

NLO computation of diffractive di-hadron production in the shockwave framework

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Diffraction and low-x
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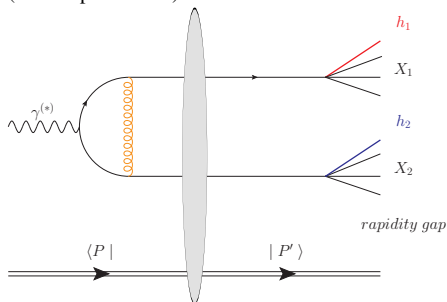
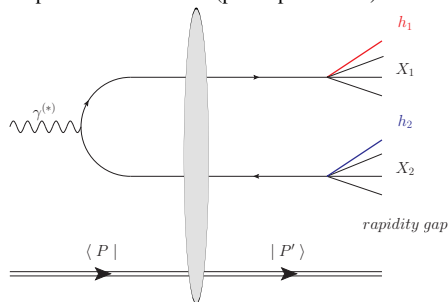
The process of interest

The process studied at NLO :

$$\gamma^{(*)}(p_\gamma) + P(p_0) \rightarrow h_1(p_{h1}) + h_2(p_{h2}) + X + P(p'_0) \quad (X = X_1 + X_2)$$

Rapidity gap between $(h_1 h_2 X)$ and $P'(p'_0)$.

The photon could be real (photo-production) or virtual (electro-production).



Parametrization of the matrix element of the dipole operator:

$$\left\langle P' (p'_0) \left| T \left(\text{Tr} \left(U_{\frac{z_\perp}{2}} U_{-\frac{z_\perp}{2}}^\dagger \right) - N_c \right) \right| P (p_0) \right\rangle \equiv 2\pi \delta(p_{00'}) F(z_\perp).$$

Its Fourier transform is

$$\int d^d z_\perp e^{i(z_\perp \cdot p_\perp)} F(z_\perp) \equiv \mathbf{F}(p_\perp).$$

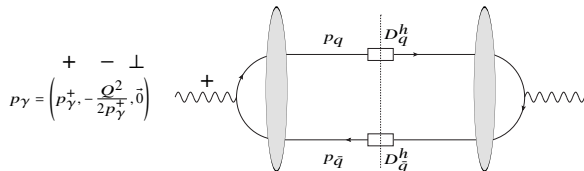
Hybrid factorization:

- Collinear factorization: Hard scale with $\Lambda_{QCD}^2 \ll \vec{p}_{h_1}^2 \sim \vec{p}_{h_2}^2$.
Constraint $\vec{p}^2 \gg \vec{p}_{h_{1,2}}^2$ with \vec{p} , the relative transverse momentum of the two hadrons.
 \Rightarrow Use of one hadron fragmentation function (FF) only to describe hadronization.
- High-energy k_t -factorization with the shockwave formalism.
Partonic process at LO and NLO has been calculated in [arXiv:1606.00419](https://arxiv.org/abs/1606.00419) [[hep-ph](#)].
- We also need $\vec{p}_{h_1}^2 \sim \vec{p}_{h_2}^2 < Q_s^2$ to be in the saturation region.

LO cross-section

Sudakov decomposition : $p_i^\mu = x_i p_\gamma^+ n_1^\mu + \frac{\vec{p}_\perp^2}{2x_i p_\gamma^+} n_2^\mu + p_\perp^\mu$.

n_1^μ, n_2^μ are light-cone vectors: $n_1^2 = n_2^2 = 0$.

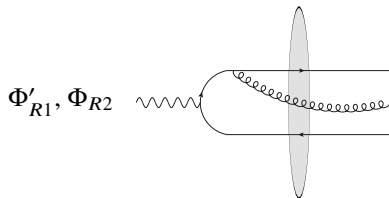
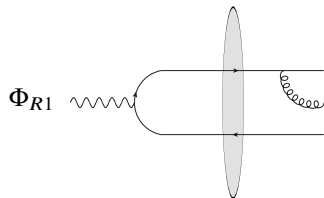
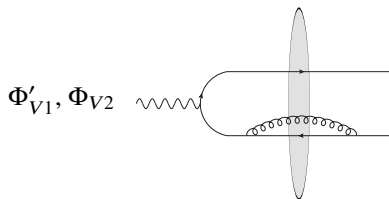
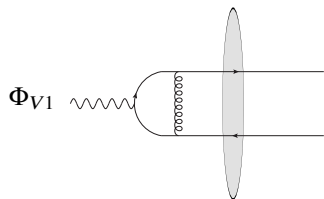


Using collinearity $(p_q^+, \vec{p}_q) = (x_q/x_{h_1})(p_{h_1}^+, \vec{p}_{h_1})$ and $(p_{\bar{q}}^+, \vec{p}_{\bar{q}}) = (x_{\bar{q}}/x_{h_2})(p_{h_2}^+, \vec{p}_{h_2})$:

$$\frac{d\sigma_{0JI}^{h_1 h_2}}{dx_{h_1} dx_{h_2} d^d p_{h_1} d^d p_{h_2}} = \sum_q \int_{x_{h_1}}^1 \frac{dx_q}{x_q} \int_{x_{h_2}}^1 \frac{dx_{\bar{q}}}{x_{\bar{q}}} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d D_q^{h_1} \left(\frac{x_{h_1}}{x_q}\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}\right) \frac{d\hat{\sigma}_{JI}}{dx_q dx_{\bar{q}} d^d p_q d^d p_{\bar{q}}} + (h_1 \leftrightarrow h_2)$$

