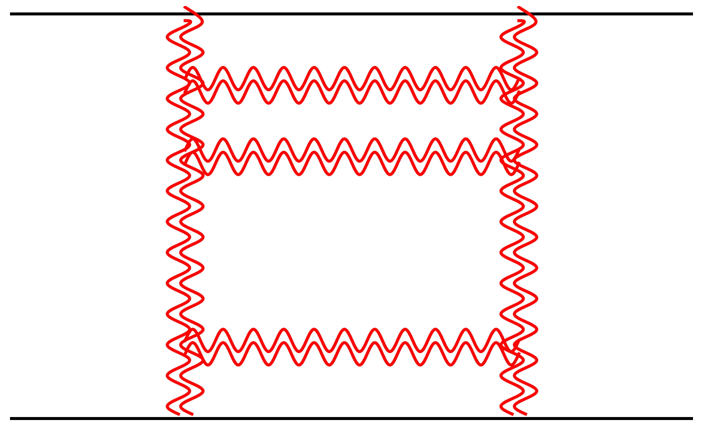
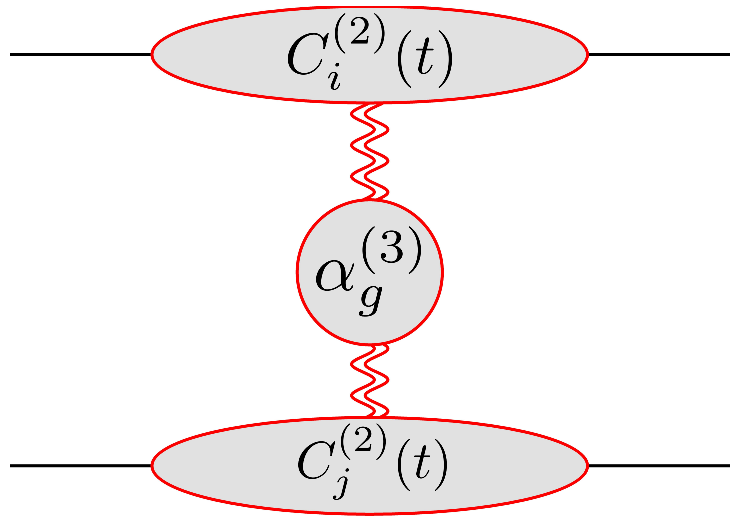
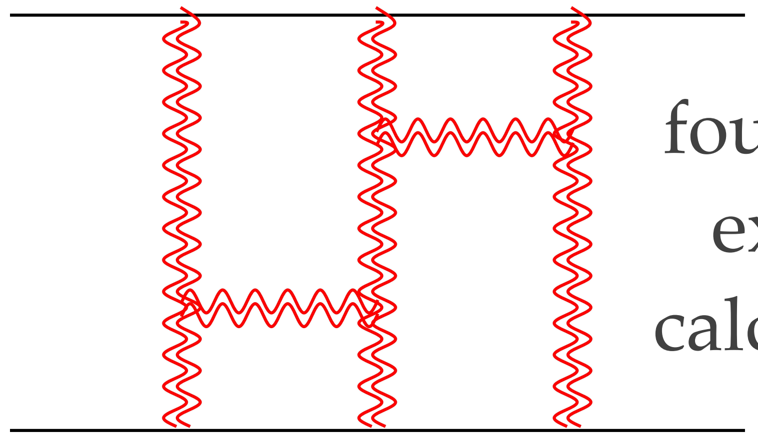
- High-energy limits constrain infrared singularities through four loops

  [Caron-Huot ’13; Falcioni, Gardi, Maher, **CM**, Vernazza ’21]

  - Useful in *bootstrapping* the full structure in general kinematics

    [Almelid, Duhr, Gardi, McLeod, White ’17]

Summary

	Real	Imaginary
LL $\alpha_s^n L^n$	 one-loop Regge trajectory	✗
NLL $\alpha_s^n L^{n-1}$	 two-loop Regge trajectory one-loop impact factors	 infrared divergences resummed finite terms to arbitrary order resummation unknown
NNLL $\alpha_s^n L^{n-2}$	  four loops explicit calculation	Not yet developed

Outlook

- Resum full imaginary amplitude at NLL
  - Interpretation in the complex angular momentum plane?
- Five-loop and beyond for the real NNLL
- Understand imaginary part at NNLL
- 2 -> 3 amplitudes
  - Significant interest for fixed order computations
  - How to organise? What can we extract? Phenomenological uses?

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Thank you

Backup slides

Formulating highly energetic partons as Wilson lines

[Caron-Huot '13; Caron-Huot, Gardi, Vernazza '17]

"eikonal approximation"

Fourier conjugate of t

$$U(z_\perp) = \mathbf{P}\exp\left[ig_s \mathbf{T}^a \int_{-\infty}^{\infty} dx^+ A_+^a(x^+, x^- = 0, z_\perp)\right]$$

regulate rapidity divergences by tilting Wilson-line off the light cone

$$\eta = \frac{1}{2}\log\frac{p_+}{p_-}$$

$$T^a_{i,L} = [\mathbf{T}^a U(z_i)]\frac{\delta}{\delta U(z_i)}$$

$$T^a_{i,R} = [U(z_i)\mathbf{T}^a]\frac{\delta}{\delta U(z_i)}$$

our parton is a collection of such Wilson lines

evolves according to Balitsky-JIMWLK

$$\frac{d}{d\eta}|\psi_i\rangle = -H|\psi_i\rangle \qquad H = \frac{\alpha_s}{2\pi^2}\int d^d z_i d^d z_j d^d z_0 \frac{z_{0i}\cdot z_{0j}}{(z_{0i}^2 z_{0j}^2)^{1-\epsilon}}\left\{T^a_{i,L}T^a_{j,L} + T^a_{i,R}T^a_{j,R} - U^{ab}_{\text{ad}}(z_0)\left(T^a_{i,L}T^b_{j,R} + T^a_{j,L}T^b_{i,R}\right)\right\}$$

