

Introduction and motivations

BFKL resummation

LO Higgs impact factor

Higgs NLO impact factor

NLO Higgs impact factor: Real corrections

NLO Higgs impact factor: Virtual corrections

Cancellation of divergences

Higgs plus jet

Higgs p_T -distribution

Conclusions and outlook

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- Record energies in the center-of-mass reachable by modern and future colliders allow us to study strong interactions in so far unexplored kinematic regions
- **Semi-hard** collision process \rightarrow stringent *scale hierarchy*

$$s \gg Q^2 \gg \Lambda_{\text{QCD}}^2, \quad Q^2 \text{ a hard scale,}$$

 **Regge kinematic region**

$$\alpha_s(Q^2) \ln\left(\frac{s}{Q^2}\right) \sim 1 \implies \text{all-order } \mathbf{resummation} \text{ needed}$$

- The **Balitskii-Fadin-Kuraev-Lipatov (BFKL)** approach is the general framework for this *high-energy* resummation
 - Leading-Logarithmic-Approximation (**LLA**): $(\alpha_s \ln s)^n$
 - Next-to-Leading-Logarithmic-Approximation (**NLLA**): $\alpha_s(\alpha_s \ln s)^n$
- Progress on **NNLLA**

[Sunday's talk by E. Byrne and Thursday's talk by C. Milloy]

Factorization scheme for hadronic impact factors

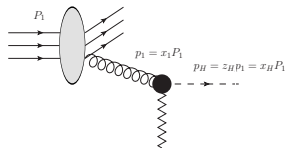
- Infrared safety of impact factor for colorless particle

[V. S. Fadin, A. D. Martin (1999)]

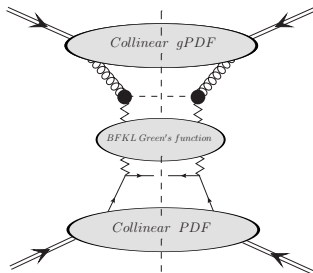
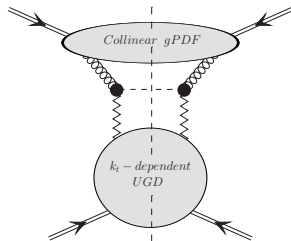
- Impact factors of colored particles afflicted by *infrared singularities*

$$p_H = z_H p_1 + \frac{m_H^2 + \vec{p}_H^2}{z_H s} p_2 + p_{H,\perp}$$

$$\frac{d\Phi_{PP}^H}{dx_H d^2\vec{p}_H} = \int_{x_H}^1 \frac{dz_H}{z_H} f_g\left(\frac{x_H}{z_H}\right) \frac{d\Phi_{gg}^H}{dz_H d^2\vec{p}_H}$$

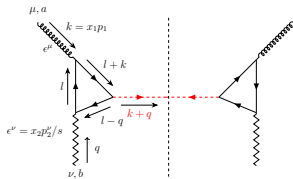


- Hybrid factorization(s)



LO Higgs impact factor

- Gluon-Reggeon \rightarrow Higgs (through the top quark loop)
- Off-shell t -channel gluon



- **LO impact factor**

$$\frac{d\Phi_{PP}^{\{H\}(0)}}{dx_H d^2\vec{p}_H} = \frac{\alpha_s^2}{v^2} \frac{\vec{q}^2 |\mathcal{F}(m_t, m_H, \vec{q}^2)|^2}{128\pi^2 \sqrt{2(N^2 - 1)}} f_g(x_H) \delta^{(2)}(\vec{p}_H - \vec{q})$$

↓ Infinite top-mass limit

$$\frac{d\Phi_{PP}^{\{H\}(0)}}{dx_H d^2\vec{p}_H} = \frac{g_H^2 \vec{q}^2 f_g(x_H) \delta^{(2)}(\vec{q} - \vec{p}_H)}{8\sqrt{N^2 - 1}}$$

- The study can be upgraded to **Next-to-Leading Order (NLO)**, in the limit $m_t \rightarrow \infty$, by using the effective lagrangian

$$\mathcal{L}_{\text{ggH}} = -\frac{1}{4} \mathbf{g}_H \mathbf{F}_{\mu\nu}^a \mathbf{F}^{\mu\nu, a} \mathbf{H} \quad g_H = \frac{\alpha_s}{3\pi v} + \mathcal{O}(\alpha_s^2)$$

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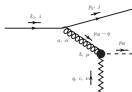
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NLO Higgs impact factor: Real corrections

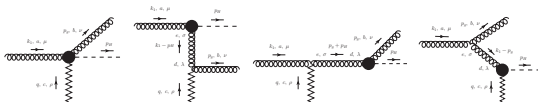
- Quark initiated contribution



$$d\Phi_{qq}^{\{Hq\}} \sim \left[\frac{4(1-z_H)(\vec{r} \cdot \vec{q})^2 + z_H^2 \vec{q}^2 \vec{r}^2}{z_H \vec{r}^2} \right]$$

Rapidity divergence absent $\implies s_\Lambda \rightarrow \infty$ Collinear divergence: $r \equiv (\vec{q} - \vec{p}_H) \rightarrow \vec{0}$

- Gluon initiated contribution



$$d\Phi_{gg}^{\{Hg\}} \sim \left\{ \frac{\vec{q}^2 z_H}{(1-z_H)\vec{r}^2} + \frac{\vec{q}^2}{\vec{r}^2} \left[z_H(1-z_H) + 2(1-\epsilon) \frac{1-z_H}{z_H} \frac{(\vec{q} \cdot \vec{r})^2}{\vec{q}^2 \vec{r}^2} \right] \right\} \\ \times \theta \left(s_\Lambda - \frac{(1-z_H)m_H^2 + \vec{\Delta}^2}{z_H(1-z_H)} \right) + \text{finite}$$

Rapidity divergence $\implies s_\Lambda$ still present Soft divergence: $z_H \rightarrow 1$, $\vec{r} \rightarrow \vec{0}$
 Collinear divergence: $\vec{r} \rightarrow \vec{0}$ $\vec{\Delta} = \vec{p}_H - z_H \vec{q}$

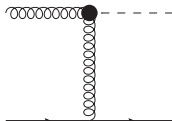
- Agreement with calculation within Lipatov effective action framework

[M. Hentschinski, K. Kutak, A. van Hameren (2021)]

NLO Higgs impact factor: Virtual corrections

- 1-loop ggH effective vertex

$$\Gamma_{\{H\}g}^{ac(1)}(q) = \Gamma_{\{H\}g}^{ac(0)}(q) [1 + \delta_{\text{NLO}}]$$



- General strategy: Comparison of a suitable amplitude (in the high-energy limit) with the expected *Regge form*

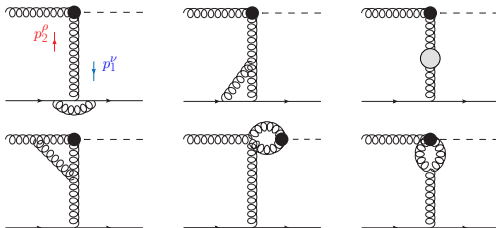
$$\begin{aligned} \mathcal{A}_{gq \rightarrow Hq}^{(8,-)} &= \Gamma_{\{H\}g}^{ac} \frac{s}{t} \left[\left(\frac{s}{-t} \right)^{\omega(t)} + \left(\frac{-s}{-t} \right)^{\omega(t)} \right] \Gamma_{qq}^c \approx \Gamma_{\{H\}g}^{ac(0)} \frac{2s}{t} \Gamma_{qq}^{c(0)} \\ &+ \Gamma_{\{H\}g}^{ac(0)} \frac{s}{t} \omega^{(1)}(t) \left[\ln \left(\frac{s}{-t} \right) + \ln \left(\frac{-s}{-t} \right) \right] \Gamma_{qq}^{c(0)} + \Gamma_{\{H\}g}^{ac(0)} \frac{2s}{t} \Gamma_{qq}^{c(1)} + \Gamma_{\{H\}g}^{ac(1)} \frac{2s}{t} \Gamma_{qq}^{c(0)} \end{aligned}$$