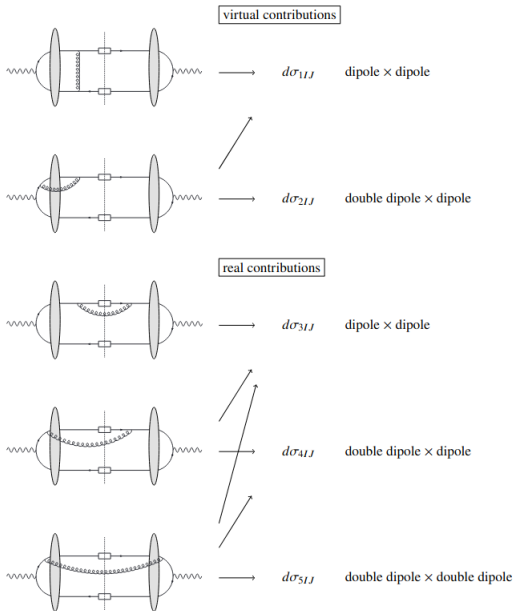
J, I labels the photon polarization for respectively the complex conjugated amplitude and the amplitude.

NLO impact factor before fragmentation



Rapidity divergence $x_g \rightarrow 0$ in Φ_{V2} . Removed with $\Phi_0 \otimes \mathcal{K}_{B-JIMWLK}$
 $\Rightarrow \check{\Phi}_{V2} = \Phi_{V2} - \Phi_0 \otimes \mathcal{K}_{B-JIMWLK}$ is finite

NLO cross-section in a nutshell and divergences



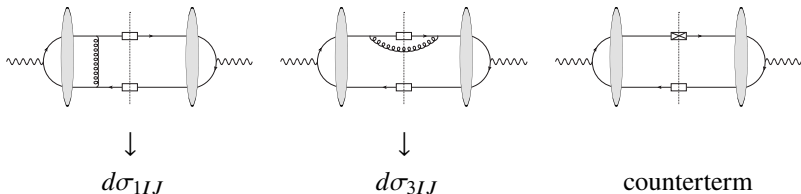
NLO cross-section in a nutshell and divergences

IR divergences to deal with:

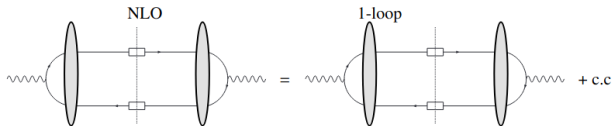
- Collinear divergences $\vec{p}_g \propto \vec{p}_q$ or $\vec{p}_g \propto \vec{p}_{\bar{q}}$
- Soft divergences where $x_g \rightarrow 0$ and $p_{g\perp} = x_g u_\perp \sim x_g \rightarrow 0$ where u_\perp of order p_T .

Regularization with dimensional regularization $D = 2 + d = 4 + 2\epsilon$ and longitudinal cut-off $|x_g| > \alpha$.

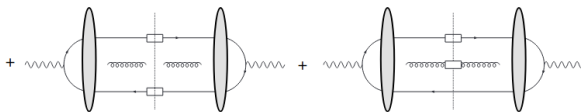
- We prove the cancellation of divergences between the divergent part of $d\sigma_{3JJ}$, counterterms from FF renormalization and $d\sigma_{1JJ}$.
- The finite terms are extracted.



NLO cross-section in a nutshell and divergences

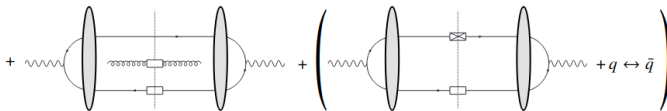


(a) : soft + collinear



(b) : soft + collinear

(c) : collinear



(d) : collinear

(e) : collinear from counterterm

Counterterm from renormalization and evolution equation of FF : diagram (e)

$$\begin{aligned}
 & \text{bare} \qquad \qquad \text{dressed} \\
 D_q^{h_1} \left(\frac{x_{h_1}}{x_q} \right) &= D_q^{h_1} \left(\frac{x_{h_1}}{x_q}, \mu_F \right) - \frac{\alpha_s}{2\pi} \left(\frac{1}{\hat{\epsilon}} + \ln \frac{\mu_F^2}{\mu^2} \right) \int_{\frac{x_{h_1}}{x_q}}^1 \frac{d\beta_1}{\beta_1} \\
 & \times \left[P_{qq}(\beta_1) D_q^{h_1} \left(\frac{x_{h_1}}{x_q \beta_1}, \mu_F \right) + P_{gq}(\beta_1) D_g^{h_1} \left(\frac{x_{h_1}}{\beta_1 x_q}, \mu_F \right) \right]
 \end{aligned}$$

with the usual DGLAP splitting functions

$$P_{qq}(\beta_1) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right]$$

$$P_{gq}(\beta_1) = C_F \frac{1+(1-z)^2}{z}$$

+ prescription: $\int_a^1 d\beta \frac{F(\beta)}{(1-\beta)_+} = \int_a^1 d\beta \frac{F(\beta)-F(1)}{1-\beta} - \int_0^a d\beta \frac{F(1)}{1-\beta}$

$$\frac{1}{\hat{\epsilon}} = \frac{\Gamma(1-\epsilon)}{(4\pi)^\epsilon \epsilon} \sim \frac{1}{\epsilon} + \gamma_E - \ln(4\pi)$$

