Higher-point functions in $\mathcal{N} = 4$ SYM from an integrable system

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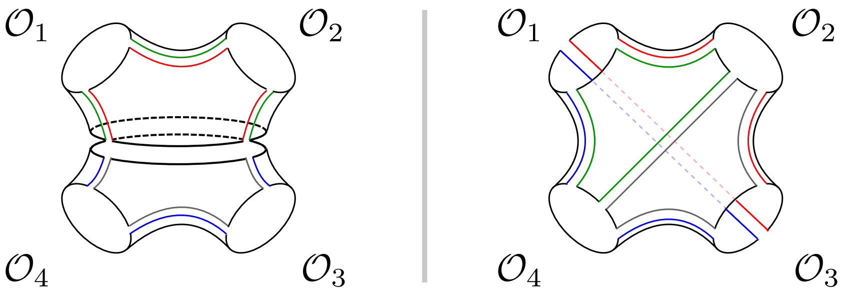
Review of B. Eden,Y. Jiang, D. le Plat, and A. Sfondrini, hep-th/1611.95436, 1710.10212

Introduction

- The **spectrum problem** in AdS/CFT is fairly well-understood (anomalous dimensions of composite operator $\sim$ energy levels of strings).
- **Three-point** functions by **hexagon operators** [Basso, Komatsu, Vieira (2015)]
- **Four-point** functions by **hexagon tilings** [Eden, Sfondrini (2016)], [Fleury, Komatsu (2016)]
- This is a promising **non-OPE** approach: no sum over intermediate physical states.



Second paper:

- **Non-planar** corrections and **multi-trace operators** by tiling **higher Riemann surfaces**
- Need to import colour factors into the **integrability** framework.

BMN operators [Berenstein, Maldacena, Nastase (2002)], [Minahan, Zarembo (2002)]

Half-BPS operators:

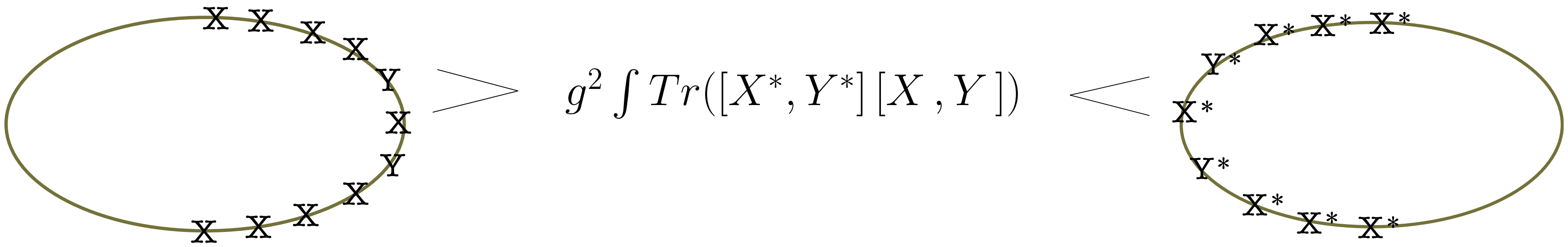
$$\mathcal{O}_L = \frac{1}{\sqrt{LN^L}} \mathrm{Tr}(Z^L)$$

with Z a scalar field.

$SU(2)$ sector **BMN-operators** with two scalar **excitations**:

$$\mathcal{O}_L^k = \mathrm{Tr}(Z^{L-k-2} Y Z^k Y)\,, \qquad Y \neq \bar{Z}\,.$$

Two-point function at $\mathbf{g^2}$:



Length 4: basis $\{\mathcal{O}_4^0, \mathcal{O}_4^1\}$. Mixing matrix (eigenvalues: **anomalous dimensions**)

$$\Gamma_1/N = \begin{pmatrix} 2 & -4 \\ -2 & 4 \end{pmatrix}\,, \qquad \gamma_{1,0} = 0,\ \underline{v}_0 = 2\,\mathcal{O}_4^0 + \mathcal{O}_4^1 = \mathrm{Tr}(Y\partial Z)^2 \mathrm{Tr}(Z^4)\,, \qquad \gamma_{1,1} = 6,\ \underline{v}_1 = \mathcal{O}_4^0 - \mathcal{O}_4^1$$