- Virtual corrections** to the impact factor

$$\begin{aligned} \frac{d\Phi_{gg}^{\{H\}(1)}}{dz_H d^2\vec{p}_H} &= \frac{d\Phi_{gg}^{\{H\}(0)}}{dz_H d^2\vec{p}_H} \frac{\bar{\alpha}_s}{2\pi} \left(\frac{\vec{q}^2}{\mu^2} \right)^{-\epsilon} \left[-\frac{C_A}{\epsilon^2} - \frac{C_A}{\epsilon} \ln \left(\frac{\vec{q}^2}{s_0} \right) + \frac{\beta_0}{2\epsilon} \right. \\ &\left. + 11 - \frac{5n_f}{9} + C_A \left(2 \Re e \left(\text{Li}_2 \left(1 + \frac{m_H^2}{\vec{q}^2} \right) \right) + \frac{\pi^2}{3} + \frac{67}{18} \right) \right] \end{aligned}$$

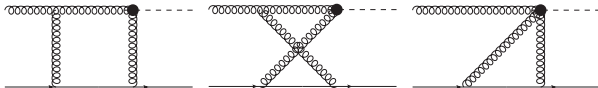
NLO Higgs impact factor: Virtual corrections

- Single gluon in the t -channel diagrams



Gribov's trick: $g^{\rho\nu} = g_{\perp\perp}^{\rho\nu} + 2 \frac{p_1^\rho p_2^\nu + p_1^\nu p_2^\rho}{s} \rightarrow 2s \frac{p_1^\nu}{s} \frac{p_2^\rho}{s}$

- Two gluons in the t -channel diagrams



Dimension-5 operator in $\mathcal{L} = \mathcal{L}_{QCD} + \mathcal{L}_{ggh} \rightarrow$ **Gribov's trick violation**

$$g^{\rho\nu} = g_{\perp\perp}^{\rho\nu} + 2 \frac{p_1^\rho p_2^\nu + p_1^\nu p_2^\rho}{s} \rightarrow 2s \frac{p_1^\nu}{s} \frac{p_2^\rho}{s} + g_{\perp\perp}^{\rho\nu}$$

Showing cancellation of divergences

- **BFKL cross section**

$$\sigma_{AB} = \frac{1}{(2\pi)^{D-2}} \int \frac{d^{D-2}q_1}{\bar{q}_1^2} \int \frac{d^{D-2}q_2}{\bar{q}_2^2} \\ \times \Phi_{AA}(\bar{q}_1; s_0) \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \left[\left(\frac{s}{s_0} \right)^\omega G_\omega^{(0)}(\bar{q}_1, \bar{q}_2) \right] \Phi_{BB}(-\bar{q}_2; s_0)$$

- Spectral representation of the Green function at LO

$$G_\omega^{(0)}(\bar{q}_1, \bar{q}_2) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} d\nu \frac{\phi_\nu^n(\bar{q}_1^2) \phi_\nu^{n*}(\bar{q}_2^2)}{\omega - \frac{\alpha_s C_A}{\pi} \chi(n, \nu)},$$

- **Projection** onto the eigenfunction of the BFKL kernel

$$\int \frac{d^{2-2\epsilon}q}{\pi\sqrt{2}} (\bar{q}^2)^{i\nu - \frac{3}{2}} e^{in\phi} \Phi_{AA}^{(0)}(\bar{q}) \equiv \Phi_{AA}^{(0)}(n, \nu)$$

- LO projected impact factor

$$\frac{d\Phi_{PP}^{\{H\}}(0)(x_H, \vec{p}_H, n, \nu)}{dx_H d^2\vec{p}_H} = \frac{g_H^2}{8(1-\epsilon)\sqrt{N^2-1}} \frac{(\vec{p}_H^2)^{i\nu - \frac{1}{2}} e^{in\phi_H}}{\pi\sqrt{2}} f_g(x_H)$$

- Scheme for *cancellation of divergences*

- i.* Rapidity divergence \rightarrow removed by the BFKL counterterm
- ii.* UV divergences \rightarrow QCD coupling renormalization
- iii.* Soft divergence \rightarrow cancelled in the real plus virtual combination
- iiii.* Surviving initial-state IR divergences \rightarrow gPDF renormalization

Cancellation of divergences in the (n, ν) -space

- UV counterterm $d\Phi_{PP}^{\{H\}} \Big|_{\alpha_s \text{ c.t.}} = d\Phi_{PP}^{\{H\}(0)} \frac{\bar{\alpha}_s}{2\pi} \left[-\frac{\beta_0}{\epsilon} \right] + \text{finite}$
- gPDF counterterm

$$d\Phi_{PP}^{\{H\}} \Big|_{P_{qg} \text{ c.t.}} = \frac{d\Phi_{PP}^{\{H\}(0)} \bar{\alpha}_s}{f_g(x_H) 2\pi} \left[\frac{1}{\epsilon} P_{gq} \otimes \sum_{a=q\bar{q}} f_a \right] + \text{finite}$$

$$d\Phi_{PP}^{\{H\}} \Big|_{P_{gg} \text{ c.t.}} = \frac{d\Phi_{PP}^{\{H\}(0)} \bar{\alpha}_s}{f_g(x_H) 2\pi} \left[\frac{1}{\epsilon} \tilde{P}_{gg} \otimes f_g + \frac{1}{2} \frac{\beta_0}{\epsilon} f_g(x_H) \right] + \text{finite}$$

- Real quark contribution

$$d\Phi_{PP}^{\{Hg\}} \Big|_{\text{quark}} = \frac{d\Phi_{PP}^{\{H\}(0)} \bar{\alpha}_s}{f_g(x_H) 2\pi} \left[-\frac{1}{\epsilon} P_{gq} \otimes \sum_{a=q\bar{q}} f_a \right] + \text{finite}$$

- Real gluon contribution (BFKL counterterm subtracted)

$$d\Phi_{PP}^{\{Hq\}} \Big|_{\text{gluon}} = \frac{d\Phi_{PP}^{\{H\}(0)} \bar{\alpha}_s}{f_g(x_H) 2\pi} \left(\frac{\tilde{p}_H^2}{\mu^2} \right)^{-\epsilon} \left[\left(\frac{C_A}{\epsilon^2} + \frac{C_A}{\epsilon} \ln \left(\frac{\tilde{p}_H^2}{s_0} \right) \right) f_g(x_H) - \frac{1}{\epsilon} \tilde{P}_{gg} \otimes f_g \right] + \text{finite}$$

- Virtual corrections contribution

$$d\Phi_{PP}^{\{H\}} \Big|_{\text{virtual}} = d\Phi_{PP}^{\{H\}(0)} \frac{\bar{\alpha}_s}{2\pi} \left(\frac{\tilde{p}_H^2}{\mu^2} \right)^{-\epsilon} \left[-\frac{C_A}{\epsilon^2} - \frac{C_A}{\epsilon} \ln \left(\frac{\tilde{p}_H^2}{s_0} \right) + \frac{1}{\epsilon} \frac{\beta_0}{2} \right] + \text{finite}$$

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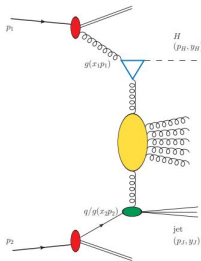
Conclusions and outlook

Higgs plus jet

- Inclusive **Higgs plus jet** production in proton-proton collision

[F.G. Celiberto, D. Yu. Ivanov, M.M.A. Mohammed, A. Papa (2021)]

- i.* Large p_T and large rapidity separation
- ii.* Full NLL Green function
- iii.* Partial NLO impact factors \rightarrow LO + RG-universal NLO corrections



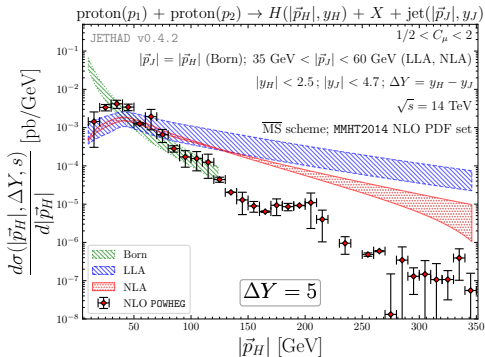
$$\begin{aligned} \frac{d\sigma_{\text{PP}}}{dy_H dy_J d|\vec{p}_H| d|\vec{p}_J| d\phi_1 d\phi_2} &= \frac{1}{(2\pi)^2} \\ &\times \int \frac{d^2\vec{q}_1}{\vec{q}_1^2} (\mathcal{V}_H^{(g)}(\vec{q}_1, s_0, x_1, \vec{p}_H) \otimes f_g(x_1)) \\ &\times \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \left(\frac{x_1 x_2 s}{s_0} \right)^\omega G_\omega(\vec{q}_1, \vec{q}_2) \\ &\times \int \frac{d^2\vec{q}_2}{\vec{q}_2^2} \left(\sum_r \mathcal{V}_J^{(p)}(\vec{q}_2, s_0, x_2, \vec{p}_J) \otimes f_r(x_2) \right) \end{aligned}$$