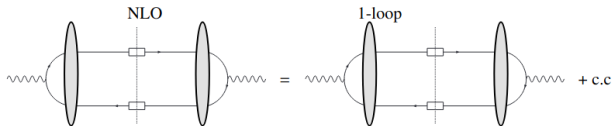
Counterterm cross-section for LL

$$\begin{aligned}
 \frac{d\sigma_{LL}^{h_1 h_2}}{dx_{h_1} dx_{h_2} d^d p_{h_1\perp} d^d p_{h_2\perp}} \Big|_{\text{ct}} &= \frac{4\alpha_{\text{em}} Q^2}{(2\pi)^{4(d-1)} N_c} \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}} x_q x_{\bar{q}} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d \delta(1-x_q-x_{\bar{q}}) \\
 &\times \mathcal{F}_{LL} \left(-\frac{\alpha_s}{2\pi}\right) \left(\frac{1}{\hat{\epsilon}} + \ln \frac{\mu_F^2}{\mu^2}\right) \sum_q Q_q^2 \left\{ \int_{\frac{x_{h_1}}{x_q}}^1 \frac{d\beta_1}{\beta_1} \left[C_F \frac{1+\beta_1^2}{(1-\beta_1)_+} D_q^{h_1} \left(\frac{x_{h_1}}{\beta_1 x_q}, \mu_F\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right) \right. \right. \\
 &+ P_{gq}(\beta_1) D_g^{h_1} \left(\frac{x_{h_1}}{\beta_1 x_q}, \mu_F\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right) \left. \right] + \int_{\frac{x_{h_2}}{x_{\bar{q}}}}^1 \frac{d\beta_2}{\beta_2} \left[P_{gq}(\beta_2) D_q^{h_1} \left(\frac{x_{h_1}}{x_q}, \mu_F\right) D_g^{h_2} \left(\frac{x_{h_2}}{\beta_2 x_{\bar{q}}}, \mu_F\right) \right. \right. \\
 &+ C_F \frac{1+\beta_2^2}{(1-\beta_2)_+} D_q^{h_1} \left(\frac{x_{h_1}}{x_q}, \mu_F\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{\beta_2 x_{\bar{q}}}, \mu_F\right) \left. \right] + 3C_F D_q^{h_1} \left(\frac{x_{h_1}}{x_q}, \mu_F\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right) \left. \right\} \\
 &+ (h_1 \leftrightarrow h_2).
 \end{aligned}$$

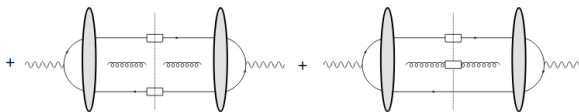
\mathcal{F}_{LL} contains the non-perturbative part: matrix elements of the dipole operators on the target states

$$\mathcal{F}_{LL} = \left| \int d^d p_{2\perp} \frac{\mathbf{F} \left(\frac{x_q}{2x_{h_1}} p_{h_1\perp} + \frac{x_{\bar{q}}}{2x_{h_2}} p_{h_2\perp} - p_{2\perp} \right)}{\left(\frac{x_{\bar{q}}}{x_{h_2}} \vec{p}_{h_2} - \vec{p}_2 \right)^2 + x_q x_{\bar{q}} Q^2} \right|^2.$$

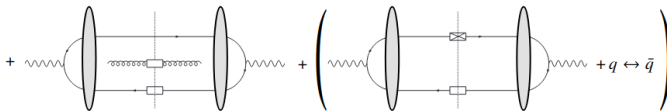
Diagrams (b), (c), (d)



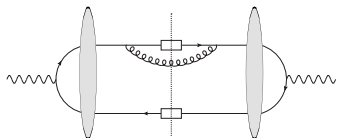
(a) : soft + collinear



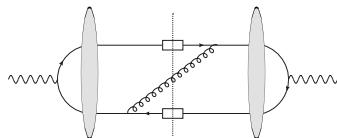
(c) : collinear



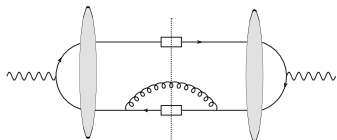
Divergent diagrams in diagram (b), (c), (d)



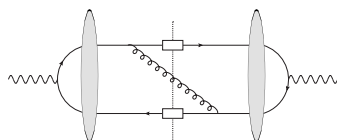
(1) : soft + collinear (qg)



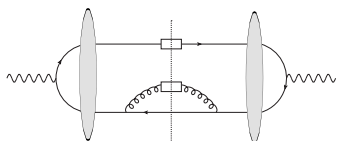
(2) : soft



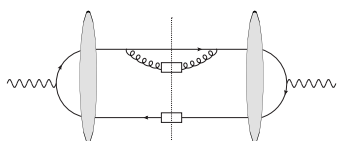
(3) : soft + collinear ($\bar{q}g$)



(4) : soft



(5) : collinear ($\bar{q}g$)



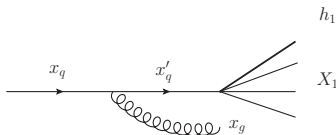
(6) : collinear (qg)

Important points for the calculation of collinear divergences

- Change variables to have longitudinal momentum fraction expressed wrt to the parent parton rather than the photon.
This is to be able to compare to the counterterm.