Spin chains and Bethe ansatz

- **Planar single-trace** operators
- **Heisenberg chain:** identify $X = \downarrow$, $Y = \uparrow$, the four-vertex as $\mathbb{I} - \mathbb{P}$.
- **momentum/rapidity** $u_i = \frac{1}{2}\cot\left(\frac{p_i}{2}\right)$ for each excitation

Shift operator and **scattering matrix**

$$\exp(i\,p) = \left(\frac{u+\frac{i}{2}}{u-\frac{i}{2}}\right), \qquad S(u_2,u_1) = \frac{u_2-u_1+i}{u_2-u_1-i}\,.$$

Moving the first magnon once around the chain implies the **Bethe equations**

$$\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^L \frac{u_k-u_j+i}{u_k-u_j-i} = 1\,.$$

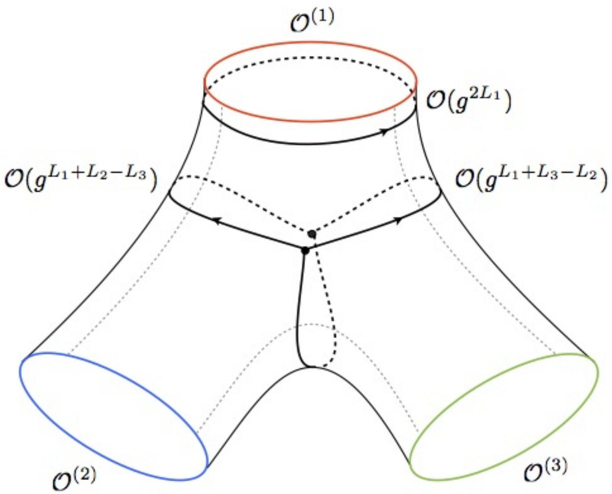
Momentum conservation: $p_1+p_2=0 \Leftrightarrow u_1+u_2=0 \Rightarrow$

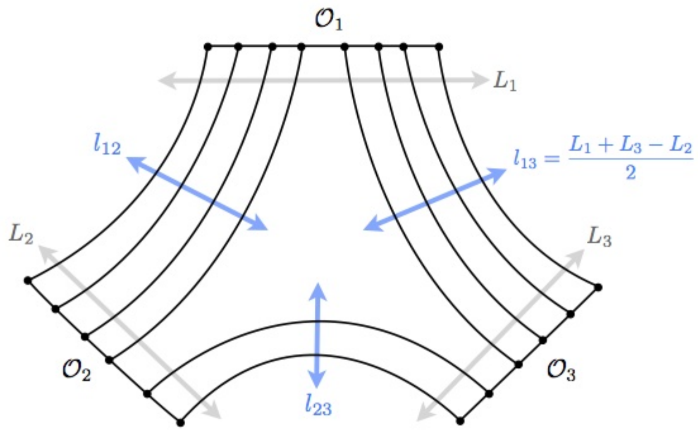
$$\left(\frac{u_1+\frac{i}{2}}{u_1-\frac{i}{2}}\right)^{L-1} = 1\,, \qquad u_2 = -u_1\,.$$