- Hadronic cross section expanded in **azimuthal coefficients**

$$\frac{d\sigma_{\text{PP}}}{dy_H dy_J d|\vec{p}_H| d|\vec{p}_J| d\phi_1 d\phi_2} = \frac{1}{(2\pi)^2} \left[C_0 + 2 \sum_{n=1}^{\infty} \cos(n\phi) C_n \right]$$

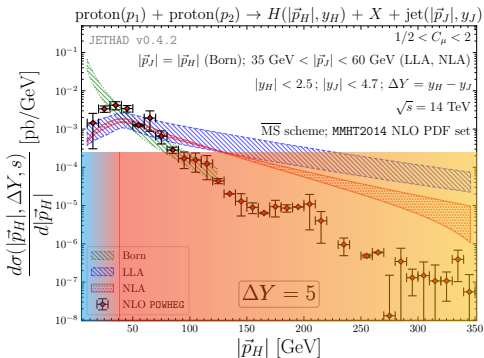
Higgs p_T -distribution

$$\frac{d\sigma(|\vec{p}_H|, \Delta Y, s)}{d|\vec{p}_H|d\Delta Y} = \int_{p_J^{min}}^{p_J^{max}} d|\vec{p}_J| \int_{y_H^{min}}^{y_H^{max}} dy_H \int_{y_J^{min}}^{y_J^{max}} dy_J \delta(y_H - y_J - \Delta Y) C_0$$



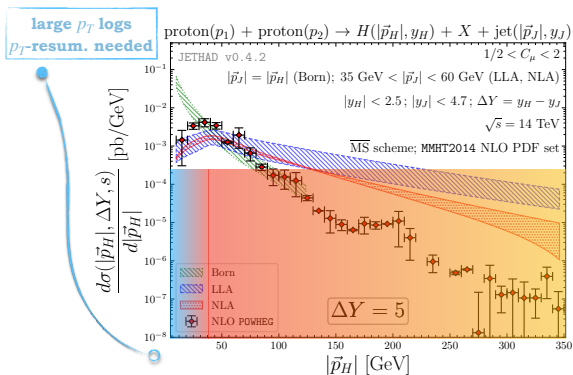
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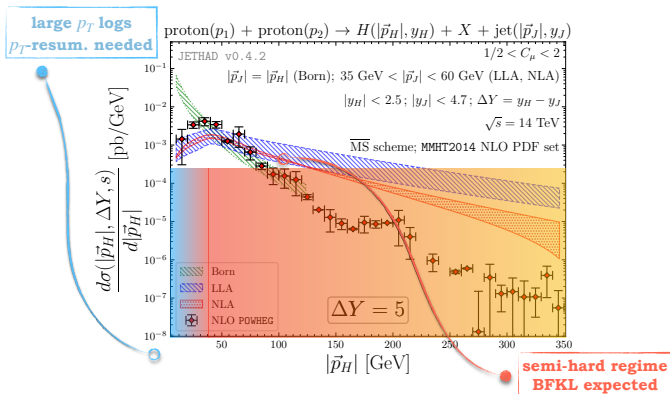
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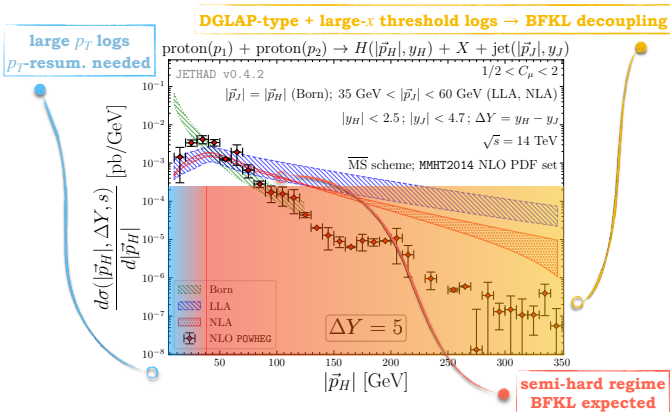
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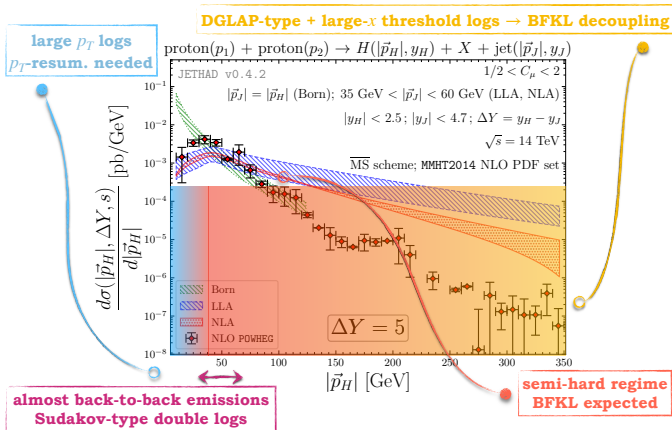
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$$\frac{d\sigma(|\vec{p}_H|, \Delta Y, s)}{d|\vec{p}_H|d\Delta Y} = \int_{p_J^{min}}^{p_J^{max}} d|\vec{p}_J| \int_{y_H^{min}}^{y_H^{max}} dy_H \int_{y_J^{min}}^{y_J^{max}} dy_J \delta(y_H - y_J - \Delta Y) C_0$$



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Conclusions

- **NLO corrections** to the forward Higgs boson impact factor has been obtained both in q_T and (n, ν) -space in the $m_t \rightarrow \infty$ limit
- **Stability of the BFKL series** under higher-order corrections and scale variations has been observed, with partial NLLA, in the inclusive forward emissions of a Higgs in association with a backward jet
- *Gribov's philosophy* for high-energy computations proposed in QCD needs to be *revisited* for non-renormalizable interactions

Outlook and related topics

- **Full NLL/NLO Higgs plus jet production**
- The impact of the high-energy resummation in central inclusive Higgs production at FCC center-of-mass energies is expected to be large
[M. Bonvini, S. Marzani (2018)]
[Monday's talk by M. Bonvini]
- Unified formalism to include different kinds of resummations
(BFKL+Sudakov) [B. Xiao, F. Yuan (2018)]

Thank you for the attention !

Backup

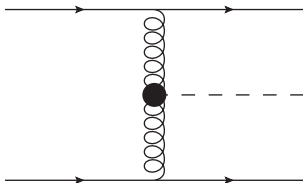
Outlook: Extension to the central production at NLO

- LO vertex with full mass corrections

[R.S. Pasechnik, O.V. Teryaev, A. Szczurek (2006)]

- Computation of real corrections quite straightforward
- Need to extract the vertex at one-loop in the central region of rapidity
- Necessity of a reference NLO two into three particle amplitude, e. g.

$$\mathcal{A}_{q+q \rightarrow q+H+q}$$



- Necessary scalar integrals known: I_4^{2m} and I_5^{1m}

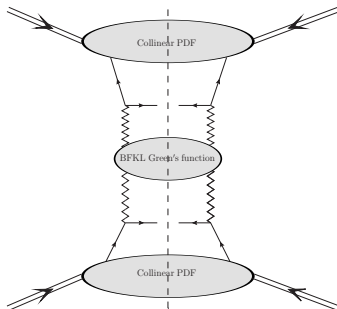
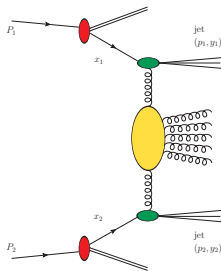
[Z. Bern, L. Dixon, D. A. Kosower (1998)]

Hybrid collinear/high-energy factorization

- Straightforward adaptation to partially inclusive processes: just restrict the summation over intermediate states

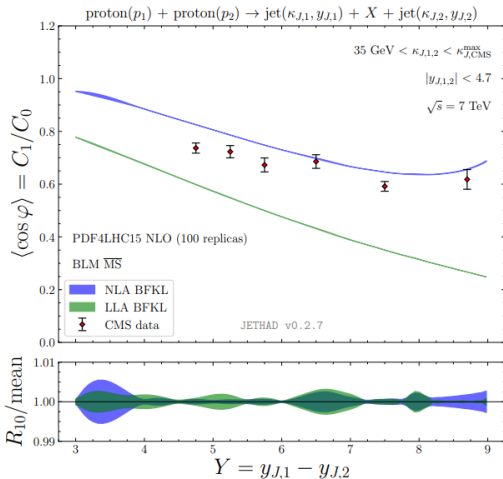
Mueller-Navelet jets

- Inclusive production of two rapidity-separated jets in proton-proton collision
- Large energy logarithms \rightarrow BFKL resummed partonic cross section
- Moderate values of parton $x \rightarrow$ collinear PDFs