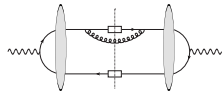
Example: To extract the collinear divergences from qg splitting, do:

$$x'_q = \beta_1 x_q,$$
$$x_g = (1 - \beta_1)x_q.$$



- To disentangle the transverse momentum of the spectator parton and be able to integrate over it without touching the non-perturbative part
⇒ Fourier transform of the matrix element of the dipole operator.

Specific case : diagram (1)



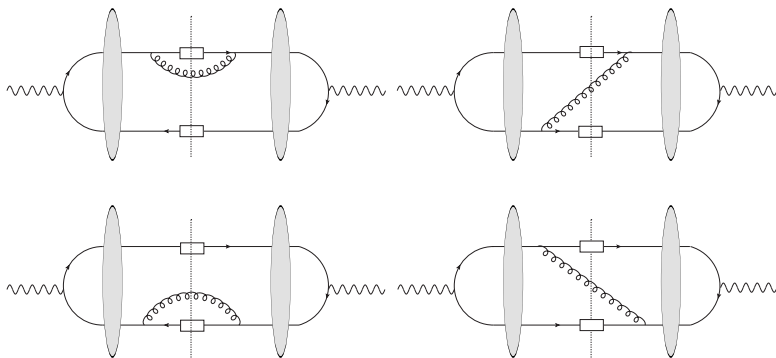
$$\begin{aligned}
 & \frac{d\sigma^{q\bar{q} \rightarrow h_1 h_1}}{dx_{h_1} dh_2 d^d p_{h_1} d^d p_{h_2}} \Big|_{\text{coll qg}} \\
 &= \frac{4\alpha_{\text{em}} Q^2}{(2\pi)^{4(d-1)} N_c} \sum_q \int_{x_{h_1}}^1 \frac{dx'_q}{x'_q} \int_\alpha^1 \frac{dx_g}{x_g} \int_{x_{h_2}}^1 \frac{dx_{\bar{q}'}}{x_{\bar{q}'}} \delta(1 - x'_q - x_{\bar{q}'} - x_g) \\
 &\times \left(\frac{x'_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}'}}{x_{h_2}}\right)^d Q_q^2 D_q^{h_1} \left(\frac{x_{h_1}}{x'_q}, \mu_F\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}'}} , \mu_F\right) \frac{\alpha_s}{\mu^{2\epsilon}} C_F \frac{d^d p_{g\perp}}{(2\pi)^d} \\
 &\times \int d^d p_{1\perp} d^d p_{2\perp} \delta\left(\frac{x'_q}{x_{h_1}} p_{h_1\perp} - p_{1\perp} + \frac{x_{\bar{q}'}}{x_{h_2}} p_{h_2\perp} - p_{2\perp} + p_{g\perp}\right) F\left(\frac{p_{12\perp}}{2}\right) \\
 &\times \int d^d p_{1'\perp} d^d p_{2'\perp} \delta\left(\frac{x'_q}{x_{h_1}} p_{h_1\perp} - p_{1'\perp} + \frac{x_{\bar{q}'}}{x_{h_2}} p_{h_2\perp} - p_{2'\perp} + p_{g\perp}\right) F^*\left(\frac{p_{1'2'\perp}}{2}\right) \\
 &\times \frac{(dx_g^2 + 4x'_q(x'_q + x_g))x_{\bar{q}'}^2(1 - x_{\bar{q}'})^2}{\left(x'_{\bar{q}}(1 - x'_{\bar{q}})Q^2 + \left(\frac{x'_{\bar{q}}}{x_{h_2}} \vec{p}_{h_2} - \vec{p}_2\right)^2\right) \left(x'_{\bar{q}}(1 - x'_{\bar{q}})Q^2 + \left(\frac{x'_{\bar{q}}}{x_{h_2}} \vec{p}_{h_2} - \vec{p}_{2'}\right)^2\right)} \left(x'_q \vec{p}_g - x_g \frac{x'_q}{x_{h_1}} \vec{p}_{h_1}\right)^2 \\
 &+ (h_1 \leftrightarrow h_2)
 \end{aligned}$$