Energy or anomalous dimension

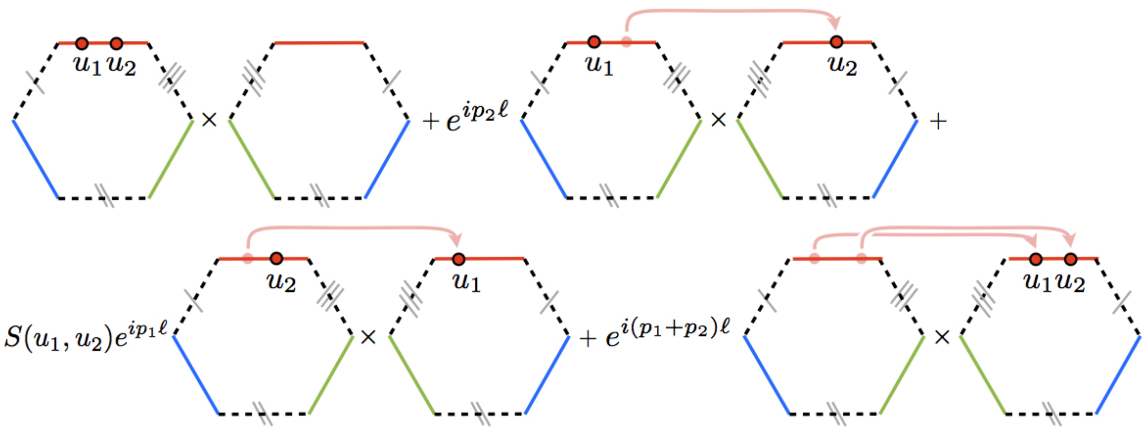
$$\gamma_1 = \sum_{i=1}^{2} \frac{1}{u_i^2+\frac{1}{4}}\,.$$

Three-point functions by hexagons [Basso, Komatsu, Vieira (2015)]





- **Input**: **Bethe roots** characterising the three **operators**.
- Split into **top** and **bottom hexagon. Entangled state:**



$$\mathcal{A} = \sum_{\alpha\cup\bar\alpha=\{u_i\}} w(\alpha,\bar\alpha)(-1)^{|\alpha|}\, \mathfrak{h}_{Y\ldots Y}(\alpha)\, \mathfrak{h}_{Y\ldots Y}(\bar\alpha)$$

$$\mathfrak{h}_{Y_1\ldots Y_n} = \Big(\prod_{i<j} h_{ij}\Big)\, \langle\, \phi^1_n \ldots \phi^1_1 \,|\, S \,|\, \bar\phi^{2'}_1 \ldots \bar\phi^{2'}_n \,\rangle\,, \qquad h_{12} = \frac{x_1^- - x_2^-}{x_1^- - x_2^+}\, \frac{1 - 1/(x_1^- x_2^+)}{1 - 1/(x_1^+ x_2^+)}\, \frac{1}{\sigma_{12}}\,,$$

$$x^\pm = x(u \pm i/2), \quad x(u) = \frac{u}{2}\big(1 + \sqrt{1 - 2\,g^2/u^2}\big)\,.$$

- Use **S** on **left or right**: $Y \to \phi^1 \bar\phi^{2'}$, $S\,|\bar\phi^{2'}_1 \bar\phi^{2'}_2 \bar\phi^{2'}_3\rangle = A_{12}A_{13}A_{23}\,|\bar\phi^{2'}_3 \bar\phi^{2'}_2 \bar\phi^{2'}_1\rangle$. [Beisert (2005)]
- Contract using $\langle \psi^\alpha | \bar\psi^{\dot\beta}\rangle = \epsilon^{\alpha\dot\beta}$, $\langle \phi^a | \bar\phi^{b'}\rangle = \epsilon^{ab'}$.

- **Lüscher** corrections: X_a **bound state** in **mirror kinematics**.