- Hybrid formalism: can be extended to several type of semi-hard reactions

Muller-Navelet: Theory vs Experiment



[B. Ducloué, L. Szymanowski, S. Wallon (2013)]

[F. Caporale, D.Yu. Ivanov, B. Murdaca, A. Papa (2014)]

In this slide: [F.G. Celiberto (2021)]

Mueller-Navelet: Theory vs Experiment

- CMS @7Tev with symmetric p_T -ranges, only! [CMS collaboration (2016)]
- LHC kinematic **domain** in between the sectors described by BFKL and DGLAP approaches
- Clearer manifestation of high-energy signatures expected at increasing energies (higher hadronic center-of-mass energy or higher rapidity difference between tagged jets)
- Need for more exclusive final states as well as more sensitive observables

Partially inclusive processes in NLA

- Mueller-Navelet jet production

[J. Bartels, D. Colferai, G.P. Vacca (2003)]

[F. Caporale, D.Yu. Ivanov, B. Murdaca, A. Papa, A. Perri (2011)]

[D.Yu. Ivanov, A. Papa (2012)]

[D. Colferai, A. Niccoli (2015)]

[B. Ducloué, L. Szymanowski, S. Wallon (2013,2014)]

[F. Caporale, D.Yu. Ivanov, B. Murdaca, A. Papa (2014)]

[F.G. Celiberto, D.Yu. Ivanov, B. Murdaca, A. Papa (2015)]

- Light hadron-light hadron production

[D.Yu. Ivanov, A. Papa (2012)]

[F.G. Celiberto, D.Yu. Ivanov, B. Murdaca, A. Papa (2016, 2017)]

- Light hadron-jet production

[A.D. Bolognino, F.G. Celiberto, D.Yu. Ivanov, M.M.A. Mohammed, A. Papa (2018)]

- Heavy-light hadrons in VFNS

[F.G. Celiberto, M.F, D.Yu. Ivanov, M.M.A. Mohammed, A. Papa (2021)]

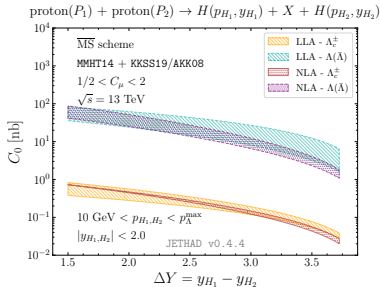
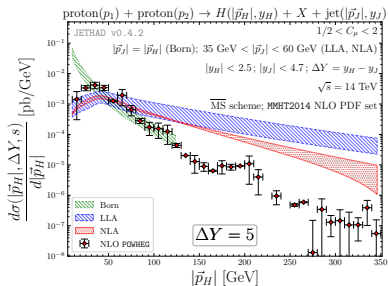
[F.G. Celiberto, M.F, D.Yu. Ivanov, A. Papa (2021)]

Other partially-inclusive reactions

- Three / four jet production (partial NLA)
 - [F. Caporale, G. Chachamis, B. Murdaca, A. Sabio Vera (2016)]
 - [F. Caporale, F.G. Celiberto, G. Chachamis, A. Sabio Vera (2016)]
 - [F. Caporale, F.G. Celiberto, G. Chachamis, D.G. Gomez, A. Sabio Vera (2016, 2017)]
- Drell-Yan pair - jet (partial NLA)
 - [K. Golec-Biernat, L. Motyka, T. Stebel (2018)]
- Higgs - jet
 - Partial NLA
 - [F.G. Celiberto, D.Yu. Ivanov, M.M.A. Mohammed, A. Papa (2020)]
 - Full NLA, $m_t \rightarrow \infty$ limit
 - [F.G. Celiberto, D.Yu. Ivanov, M.F., M.M.A. Mohammed, A. Papa (2022, in preparation)]
- Heavy-quark pair photo/hadro-production (partial NLA)
 - [F.G. Celiberto, D.Yu. Ivanov, B. Murdaca, A. Papa (2017)]
 - [A.D. Bolognino, F.G. Celiberto, M.F., D.Yu. Ivanov, A. Papa (2019)]
- J/Ψ - jet production (partial NLA)
 - [R. Boussarie, B. Ducloué, L. Szymanowski, S. Wallon (2018)]

Stabilization effects

- Stabilization effects in Higgs and heavy flavor production
- Λ -baryon FFs
 - heavy species $\rightarrow \Lambda_c$
KKSS19 [B.A. Kniehl, G. Kramer, I. Schienbein, H. Spiesberger (2020)]
 - light species $\rightarrow \Lambda$
AKK08 [S.Albino, B.A. Kniehl, and G. Kramer (2008)]



[F.G. Celiberto, D.Yu. Ivanov, M.M.A. Mohammed, A. Papa (2020)]

[F.G. Celiberto, D.Yu. Ivanov, M. F., A. Papa (2021)]

What is the BFKL resummation?

- The **Balitsky-Fadin-Kuraev-Lipatov (BFKL)** approach is the general framework for the resummation of energy-type logarithms
 - Leading-Logarithm-Approximation (LLA): $(\alpha_s \ln s)^n$
 - Next-to-Leading-Logarithm-Approximation (NLLA):
 $\alpha_s (\alpha_s \ln s)^n$

In which contexts can BFKL approach be applied?

- **Semi-hard** collision processes, featuring the scale hierarchy

$$s \gg Q^2 \gg \Lambda_{\text{QCD}}^2, \quad Q^2 \text{ a hard scale,}$$

$$\alpha_s(Q^2) \ln\left(\frac{s}{Q^2}\right) \sim 1 \implies \text{all-order resummation needed}$$

- **UGD sector**

The evolution of the **Unintegrated gluon density**,

$$\mathcal{F}(x, \vec{k}) \quad \text{t.c.} \quad f^g(x, Q^2) = \int \frac{d^2 \vec{k}}{\pi \vec{k}^2} \mathcal{F}(x, \vec{k}) \theta(Q^2 - \vec{k}^2)$$

as a function of $\ln(1/x) = \ln(s/Q^2)$, is governed by BFKL:

$$\frac{\partial \mathcal{F}}{\partial \ln(1/x)} = \mathcal{F} \otimes \mathcal{K}$$

Before QCD

- Assumptions on S -matrix ($S_{ab} = \langle b_{out} | a_{in} \rangle$):
 - **Lorentz invariance:**
It can be expressed as a function of Lorentz invariant scalar product, e.g (s, t) for $2 \rightarrow 2$ particle scattering.
 - **Analitycity**
Causality \rightarrow Analytic function with only those singularity required by unitarity.
 - **Unitarity**

Cutkosky rules

$$2\Im \mathcal{A}_{ab} = (2\pi)^4 \delta^4 \left(\sum_a p_a - \sum_b p_b \right) \sum_c \mathcal{A}_{ac} \mathcal{A}_{cb}^\dagger$$

Optical theorem

$$2\Im \mathcal{A}_{aa}(s, 0) = F \sigma_{tot}$$

- Unitarity \rightarrow relates the imaginary parts of amplitudes to sum of products of other amplitudes, **dispersion relations** \rightarrow reconstruct the corresponding real parts
- More in general **subtract dispersion relation** \rightarrow we must know the asymptotic behaviour of amplitudes \rightarrow **Regge theory**

- Asymptotic behavior of amplitudes in the Regge region:

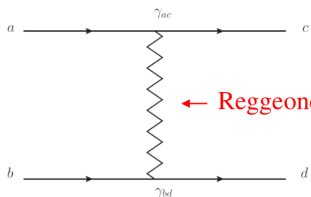
$$\mathcal{A}(s, t) \xrightarrow{s \gg |t|} \frac{\eta + e^{-i\pi\alpha(t)}}{2} \beta(t) s^{\alpha(t)}$$

- Definition of “**Reggeization**”

A particle of mass M and spin J is said to “Reggeize” if the amplitude, \mathcal{A} , for a process involving the exchange in the t -channel of the quantum numbers of that particle behaves asymptotically in s as

$$\mathcal{A} \propto s^{\alpha(t)}$$

where $\alpha(t)$ is the trajectory and $\alpha(M^2) = J$, so that the particle itself lies on the trajectory.