Results for diagram (1)

$$\begin{aligned}
 & \left. \frac{d\sigma_{3LL}^{q\bar{q} \rightarrow h_1 h_2}}{dx_{h_1} dx_{h_2} dp_{h_1\perp} d^d p_{h_2\perp}} \right|_{\text{coll. qg div}} \\
 &= \frac{4\alpha_{\text{em}} Q^2}{(2\pi)^{4(d-1)} N_c} \sum_q \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}} x_q x_{\bar{q}} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d \delta(1-x_q-x_{\bar{q}}) \\
 &\times \int d^d p_{2\perp} \int d^d z_{1\perp} \frac{e^{iz_{1\perp} \cdot \left(\frac{x_q}{2x_{h_1}} p_{h_1\perp} + \frac{x_{\bar{q}}}{2x_{h_2}} p_{h_2\perp} - p_{2\perp}\right)} F(z_{1\perp})}{x_q x_{\bar{q}} Q^2 + \left(\frac{x_{\bar{q}}}{x_{h_2}} \vec{p}_{h_2} - \vec{p}_2\right)^2} \\
 &\times \int d^d p_{2'\perp} \int d^d z_{2\perp} \frac{e^{-iz_{2\perp} \cdot \left(\frac{x_q}{2x_{h_1}} p_{h_1\perp} + \frac{x_{\bar{q}}}{2x_{h_2}} p_{h_2\perp} - p_{2'\perp}\right)} F^*(z_{2\perp})}{x_q x_{\bar{q}} Q^2 + \left(\frac{x_{\bar{q}}}{x_{h_2}} \vec{p}_{h_2} - \vec{p}_{2'}\right)^2} \\
 &\times \frac{\alpha_s}{2\pi} \frac{1}{\hat{\epsilon}} Q_q^2 \left[\int_{\frac{x_{h_1}}{x_q}}^1 \frac{d\beta_1}{\beta_1} C_F \frac{1+\beta_1^2}{(1-\beta_1)_+} D_q^{h_1} \left(\frac{x_{h_1}}{\beta_1 x_q}, \mu_F\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right) \right. \\
 &+ \int_{\frac{x_{h_1}}{x_q}}^{1-\frac{\alpha}{x_{\bar{q}}}} d\beta_1 C_F \frac{2}{1-\beta_1} \left(\frac{c_0^2}{\left(\frac{z_{1\perp} - z_{2\perp}}{2}\right)^2 \mu^2} \right)^\epsilon D_q^{h_1} \left(\frac{x_{h_1}}{x_q}, \mu_F\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right) \\
 &\left. - 2C_F \ln \left(1 - \frac{x_{h_1}}{x_q}\right) D_q^{h_1} \left(\frac{x_{h_1}}{x_q}, \mu_F\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right) \right] + (h_1 \leftrightarrow h_2).
 \end{aligned}$$

Cancellation with counterterm. The **second term** is to be removed: double-counting with soft contribution. The **third term**, from the introduction of the + prescription, will be removed with a term in the soft contribution.

Soft limit



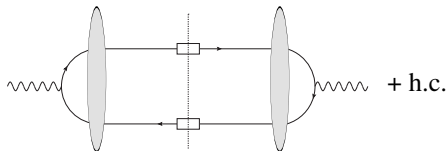
Rescaling $\vec{p}_g = x_g \vec{u}$ to isolate the divergences in the form $\int_\alpha^1 \frac{dx_g}{x_g^{3-d}}$ and putting $x_g \rightarrow 0$ in the rest of the integrand.

Soft divergence

$$\begin{aligned}
 \left. \frac{d\sigma_{3LL}}{dx_{h_1} dh_2 d^d p_{h_1\perp} dp_{h_2\perp}} \right|_{\text{soft div}} &= \frac{4\alpha_{\text{em}} Q^2}{(2\pi)^{4(d-1)} N_c} \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}} x_q x_{\bar{q}} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d \\
 &\times \delta(1 - x_q - x_{\bar{q}}) \sum_q Q_q^2 D_q^{h_1} \left(\frac{x_{h_1}}{x_q}, \mu_F\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right) \mathcal{F}_{LL} \\
 &\times \frac{\alpha_s C_F}{2\pi} \frac{1}{\hat{\epsilon}} \left[-4 \ln \alpha + 2 \ln x_q + 2 \ln \left(1 - \frac{x_{h_1}}{x_q}\right) - 4\epsilon \ln^2 \alpha \right. \\
 &\left. - 4\epsilon \ln \alpha \ln \left(\frac{\left(\frac{\vec{p}_{h_1}}{x_{h_1}} - \frac{\vec{p}_{h_2}}{x_{h_2}}\right)^2}{\mu^2} \right) + 2 \ln x_{\bar{q}} + 2 \ln \left(1 - \frac{x_{h_2}}{x_{\bar{q}}}\right) \right]
 \end{aligned}$$

Cancellation with the residual divergence from the collinear term.