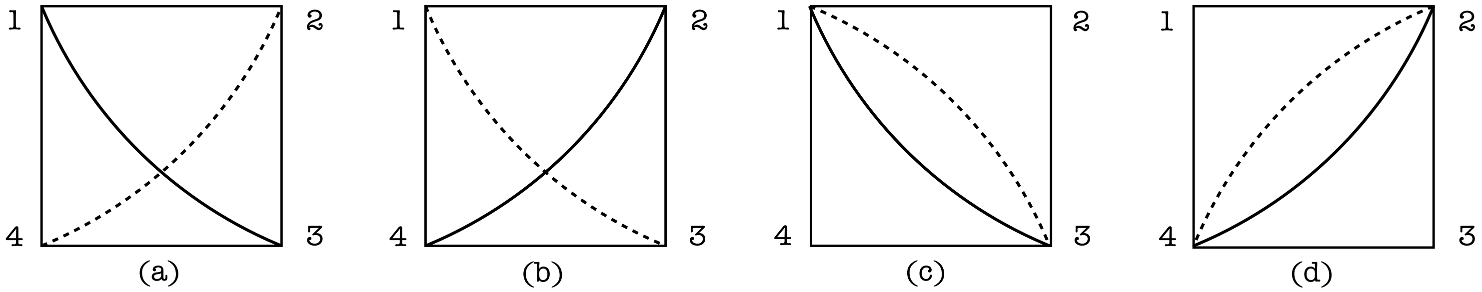
$$\delta\mathcal{A} = \sum_{a>0} \int \frac{du}{2\pi}\, \mu(u)\; \omega(\alpha,\bar\alpha)\; \mathfrak{h}_{XY\ldots Y}(u^{n\gamma},\alpha)\; \mathfrak{h}_{Y\ldots Y\bar X}(\bar\alpha, u^{-n\gamma})\,, \qquad \mu(u) = \frac{a\,(g^2)^{l+1}}{(2\,u^{+a}\,u^{-a})^{l+2}} + \ldots$$

- Mirror measure: **bridge length** as **exponent of $\mathbf{g^2}$**.
- Further **g**-dependence in $\mathbf{x}^\pm$.

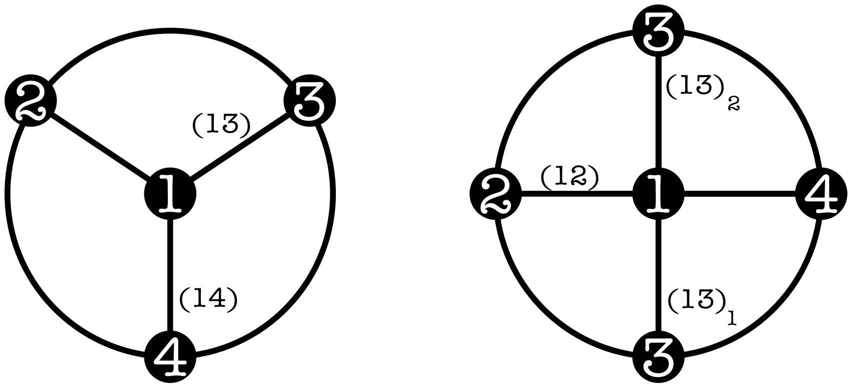
Soft cushions

- **Planar $SU(N)$ tree diagrams**
- Operators: **beads on a ring. Connect on** the surface of a **sphere**.
- **Tiling by hexagons** [Basso,Komatsu,Vieira (2015)]

Four classes of **four-point graphs** with **edge widths/bridge lengths** $\{l_{ij}\}$:



Looking onto point 1:



BMN-(BPS)^3 four-point functions

Due to flavour

$$\langle \mathcal{B}^k_{L_1}(x_1)\, \mathcal{O}_{L_2}(x_2)\, \mathcal{O}_{L_3}(x_3)\, \mathcal{O}_{L_4}(x_4) \rangle \;=\; 0\,.$$

Co-moving vacuum [Drukker, Plefka (2009)]

$$\hat{\mathcal{O}}_L = \frac{1}{\sqrt{LN^L}}\,\mathrm{Tr}(\hat{Z}^L)\,, \qquad \hat{Z} = Z + a^2\bar{Z} + a(Y - \bar{Y})\,, \qquad x_i^\nu = \delta_3^\nu\, a_i$$

with **a** the **position** along a **line** in Minkowski space.