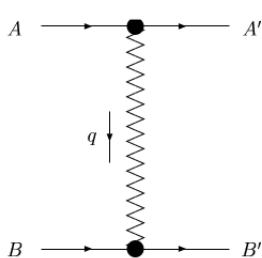
The Reggeized gluon

Elastic scattering process $A + B \rightarrow A' + B'$

- **Gluon quantum numbers** in the t -channel
- **Regge limit**: $s \simeq -u \rightarrow \infty$, t fixed (i.e not growing with s)
- All-order resummation:

leading logarithmic approximation (LLA): $(\alpha_s \ln s)^n$

next-to-leading logarithmic approximation (NLA): $\alpha_s (\alpha_s \ln s)^n$



$$(\mathcal{A})_{AB}^{A'B'} = \Gamma_{A'A}^c \left[\left(\frac{-s}{-t} \right)^{j(t)} - \left(\frac{s}{-t} \right)^{j(t)} \right] \Gamma_{B'B}^c$$

$$j(t) = 1 + \omega(t), \quad j(0) = 1$$

$j(t)$ - Reggeized gluon trajectory

$$\Gamma_{A'A}^c = g \langle A' | T^c | A \rangle \Gamma_{A'A}$$

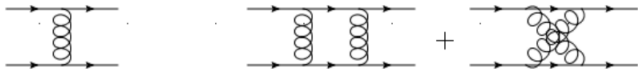
T^c - fundamental(quarks) or adjoint(gluons)

[Ya. Ya. Balitsky, V.S. Fadin, L.N. Lipatov (1979)]

$$\Gamma_{A'A}^{(0)} = \delta_{\lambda_{A'}, \lambda_A}, \quad \omega^{(1)}(t) = \frac{g^2 t}{(2\pi)^{(D-1)}} \frac{N}{2} \int \frac{d^{D-2} k_{\perp}}{k_{\perp}^2 (q - k)_{\perp}^2} = -g^2 \frac{N\Gamma(1-\epsilon)}{(4\pi)^{2+\epsilon}} \frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)} (\vec{q}^2)^{\epsilon}$$

- LLA

The Reggeized gluon

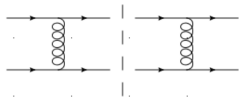


$$\mathcal{A}_0^8 = 8\pi\alpha_s \frac{s}{t} \delta_{\lambda_{A'}\lambda_A} \delta_{\lambda_{B'}\lambda_B} G_0^8$$

$$\mathcal{A}_1^8 = \mathcal{A}_0^8 \omega(t) \ln\left(\frac{s}{\vec{q}^2}\right)$$

$$\mathcal{A}^8 = \mathcal{A}_0^8 \left[1 + \omega(t) \ln\left(\frac{s}{\vec{q}^2}\right) + \frac{1}{2} \left(\omega(t) \ln\left(\frac{s}{\vec{q}^2}\right) \right)^2 + \dots \right] \rightarrow \mathcal{A}^8 = \mathcal{A}_0^8 \left(\frac{s}{\vec{q}^2} \right)^{\omega(t)}$$

$$\Im \mathcal{A}_1 = \frac{1}{2} \int d(P.S.^2) \mathcal{A}_0^8(k) \mathcal{A}_0^8(k-q)$$



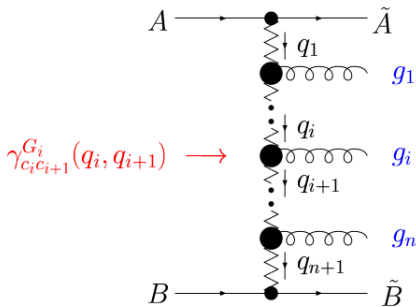
The integration that appears in $\omega(t)$ is the residue of that over the phase space.
The terms in the denominator come from the propagators.

► NLLA

[V.S. Fadin, R. Fiore, M.G. Kozlov, A.V. Reznichenko (2006)]

BFKL in LLA

Inelastic scattering process $A + B \rightarrow \tilde{A} + \tilde{B} + n$ in the LLA



Leading logarithms resummation

↓
Multi-Regge kinematics (MRK)

$$\Re \mathcal{A}_{AB}^{\tilde{A}\tilde{B}+n} = 2s \Gamma_{\tilde{A}A}^{c_1} \left(\prod_{i=1}^n \gamma_{c_i c_{i+1}}^{P_i}(q_i, q_{i+1}) \left(\frac{s_i}{s_0} \right)^{\omega(t_i)} \frac{1}{t_i} \right) \frac{1}{t_{n+1}} \left(\frac{s_{n+1}}{s_0} \right)^{\omega(t_{n+1})} \Gamma_{\tilde{B}B}^{c_{n+1}}$$

- s_0 -energy scale, arbitrary in LLA.
- Terms that contain fermions in intermediate states are suppressed in relation to those that involve the exchange of gluons
- “Vertical” gluons become Reggeized due to radiative corrections (“ladders within ladders”)

Multi-Regge kinematics

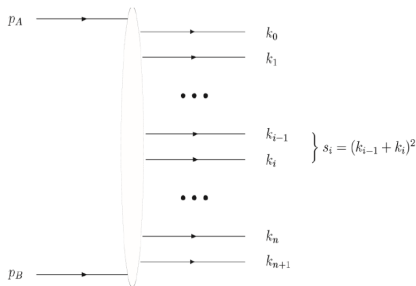
Multi-Regge kinematics

- Sudakov decomposition for the produced particles: $k_i = z_i p_1 + \lambda_i p_2 + k_{i\perp}$

- Transverse momenta of the produced particles are limited
- Their Sudakov variables z_i and λ_i , are strongly ordered:

$$z_0 \gg z_1 \gg \dots \gg z_n \gg z_{n+1}$$

$$\lambda_{n+1} \gg \lambda_n \gg \dots \gg \lambda_1 \gg \lambda_0$$

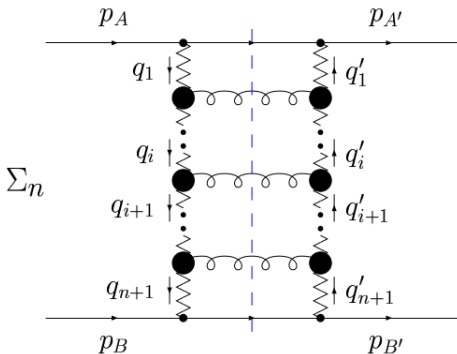


- Leading logarithms come from the integration over the longitudinal momenta of the produced particles
- In the LLA, where each added particle contributes only one $\ln s$, only this kinematics counts

BFKL in LLA

Amplitude $A + B \rightarrow A' + B'$ in the LLA via Cutkosky rules

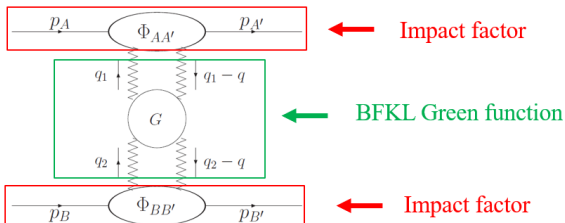
$$\Im \mathcal{A}_{AB}^{A'B'} = \frac{1}{2} \sum_{n=0}^{\infty} \sum_f \int \mathcal{A}_{AB}^{\tilde{A}\tilde{B}+n} \left(\mathcal{A}_{A'B'}^{\tilde{A}\tilde{B}+n} \right)^* d\Phi_{\tilde{A}\tilde{B}+n}$$



$$\mathcal{A}_{AB}^{A'B'} = \sum_{\mathcal{R}} (\mathcal{A}_{\mathcal{R}})_{AB}^{A'B'} \quad \mathcal{R} = 1(\text{singlet}), 8(\text{octet}), \dots$$

BFKL resummation

- Diffusion $A + B \rightarrow A' + B'$ in the **Regge kinematic region**
- Gluon Reggeization
- BFKL factorization for $\Im \mathcal{A}_{AB}^{A'B'}$: convolution of a **Green function** (process independent) with the **Impact factors** of the colliding particles (process dependent).