Target | Projectile

The amplitude is then written as $\mathcal{M}_{ij\to ij} \sim \langle\psi_j|e^{-HL}|\psi_i\rangle$

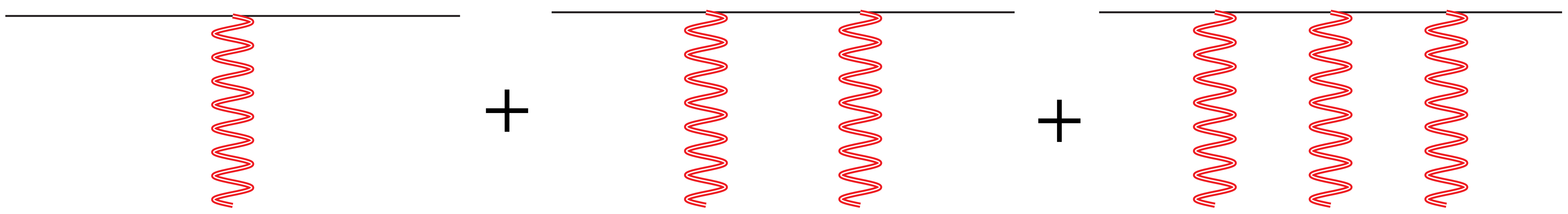
Evolve so that they are at equal rapidity

Target and projectile separated by some $z_\perp^2 \leftrightarrow t$

Reggeon field

Expand Wilson line in *Reggeons*

$$U = \exp\left[ig_s \mathbf{T}^a W^a\right] \sim$$



Only odd/even number of Reggeons contribute to the odd/even amplitude

Explicit momentum-space Hamiltonians

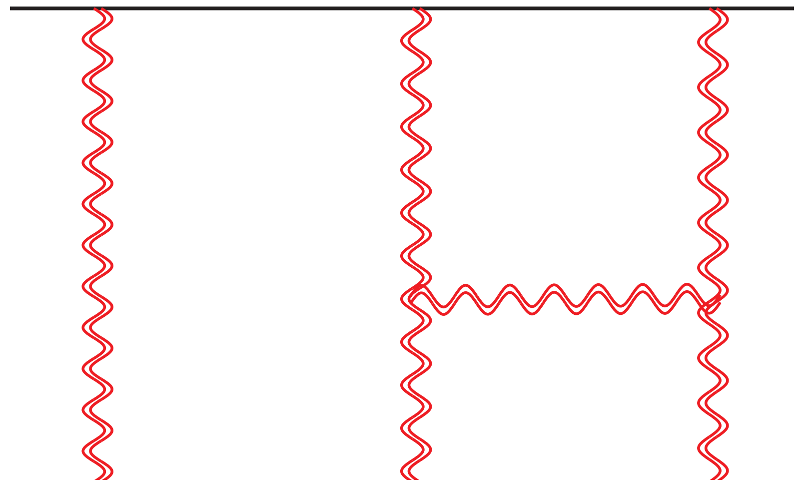
The diagonal transitions are

dresses one Reggeon with the trajectory

$$H_{k\to k} = -\int d^d p\, C_A \alpha_g(p) W^a(p) \frac{\delta}{\delta W^a(p)}$$

$$+\,\alpha_s \int d^d q d^d p_1 d^d p_2\, H_{22}(q;p_1,p_2) W^x(p_1+q) W^y(p_2-q) (F^x F^y)^{ab} \frac{\delta}{\delta W^a(p_1)} \frac{\delta}{\delta W^b(p_1)}$$

adds a *rung* between two Reggeons



with kernel $H_{22}(q;p_1,p_2) = \frac{(p_1+p_2)^2}{p_1^2 p_2^2} - \frac{(p_1+q)^2}{p_1^2 q^2} - \frac{(p_2-q)^2}{q^2 p_2^2}$

Source of the difficulty at NNLL. Three Reggeons spoil the symmetry between colour and kinematics, which is there for two Reggeons (NLL).

The off-diagonal transitions are

$$H_{1\to 3} = \alpha_s^2 \int d^d p_1 d^d p_2 d^d p\ \mathrm{tr}\left[F^a F^b F^c F^d\right] W^b(p_1) W^c(p_2) W^d(p_3) H_{13}(p_1,p_2,p_3) \frac{\delta}{\delta W^a(p)}$$

with kernel $H_{13}(p_1,p_2,p_3) = \frac{r_\Gamma}{3\epsilon}\left[\left(\frac{\mu^2}{(p_1+p_2+p_3)^2}\right)^\epsilon + \left(\frac{\mu^2}{p_2^2}\right)^\epsilon - \left(\frac{\mu^2}{(p_1+p_2)^2}\right)^\epsilon - \left(\frac{\mu^2}{(p_2+p_3)^2}\right)^\epsilon\right]$