Virtual corrections



1-loop

Cancellation between **virtual corrections** and **soft** and with **counterterm**.

$$\begin{aligned}
 \left. \frac{d\sigma_{LL}^{h_1 h_2}}{dx_{h_1} dh_2 d^d p_{h_1 \perp} d^d p_{h_2 \perp}} \right|_{\text{virtual div}} &= \frac{4\alpha_{\text{em}} Q^2}{(2\pi)^{4(d-1)} N_c} \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}} x_q x_{\bar{q}} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d \\
 &\times \sum_q Q_q^2 D_q^{h_1} \left(\frac{x_{h_1}}{x_q}, \mu_F\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right) \mathcal{F}_{LL} \\
 &\times \frac{\alpha_s}{2\pi} C_F \frac{1}{\hat{\epsilon}} \left[-4\epsilon \ln(\alpha) \ln \left(\frac{\mu^2}{\left(\frac{\vec{p}_{h_2}}{x_{h_2}} - \frac{\vec{p}_{h_1}}{x_{h_1}}\right)^2} \right) + 4 \ln(\alpha) \right. \\
 &\left. + 4\epsilon \ln^2(\alpha) - 2 \ln(x_q x_{\bar{q}}) + 3 \right]
 \end{aligned}$$

Conclusions

- Computation of the NLO cross-section to the semi-inclusive diffractive production of a pair of hadrons with large p_T , such that $\Lambda_{QCD}^2 \ll p_T^2$.
- Saturation window: $p_T^2 < Q_s^2$
- Process can be either a photo-production or electro-production. The results are applicable to ultra-peripheral collisions at the LHC (especially at the LHCb) or at the EIC.
- Full cancellation of divergences has been observed between real corrections, virtual ones and counterterm from FF renormalization.
- Expressions of the detailed finite cross-sections will be found in the to-be-published paper, in general kinematics (Q^2, t) .
- The phenomenological application of the NLO computation will take time due to the complexity of the analytical results.

Thank you for your attention!

$$\frac{d\sigma_{0LL}^{h_1 h_2}}{dx_{h_1} dx_{h_2} d^d p_{h_1\perp} d^d p_{h_2\perp}} = \frac{4\alpha_{\text{em}} Q^2}{(2\pi)^{4(d-1)} N_c} \sum_q \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}} x_q x_{\bar{q}} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d$$

$$\times \delta(1 - x_q - x_{\bar{q}}) Q_q^2 D_q^{h_1} \left(\frac{x_{h_1}}{x_q}\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}\right) \mathcal{F}_{LL} + (h_1 \leftrightarrow h_2),$$

where

$$\mathcal{F}_{LL} = \left| \int d^d p_{2\perp} \frac{\mathbf{F} \left(\frac{x_q}{2x_{h_1}} p_{h_1\perp} + \frac{x_{\bar{q}}}{2x_{h_2}} p_{h_2\perp} - p_{2\perp} \right)}{\left(\frac{x_{\bar{q}}}{x_{h_2}} \vec{p}_{h_2} - \vec{p}_2 \right)^2 + x_q x_{\bar{q}} Q^2} \right|^2.$$

Another equivalent representation exists for the LO cross-section with, instead of \mathcal{F}_{LL} , we use :

$$\tilde{\mathcal{F}}_{LL} = \left| \int d^d p_{1\perp} \frac{\mathbf{F} \left(- \left(\frac{x_q}{2x_{h_1}} p_{h_1\perp} + \frac{x_{\bar{q}}}{2x_{h_2}} p_{h_2\perp} - p_{1\perp} \right) \right)}{\left(\frac{x_q}{x_{h_1}} \vec{p}_{h_1} - \vec{p}_1 \right)^2 + x_q x_{\bar{q}} Q^2} \right|^2.$$

The rest stays the same, apart from this change. This other representation is used when studying divergences from the \bar{q} FF renormalisation and the collinear $\bar{q}g$ divergence.

Definition of the non-perturbative functions

- Definition of the matrix element of the dipole operator

$$\left\langle P' (p'_0) \left| T \left(\text{Tr} \left(U_{\frac{z_\perp}{2}} U_{-\frac{z_\perp}{2}}^\dagger \right) - N_c \right) \right| P (p_0) \right\rangle \equiv 2\pi \delta (p_{00}^-) F (z_\perp)$$

Its Fourier transform is

$$\int d^d z_\perp e^{i(z_\perp \cdot p_\perp)} F (z_\perp) \equiv \mathbf{F} (p_\perp).$$

- Definition of the matrix element of the double dipole operator

$$\begin{aligned} & \left\langle P' (p'_0) \left| \left(\text{tr} \left(U_{\frac{z_\perp}{2}} U_x^\dagger \right) \text{tr} \left(U_x U_{-\frac{z_\perp}{2}}^\dagger \right) - N_c \text{tr} \left(U_{\frac{z_\perp}{2}} U_{-\frac{z_\perp}{2}}^\dagger \right) \right) \right| P (p_0) \right\rangle \\ & \equiv 2\pi \delta (p_{00}^-) \tilde{F} (z_\perp, x_\perp) \end{aligned}$$

with the following Fourier transform

$$\int d^d z_\perp d^d x_\perp e^{i(p_\perp \cdot x_\perp) + i(z_\perp \cdot q_\perp)} \tilde{F} (z_\perp, x_\perp) \equiv \tilde{\mathbf{F}} (q_\perp, p_\perp).$$