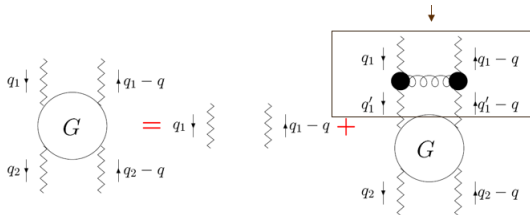


$$\begin{aligned}
 \Im \mathcal{A}_{AB}^{A'B'} &= \frac{s}{(2\pi)^{D-2}} \int \frac{d^{D-2} q_1}{\vec{q}_1^2 (\vec{q}_1 - \vec{q})^2} \frac{d^{D-2} q_2}{\vec{q}_2^2 (\vec{q}_2 - \vec{q})^2} \\
 &\times \sum_{\nu} \Phi_{A'A}^{(R,\nu)}(\vec{q}_1, \vec{q}, s_0) \int \frac{d\omega}{2\pi i} \left[\left(\frac{s}{s_0} \right)^{\omega} G_{\omega}^{(R)}(\vec{q}_1, \vec{q}_2; \vec{q}) \right] \Phi_{B'B}^{(R,\nu)}(-\vec{q}_2, \vec{q}, s_0)
 \end{aligned}$$

BFKL resummation

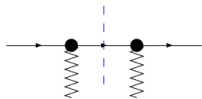
- $G_{\omega}^{(R)}(\vec{q}_1, \vec{q}_2; \vec{q})$ -Mellin transform of the Green function for the Reggeon-Reggeon scattering

$$\omega G_{\omega}^{(R)}(\vec{q}_1, \vec{q}_2; \vec{q}) = \vec{q}_1^2 (\vec{q}_1 - \vec{q})^2 \delta^{(D-2)}(\vec{q}_1 - \vec{q}_2) + \int \frac{d^{D-2} q'_1}{\vec{q}_1'^2 (\vec{q}_1' - \vec{q})^2} \mathcal{K}^{(R)}(\vec{q}_1, \vec{q}_1'; \vec{q}) G_{\omega}^{(R)}(\vec{q}_1', \vec{q}_2; \vec{q})$$



- $\Phi_{P'P}^{(R,\nu)}$ - LO impact factor in the t -channel color state (R, ν)

$$\Phi_{P'P}^{(R,\nu)} = \langle cc' | \hat{\mathcal{P}} | \nu \rangle \sum_{\{f\}} \int \frac{ds_{PR}}{2\pi} d\rho_f \Gamma_{\{f\}P}^c (\Gamma_{\{f\}P'}^{c'})^*$$



BFKL resummation

- **BFKL equation:** $\vec{q}^2 = 0$ and singlet color state representation

[Ya. Ya. Balitsky, V. S. Fadin, E.A Kuraev, L.N Lipatov (1975)]

$$\text{Redefinition : } G_\omega(\vec{q}_1, \vec{q}_2) \equiv \frac{G_\omega^{(0)}(\vec{q}_1, \vec{q}_2, 0)}{\vec{q}_1^2 \vec{q}_2^2}, \quad \mathcal{K}(\vec{q}_1, \vec{q}_2) \equiv \frac{\mathcal{K}^{(0)}(\vec{q}_1, \vec{q}_2, 0)}{\vec{q}_1^2 \vec{q}_2^2}$$

$$\omega G_\omega(\vec{q}_1, \vec{q}_2) = \delta^{(D-2)}(\vec{q}_1 - \vec{q}_2) + \int d^{D-2} q_r \mathcal{K}(\vec{q}_1, \vec{q}_r) G(\vec{q}_r, \vec{q}_2)$$

- **Elastic amplitude** factorization:

$$\Im \mathcal{A}_{AB}^{AB} = \frac{s}{(2\pi)^{D-2}} \int d^{D-2} q_1 d^{D-2} q_2 \\ \times \frac{\Phi_{AA}^{(0)}(\vec{q}_1, s_0)}{\vec{q}_1^2} \int \frac{d\omega}{2\pi i} \left[\left(\frac{s}{s_0} \right)^\omega G_\omega(\vec{q}_1, \vec{q}_2) \right] \frac{\Phi_{BB}^{(0)}(-\vec{q}_2, s_0)}{\vec{q}_2^2}$$

- **Optical Theorem:**

$$\sigma_{AB} = \frac{\Im \mathcal{A}_{AB}^{AB}}{s}$$

- Impact factor in the color singlet state:

$$\Phi_{PP}^{(0)} = \langle cc' | \hat{\mathcal{P}} | 0 \rangle \sum_{\{f\}} \int \frac{ds_{PR}}{2\pi} d\rho_f \Gamma_{\{f\}P}^c (\Gamma_{\{f\}P}^{c'})^*$$

Solution of the BFKL equation

- Let's solve the equation

$$\omega G_\omega(\vec{q}_1, \vec{q}_2) = \delta^{(D-2)}(\vec{q}_1 - \vec{q}_2) + \int d^{D-2} q_r \mathcal{K}(\vec{q}_1, \vec{q}_r) G(\vec{q}_r, \vec{q}_2)$$

$$\mathcal{K}(\vec{q}_1, \vec{q}_r) = \mathcal{K}^{(R)}(\vec{q}_1, \vec{q}_r) + 2\omega(\vec{q}_1^2)\delta^{(2)}(\vec{q}_1 - \vec{q}_r)$$

- We can see $\mathcal{K}(\vec{k}, \vec{k}')$ as the integral kernel of an operator acting on a space of complex functions (defined on a bi-dimensional vector space)

$$\hat{\mathcal{K}}[f(\vec{k})] = \int d^2 \vec{k}' \mathcal{K}(\vec{k}, \vec{k}') f(\vec{k}')$$

- We solve the eigenvalue problem for the Kernel

$$\text{Eigenvalues} \longrightarrow \omega_n(\nu) = \bar{\alpha}_s \chi_n(\nu), \quad \bar{\alpha}_s = \frac{\alpha_s N}{\pi}$$

$$\text{Eigenfunctions} \longrightarrow \phi_\nu^n(\vec{q}) = \frac{1}{\pi\sqrt{2}} (\vec{q}^2)^{-\frac{1}{2}+i\nu} e^{in\theta}$$

- Then we are able to reconstruct the $G_\omega(\vec{q}_1, \vec{q}_2)$

$$G_\omega(\vec{q}_1, \vec{q}_2) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} d\nu \left(\frac{\vec{q}_1^2}{\vec{q}_2^2} \right)^{i\nu} \frac{e^{in(\theta_1 - \theta_2)}}{2\pi^2 q_1 q_2} \frac{1}{\omega - \bar{\alpha}_s \chi(n, \nu)} \longrightarrow G_s(\vec{q}_1, \vec{q}_2) \sim s^{\omega_0}$$

$$\omega_0 = 4\bar{\alpha}_s \ln 2 \simeq 0.40 \text{ for } \alpha_s = 0.15$$

BFKL at NLLA in a nutshell

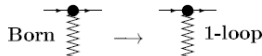
- Resummation of subleading logarithms means **new kinematics**
 1. Multi-Regge kinematics (MRK)
 2. Quasi multi-Regge kinematics (QMRK)
- Production amplitudes keep the simple factorized form

$$\Re \mathcal{A}_{AB}^{\tilde{A}\tilde{B}+n} = 2s \Gamma_{\tilde{A}\tilde{A}}^{c_1} \left(\prod_{i=1}^n \gamma_{c_i c_{i+1}}^{P_i}(q_i, q_{i+1}) \left(\frac{s_i}{s_0} \right)^{\omega(t_i)} \frac{1}{t_i} \right) \frac{1}{t_{n+1}} \left(\frac{s_{n+1}}{s_0} \right)^{\omega(t_{n+1})} \Gamma_{\tilde{B}\tilde{B}}^{c_{n+1}}$$

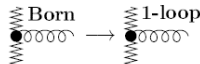
- **Multi-Regge kinematics** \rightarrow previous quantity must be calculated at 1-loop (one α_s more)

- $\omega(1) \rightarrow \omega(2)$

- $\Gamma_{P'P}^c(\text{Born}) \rightarrow \Gamma_{P'P}^c(\text{1-loop})$



- $\gamma_{c_i c_{i+1}}^{G_i}(\text{Born}) \rightarrow \gamma_{c_i c_{i+1}}^{G_i}(\text{1-loop})$

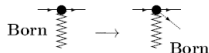


[V.S. Fadin, L.N. Lipatov (1989)]

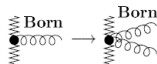
BFKL at NLLA in a nutshell

- **Quasi multi-Regge kinematics** \rightarrow A pair of particles, but only one!, may have longitudinal Sudakov variables of the same order (one logarithm less)

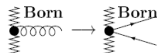
- $\Gamma_{P'P}^c(\text{Born}) \rightarrow \Gamma_{\{f\}P}^c(\text{Born})$



- $\gamma_{c_i c_{i+1}}^{G_i}(\text{Born}) \rightarrow \gamma_{c_i c_{i+1}}^{Q\bar{Q}}(\text{Born})$

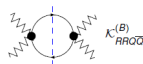
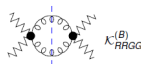


- $\gamma_{c_i c_{i+1}}^{G_i}(\text{Born}) \rightarrow \gamma_{c_i c_{i+1}}^{GG}(\text{Born})$



- 3 new contributions to the kernel

$$\mathcal{K} = \mathcal{K}_{RRG}^{\text{Born}} + \mathcal{K}_{RRG}^{1\text{-loop}} + \mathcal{K}_{RRGG}^{\text{Born}} + \mathcal{K}_{RRQ\bar{Q}}^{\text{Born}}$$



NLO Higgs impact factor: Real corrections

- NLO definition of the impact factor

$$\Phi_{AA}(\vec{q}_1; s_0) = \left(\frac{s_0}{\vec{q}_1^2} \right)^{\omega(-\vec{q}_1^2)} \sum_{\{f\}} \int \theta(s_\Lambda - s_{AR}) \frac{ds_{AR}}{2\pi} d\rho_f \Gamma_{\{f\}A}^c \left(\Gamma_{\{f\}A}^{c'} \right)^* \langle cc' | \hat{\mathcal{P}}_0 | 0 \rangle$$

$$- \frac{1}{2} \int d^{D-2} q_2 \frac{\vec{q}_1^2}{\vec{q}_2^2} \Phi_{AA}^{(0)}(\vec{q}_2) \mathcal{K}_r^{(0)}(\vec{q}_2, \vec{q}_1) \ln \left(\frac{s_\Lambda^2}{s_0(\vec{q}_2 - \vec{q}_1)^2} \right)$$

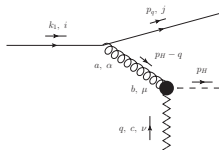
$s_\Lambda \rightarrow$ rapidity regulator

$\omega(-\vec{q}_1^2) \rightarrow$ 1-loop Regge trajectory

- Quark initiated contribution

$$\frac{d\Phi_{qq}^{\{Hq\}}(z_H, \vec{p}_H, \vec{q})}{dz_H d^2\vec{p}_H} = \frac{g^2 g_H^2 \sqrt{N^2 - 1}}{16N(2\pi)^{D-1} z_H} \left[\frac{4(1 - z_H) [(\vec{q} - \vec{p}_H) \cdot \vec{q}]^2 + z_H^2 \vec{q}^2 (\vec{q} - \vec{p}_H)^2}{[(\vec{q} - \vec{p}_H)^2]^2} \right]$$

- Rapidity** divergence absent $\implies s_\Lambda \rightarrow \infty$
- Collinear** divergence: $(\vec{q} - \vec{p}_H) \rightarrow \vec{0}$
- Gauge invariance: $d\Phi_{gg}^{\{gH\}}|_{\vec{q}^2=0} \rightarrow 0$



NLO Higgs impact factor: Real corrections

- Gluon initiated contribution

$$\frac{d\Phi_{gg}^{\{H\}}(z_H, \vec{p}_H, \vec{q}; s_0)}{dz_H d^2 p_H} = \frac{g^2 g_H^2 C_A}{8(2\pi)^{D-1} (1-\epsilon) \sqrt{N^2-1}}$$

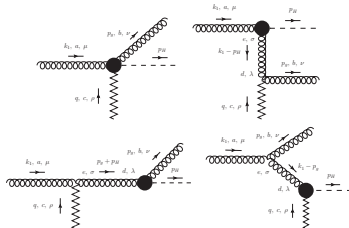
$$\times \left\{ \frac{2}{z_H(1-z_H)} \left[2z_H^2 + \frac{(1-z_H)z_H m_H^2 (\vec{q} \cdot \vec{r}) [z_H^2 - 2(1-z_H)\epsilon] + 2z_H^3 (\vec{p}_H \cdot \vec{r})(\vec{p}_H \cdot \vec{q})}{\vec{r}^2 [(1-z_H)m_H^2 + \vec{p}_H^2]} - \frac{2z_H^2(1-z_H)m_H^2}{[(1-z_H)m_H^2 + \vec{p}_H^2]} \right. \right.$$

$$\left. - \frac{(1-z_H)z_H m_H^2 (\vec{q} \cdot \vec{r}) [z_H^2 - 2(1-z_H)\epsilon] + 2z_H^3 (\vec{\Delta} \cdot \vec{r})(\vec{\Delta} \cdot \vec{q})}{\vec{r}^2 [(1-z_H)m_H^2 + \vec{\Delta}^2]} - \frac{2z_H^2(1-z_H)m_H^2}{[(1-z_H)m_H^2 + \vec{\Delta}^2]} \right]$$

$$+ \frac{(1-\epsilon)z_H^2(1-z_H)^2 m_H^4}{2} \left(\frac{1}{[(1-z_H)m_H^2 + \Delta^2]} + \frac{1}{[(1-z_H)m_H^2 + \vec{p}_H^2]} \right)^2 - \frac{2z_H^2 (\vec{p}_H \cdot \vec{\Delta})^2 - 2\epsilon(1-z_H)^2 z_H^2 m_H^4}{[(1-z_H)m_H^2 + \vec{p}_H^2] [(1-z_H)m_H^2 + \Delta^2]} \Big]$$

$$+ \frac{2\vec{q}^2}{\vec{r}^2} \left[\frac{z_H}{1-z_H} + z_H(1-z_H) + 2(1-\epsilon) \frac{(1-z_H)(\vec{q} \cdot \vec{r})^2}{z_H \vec{q}^2 \vec{r}^2} \right] \theta \left(s_\Lambda - \frac{(1-z_H)m_H^2 + \vec{\Delta}^2}{z_H(1-z_H)} \right)$$

- $\vec{\Delta} = \vec{p}_H - z_H \vec{q}$ $\vec{r} = \vec{q} - \vec{p}_H$
- **Rapidity** divergence $\rightarrow s_\Lambda$ still present
- **Soft** and **Collinear** divergences
- Gauge invariance:
 $d\Phi_{gg}^{\{gH\}}|_{\vec{q}^2=0} \rightarrow 0$



Agreement with independent calculation in the Lipatov effective action framework

[M. Hentschinski, K. Kutak, A. van Hameren (2021)]

PDF and α_s counterterms in the (n, ν) -space

- 1-loop α_s running produces the UV-counterterm

$$\frac{d\Phi_{PP}^{\{H\}}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2 \vec{p}_H} \Big|_{\text{coupling c.t.}} = \frac{d\Phi_{PP}^{\{H\}(0)}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2 \vec{p}_H} \frac{\bar{\alpha}_s}{2\pi} \left(\frac{\vec{p}_H^2}{\mu^2} \right)^{-\epsilon} \\ \times \left(\frac{11C_A}{3} - \frac{2n_f}{3} \right) \left(-\frac{1}{\epsilon} + \ln \left(\frac{\mu_R^2}{\vec{p}_H^2} \right) \right)$$

- PDF counter terms produced through DGLAP evolution equations

$$\frac{d\Phi_{PP}^{\{H\}}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2 \vec{p}_H} \Big|_{\text{P}_{qg} \text{ c.t.}} = -\frac{1}{f_g(x_H)} \frac{d\Phi_{PP}^{\{H\}(0)}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2 \vec{p}_H} \frac{\bar{\alpha}_s}{2\pi} \left(\frac{\vec{p}_H^2}{\mu^2} \right)^{-\epsilon} \\ \times \left(-\frac{1}{\epsilon} + \ln \left(\frac{\mu_F^2}{\vec{p}_H^2} \right) \right) \int_{x_H}^1 \frac{dz_H}{z_H} \left[P_{gq}(z_H) \sum_{a=q\bar{q}} f_a \left(\frac{x_H}{z_H}, \mu_F \right) \right]$$

$$\frac{d\Phi_{PP}^{\{H\}}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2 \vec{p}_H} \Big|_{\text{P}_{gg} \text{ c.t.}} = -\frac{1}{f_g(x_H)} \frac{d\Phi_{PP}^{\{H\}(0)}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2 \vec{p}_H} \frac{\bar{\alpha}_s}{2\pi} \left(\frac{\vec{p}_H^2}{\mu^2} \right)^{-\epsilon} \\ \times \left(-\frac{1}{\epsilon} + \ln \left(\frac{\mu_F^2}{\vec{p}_H^2} \right) \right) \int_{x_H}^1 \frac{dz_H}{z_H} \left[P_{gg}(z_H) f_g \left(\frac{x_H}{z_H}, \mu_F \right) \right]$$

$$P_{gq}(z) = C_F \frac{1+(1-z)^2}{z}, \quad P_{gg}(z) = 2C_A \left(\frac{z}{(1-z)_+} + \frac{(1-z)}{z} + z(1-z) \right) + \frac{11C_A - 2n_f}{6} \delta(1-z)$$