The divergent partonic LL cross-section

$$\begin{aligned}
 d\hat{\sigma}_{3LL}|_{div} &= \frac{4\alpha_{em}Q^2}{(2\pi)^{4(d-1)}N_c} Q_q^2 dx'_q dx'_q \delta(1-x'_q-x'_q-x_g) d^d p_{q\perp} d^d p_{\bar{q}\perp} \frac{\alpha_s C_F}{\mu^2\epsilon} \frac{dx_g}{x_g} \frac{d^d p_{g\perp}}{(2\pi)^d} \\
 &\times \int d^d p_{1\perp} d^d p_{2\perp} \delta(p_{q1\perp} + p_{\bar{q}2\perp} + p_{g\perp}) F\left(\frac{P_{12\perp}}{2}\right) \\
 &\times \int d^d p_{1'\perp} d^d p_{2'\perp} \delta(p_{q1'\perp} + p_{\bar{q}2'\perp} + p_{g\perp}) F^*\left(\frac{P_{1'2'\perp}}{2}\right) \\
 &\times \left\{ \frac{(dx_g^2 + 4x'_q(x'_q + x_g))}{\left(Q^2 + \frac{\vec{p}_{\bar{q}2}^2}{x'_q(1-x'_q)}\right) \left(Q^2 + \frac{\vec{p}_{\bar{q}2'}^2}{x'_q(1-x'_q)}\right) (x'_q \vec{p}_g - x_g \vec{p}_q)^2} \right. \\
 &\quad \frac{(2x_g - dx_g^2 + 4x'_q x'_q) (x'_q \vec{p}_g - x_g \vec{p}_q) \cdot (x'_q \vec{p}_g - x_g \vec{p}_q)}{\left(Q^2 + \frac{\vec{p}_{\bar{q}2'}^2}{x'_q(1-x'_q)}\right) \left(Q^2 + \frac{\vec{p}_{q1}^2}{x'_q(1-x'_q)}\right) (x'_q \vec{p}_g - x_g \vec{p}_q)^2 (x'_q \vec{p}_g - x_g \vec{p}_{\bar{q}})^2} \\
 &\quad + \frac{(dx_g^2 + 4x'_q(x'_q + x_g))}{\left(Q^2 + \frac{\vec{p}_{q1}^2}{x'_q(1-x'_q)}\right) \left(Q^2 + \frac{\vec{p}_{q1'}^2}{x'_q(1-x'_q)}\right) (x'_q \vec{p}_g - x_g \vec{p}_{\bar{q}})^2} \\
 &\quad \left. - \frac{(2x_g - dx_g^2 + 4x'_q x'_q) (x'_q \vec{p}_g - x_g \vec{p}_q) \cdot (x'_q \vec{p}_g - x_g \vec{p}_{\bar{q}})}{\left(Q^2 + \frac{\vec{p}_{q1'}^2}{x'_q(1-x'_q)}\right) \left(Q^2 + \frac{\vec{p}_{\bar{q}2}^2}{x'_q(1-x'_q)}\right) (x'_q \vec{p}_g - x_g \vec{p}_q)^2 (x'_q \vec{p}_g - x_g \vec{p}_{\bar{q}})^2} \right\}
 \end{aligned}$$

The first and third terms are associated with collinear divergences. All of them contribute to the soft divergence.

Finite term from diagram (1)

$$\begin{aligned}
 & \frac{d\sigma_{3LL}^{q\bar{q} \rightarrow h_1 h_2}}{dx_{h_1} dx_{h_2} dp_{h_1\perp} d^d p_{h_2\perp}} \Big|_{\text{coll. qg fin}} \\
 &= \frac{4\alpha_{\text{em}} Q^2}{(2\pi)^{4(d-1)} N_c} \sum_q \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}} x_q x_{\bar{q}} \delta(1-x_q-x_{\bar{q}}) \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d \\
 &\times \int d^d p_{2\perp} \int d^d z_{1\perp} \frac{e^{i z_{1\perp} \cdot \left(\frac{x_q}{2x_{h_1}} p_{h_1\perp} + \frac{x_{\bar{q}}}{2x_{h_2}} p_{h_2\perp} - p_{2\perp}\right)}}{x_q x_{\bar{q}} Q^2 + \left(\frac{x_{\bar{q}}}{x_{h_2}} \vec{p}_{h_2} - \vec{p}_{2\perp}\right)^2} F(z_{1\perp}) \\
 &\times \int d^d p_{2'\perp} \int d^d z_{2\perp} \frac{e^{-i z_{2\perp} \cdot \left(\frac{x_q}{2x_{h_1}} p_{h_1\perp} + \frac{x_{\bar{q}}}{2x_{h_2}} p_{h_2\perp} - p_{2'\perp}\right)}}{x_q x_{\bar{q}} Q^2 + \left(\frac{x_{\bar{q}}}{x_{h_2}} \vec{p}_{h_2} - \vec{p}_{2'\perp}\right)^2} F^*(z_{2\perp}) \\
 &\times \frac{\alpha_s C_F}{2\pi} \left\{ \int_{\frac{x_{h_1}}{x_q}}^1 \frac{d\beta_1}{\beta_1} Q^2 D_q^{h_1} \left(\frac{x_{h_1}}{\beta_1 x_q}, \mu_F\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right) \right. \\
 &\times \left[\ln \left(\frac{c_0^2}{\left(\frac{z_{1\perp} - z_{2\perp}}{2}\right)^2 \mu^2} \right) \frac{1 + \beta_1^2}{(1 - \beta_1)_+} + \frac{(1 - \beta_1)^2 + 2(1 + \beta_1^2) \ln \beta_1}{(1 - \beta_1)} \right] \\
 &\left. - 2 \ln \left(1 - \frac{x_{h_1}}{x_q} \right) \ln \left(\frac{c_0^2}{\left(\frac{z_{1\perp} - z_{2\perp}}{2}\right)^2 \mu^2} \right) D_q^{h_1} \left(\frac{x_{h_1}}{x_q}, \mu_F\right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right) \right\} + (h_1 \leftrightarrow h_2).
 \end{aligned}$$