Real quark and virtual contribution in the (n, ν) -space

- BFKL counterterm + Rapidity divergent part in the real gluon NLO contribution in the (n, ν) -space

$$\frac{d\Phi_{PP}^{\text{BFKL}}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2 p_H} = \frac{d\Phi_{PP}^{\{H\}(0)}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2 \vec{p}_H} \frac{\bar{\alpha}_s}{2\pi} \left(\frac{\vec{p}_H^2}{\mu^2} \right)^{-\epsilon} \times \left\{ \frac{C_A}{\epsilon^2} + \frac{C_A}{\epsilon} \ln \left(\frac{\vec{p}_H^2}{s_0} \right) - 2 \frac{C_A}{\epsilon} \ln(1 - x_H) + \mathcal{O}(\epsilon^0) \right\}$$

- Projection of the virtual contribution

$$\frac{d\Phi_{PP}^{\{H\}(1)}(x_H, \vec{p}_H, n, \nu; s_0)}{dx_H d^2 \vec{p}_H} = \frac{d\Phi_{PP}^{\{H\}(0)}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2 \vec{p}_H} \frac{\bar{\alpha}_s}{2\pi} \left(\frac{\vec{p}_H^2}{\mu^2} \right)^{-\epsilon} \left[-\frac{C_A}{\epsilon^2} + \frac{11C_A - 2n_f}{6\epsilon} - \frac{C_A}{\epsilon} \ln \left(\frac{\vec{p}_H^2}{s_0} \right) - \frac{5n_f}{9} + C_A \left(2 \Re e \left(\text{Li}_2 \left(1 + \frac{m_H^2}{\vec{p}_H^2} \right) \right) + \frac{\pi^2}{3} + \frac{67}{18} \right) + 11 \right]$$

- Projection of the real quark contribution

$$\frac{d\Phi_{PP}^{\{Hq\}}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2 \vec{p}_H} = \frac{1}{f_g(x_H)} \frac{d\Phi_{PP}^{\{H\}(0)}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2 p_H} \frac{\bar{\alpha}_s}{2\pi} \left(\frac{\vec{p}_H^2}{\mu^2} \right)^{-\epsilon} \times \int_{x_H}^1 \frac{dz_H}{z_H} \sum_{a=q\bar{q}} f_a \left(\frac{x_H}{z_H}, \mu_F \right) \left\{ -\frac{1}{\epsilon} C_F \left(\frac{1+(1-z_H)^2}{z_H} \right) + \mathcal{O}(\epsilon^0) \right\}$$

Real gluon contribution in the (n, ν) -space

- “Plus” term

$$\frac{d\Phi_{PP}^{\{Hg\}^{\text{plus}}}(x_H, \vec{p}_H, n, \nu; s_0)}{dx_H d^2\vec{p}_H} = -\frac{1}{f_g(x_H)} \frac{d\Phi_{PP}^{\{H\}^{(0)}}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2\vec{p}_H} \frac{\bar{\alpha}_s}{2\pi} \left(\frac{\vec{p}_H^2}{\mu^2}\right)^{-\epsilon}$$

$$\times \int_{x_H}^1 \frac{dz_H}{z_H} f_g\left(\frac{x_H}{z_H}\right) 2C_A \frac{z_H}{(1-z_H)_+} \left[\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \right]$$

- $(1 - x_H)$ -term

$$\frac{d\Phi_{PP}^{\{Hg\}^{(1-x_H)}}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2\vec{p}_H} = \frac{d\Phi_{PP}^{\{H\}^{(0)}}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2\vec{p}_H} \frac{\bar{\alpha}_s}{2\pi} \left(\frac{\vec{p}_H^2}{\mu^2}\right)^{-\epsilon}$$

$$\times \underline{2C_A \ln(1 - x_H)} \left[\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \right]$$

- Collinear part of the remaining term

$$\frac{d\Phi_{PP}^{\{Hg\}^{\text{coll}}}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2\vec{p}_H} = \frac{1}{f_g(x_H)} \frac{d\Phi_{PP}^{\{H\}^{(0)}}(x_H, \vec{p}_H, n, \nu)}{dx_H d^2\vec{p}_H} \frac{\bar{\alpha}_s}{2\pi} \left(\frac{\vec{p}_H^2}{\mu^2}\right)^{-\epsilon}$$

$$\times \int_{x_H}^1 \frac{dz_H}{z_H} f_g\left(\frac{x_H}{z_H}\right) \left\{ -\frac{1}{\epsilon} 2 C_A \left(z_H(1 - z_H) + \frac{(1 - z_H)}{z_H} \right) + \mathcal{O}(\epsilon^0) \right\}$$

- Complete cancellation of divergences $\rightarrow \epsilon = 0$

Finite part of the result in the (n, ν) -space

- The complete finite result is obtained in terms of hypergeometric functions and integrals of hypergeometric functions (with some shrewdness!), i.e.,

$$\begin{aligned}
 I_2(\gamma_1, n, \nu) &= \int \frac{d^{2-2\epsilon} \vec{q}}{\pi \sqrt{2}} (\vec{q}^2)^{i\nu - \frac{3}{2}} e^{in\phi(\vec{q}^2)} \gamma_1 \frac{1}{[(\vec{q} - \vec{p}_H)^2] [(1 - z_H)m_H^2 + (\vec{p}_H - z_H \vec{q})^2]} \\
 &= \frac{(\vec{p}_H^2)^{\frac{n}{2}} e^{in\phi_H}}{z_H^2 \sqrt{2} \pi^\epsilon} \left[\frac{\Gamma\left(\frac{5}{2} + \gamma_1 + \frac{n}{2} - i\nu + \epsilon\right) \Gamma\left(-\frac{1}{2} - \gamma_1 + \frac{n}{2} + i\nu - \epsilon\right)}{\Gamma(1 + n - \epsilon)} \right] \\
 &\times \int_0^1 d\Delta \left(\Delta + \frac{(1 - \Delta)}{z_H} \right)^n \left[\left(\Delta + \frac{(1 - \Delta)}{z_H^2} \right) \vec{p}_H^2 + \frac{(1 - \Delta)(1 - z_H)m_H^2}{z_H^2} \right]^{-\frac{5}{2} - \gamma_1 + i\nu - \frac{n}{2} - \epsilon} \\
 &\times {}_2F_1\left(-\frac{1}{2} - \gamma_1 + \frac{n}{2} + i\nu - \epsilon, \frac{5}{2} + \gamma_1 - i\nu + \frac{n}{2} + \epsilon, 1 + n - \epsilon, \zeta\right), \quad \zeta \xrightarrow{\Delta \rightarrow 1} 1
 \end{aligned}$$

- Extracting singular part

$$\begin{aligned}
 I_{2,as}(\gamma_1, n, \nu) &= \frac{(\vec{p}_H^2)^{-\frac{3}{2} - \gamma_1 + i\nu - \epsilon} e^{in\phi_H} \Gamma(1 + \epsilon)}{(1 - z_H) \sqrt{2} \pi^\epsilon} \frac{1}{(m_H^2 + (1 - z_H) \vec{p}_H^2)} \int_0^1 d\Delta (1 - \Delta)^{-\epsilon - 1} \\
 &= -\frac{1}{\epsilon} \frac{(\vec{p}_H^2)^{-\frac{3}{2} - \gamma_1 + i\nu - \epsilon} e^{in\phi_H} \Gamma(1 + \epsilon)}{(1 - z_H) \sqrt{2} \pi^\epsilon} \frac{1}{(m_H^2 + (1 - z_H) \vec{p}_H^2)}
 \end{aligned}$$

- Replacement: $I_2 = I_{2,as} + (I_2 - I_{2,as}) \equiv I_{2,as} + I_{2,reg}$