with $c_0 = 2e^{-\gamma_E}$

Finite dipole \times double-dipole contribution with fragmentation function $q\bar{q}$

$$\begin{aligned}
 & \frac{d\sigma_{4JI}^{q\bar{q} \rightarrow h_1 h_2}}{dx_{h_1} dx_{h_2} d^d p_{h_1\perp} d^d p_{h_2\perp}} \\
 &= \frac{\alpha_s \alpha_{em}}{(2\pi)^{4(d-1)} N_c} \frac{(p_0^-)^2}{s^2 x_{h_1}^2 x_{h_2}^2} \sum_q Q_q^2 \int dx_q \int dx_{\bar{q}} x_q x_{\bar{q}} D_q^{h_1} \left(\frac{x_{h_1}}{x_q}, \mu_F \right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F \right) \\
 &\times \int \frac{dx_g d^d p_{g\perp}}{x_g (2\pi)^d} \delta(1 - x_q - x_{\bar{q}} - x_g) \int d^d p_{1\perp} d^d p_{2\perp} d^d p'_{1\perp} d^d p'_{2\perp} \frac{d^d p_{3\perp} d^d p'_{3\perp}}{(2\pi)^d} \\
 &\times \delta(p_{q1\perp} + p_{\bar{q}2\perp} + p_{g3\perp}) \delta(p_{11'\perp} + p_{22'\perp} + p_{33'\perp}) (\varepsilon_{I\alpha} \varepsilon_{J\beta}^*) \\
 &\times \left[\Phi_3^\alpha(p_{1\perp}, p_{2\perp}) \Phi_4^{\beta*}(p'_{1\perp}, p'_{2\perp}, p'_{3\perp}) \mathbf{F} \left(\frac{p_{12\perp}}{2} \right) \tilde{\mathbf{F}}^* \left(\frac{p_{1'2'\perp}}{2}, p'_{3\perp} \right) \delta(p_{3\perp}) \right. \\
 &\left. + \Phi_4^\alpha(p_{1\perp}, p_{2\perp}, p_{3\perp}) \Phi_3^{\beta*}(p'_{1'\perp}, p'_{2'\perp}) \tilde{\mathbf{F}} \left(\frac{p_{12\perp}}{2}, p_{3\perp} \right) \mathbf{F}^* \left(\frac{p_{1'2'\perp}}{2} \right) \delta(p'_{3\perp}) \right] + (h_1 \leftrightarrow h_2)
 \end{aligned}$$

with $\Phi_3 = \Phi_{R1} + \Phi'_{R1}$ and $\Phi_{R2} = \Phi_4$

The finite double-dipole \times double-dipole contribution

$$\begin{aligned}
 & \frac{d\sigma_{5JI}^{q\bar{q} \rightarrow h_1 h_2}}{dx_{h_1} dx_{h_2} d^d p_{h_1} d^d p_{h_2}} \\
 &= \frac{\alpha_s \alpha_{\text{em}} (\varepsilon_I \alpha \varepsilon'_{J\beta})}{(2\pi)^{4(d-1)} (N_c^2 - 1)} \frac{(p_0^-)^2}{s^2 x_{h_1}^2 x_{h_2}^2} \sum_q Q_q^2 \int dx_q \int dx_{\bar{q}} x_q x_{\bar{q}} D_q^{h_1} \left(\frac{x_{h_1}}{x_q}, \mu_F \right) D_{\bar{q}}^{h_2} \left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F \right) \\
 &\times \int \frac{dx_g d^d p_{g\perp}}{x_g (2\pi)^d} \delta(1 - x_q - x_{\bar{q}} - x_g) \int d^d p_{1\perp} d^d p_{2\perp} d^d p'_{1\perp} d^d p'_{2\perp} \\
 &\times \int \frac{d^d p_{3\perp} d^d p'_{3\perp}}{(2\pi)^{2d}} \delta(p_{q1\perp} + p_{\bar{q}2\perp} + p_{g3\perp}) \delta(p_{11'\perp} + p_{22'\perp} + p_{33'\perp}) \\
 &\times \Phi_4^\alpha(p_{1\perp}, p_{2\perp}, p_{3\perp}) \Phi_4^{\beta*}(p'_{1\perp}, p'_{2\perp}, p'_{3\perp}) \tilde{\mathbf{F}} \left(\frac{p_{12\perp}}{2}, p_{3\perp} \right) \tilde{\mathbf{F}}^* \left(\frac{p_{1'2'\perp}}{2}, p'_{3\perp} \right) + (h_1 \leftrightarrow h_2) .
 \end{aligned}$$