$SO(6)$ covariantly

$$\hat{Z} = z^\mu \phi_\mu\,, \quad z^\mu = \big((1+a^2), 0, 0, 2a, 0, i(1-a^2)\big)\,.$$

$$\langle \hat{Z}(a_1)\hat{Z}(a_2)\rangle = \frac{(a_1-a_2)^2}{4\pi^2(a_1-a_2)^2} = \frac{1}{4\pi^2}$$

Drop $4\pi^2$:

$$\langle Z(0)\hat{Z}(a_2)\rangle = 1, \qquad \langle Y(0)\hat{Z}(a_2)\rangle = -\frac{1}{a_2}$$

- The four-point functions we wish to construct will be **homogeneous of order -2 in $\mathbf{a_i}$**.

- On a line: **one cross-ratio**

BMN operator at $a_1 = 0$.

We can write all results using

$$a_{23} = \frac{1}{a_2} - \frac{1}{a_3}, \qquad a_{34} = \frac{1}{a_3} - \frac{1}{a_4}.$$

Notation:

$$G(7^2; 2, 3, 2) = \langle \mathcal{B}_7^2(0) \mathcal{O}_2(a_2) \mathcal{O}_3(a_3) \mathcal{O}_2(a_4) \rangle$$

etc. and

$$G(\dots) = c * \underline{v}.(a_{23}^2,\ a_{23}a_{34},\ a_{34}^2).$$

e.g.

$$G(4; 2, 2, 2) = 4\sqrt{\tfrac{2}{3}}\,(1, 1, 1).(a_{23}^2,\ a_{23}a_{34},\ a_{34}^2).$$

Orbital part for the hexagon:

$$\mathfrak{h}(\alpha) \to \widehat{\mathfrak{h}}(\alpha) = (a_{23})^{|\alpha|}\,\mathfrak{h}(\alpha), \quad a_{23} = \frac{1}{a_2} - \frac{1}{a_3}$$

	C	$\underline{\mathbf{v}}$
$G(4;222)$	$4\sqrt{\frac{2}{3}}$	$(1,1,1)$
$G(4;242)$	$\frac{8}{\sqrt{3}}$	$(1,0,1)$
$G(4;233)$	$\sqrt{6}$	$(2,2,3)$
$G(5;232)$	$\sqrt{6}$	$(3,2,3)$
$G(6^{\mp};222)$	$4\sqrt{2}$	$(1,1,1)$
$G(4;235)$	$\sqrt{10}$	$(2,4,5)$
$G(4;244)$	$8\sqrt{\frac{2}{3}}$	$(1,1,2)$
$G(4;343)$	$2\sqrt{3}$	$(3,2,3)$
$G(5;252)$	$3\sqrt{10}$	$(1,0,1)$
$G(5;234)$	$2\sqrt{3}$	$(3,4,7)$
$G(5;333)$	$9\sqrt{6}$	$(1,1,1)$
$G(6^{\mp};242)$	$\frac{4(1\pm\sqrt{5})}{\sqrt{5}}$	$(2,1,2)$
$G(6^{\mp};233)$	$\frac{3(1\pm\sqrt{5})}{\sqrt{10}}$	$(4,4,6\pm\sqrt{5})$
$G(7';232)$	$2\sqrt{6}$	$(2,1,2)$
$G(7'';232)$	$6\sqrt{2}$	$(1,1,1)$

Mirror magnons as Yang-Mills lines

Coordinates of [Fleury, Komatsu (2016)]:

$$\alpha_1 = \bar{\alpha}_1 = z_1 = \bar{z}_1 = 0, \qquad \alpha_3 = \bar{\alpha}_3 = z_3 = \bar{z}_3 = 1, \qquad \alpha_4 = \bar{\alpha}_4 = z_4 = \bar{z}_4 = \infty,$$

$$\alpha_2 = \alpha, \quad \bar{\alpha}_2 = \bar{\alpha}, \qquad z_2 = z, \quad \bar{z}_2 = \bar{z}\,.$$

Hence $\mathbf{x_{i4}^2}$, $\mathbf{y_{i4}^2}$ is **scaled away** and

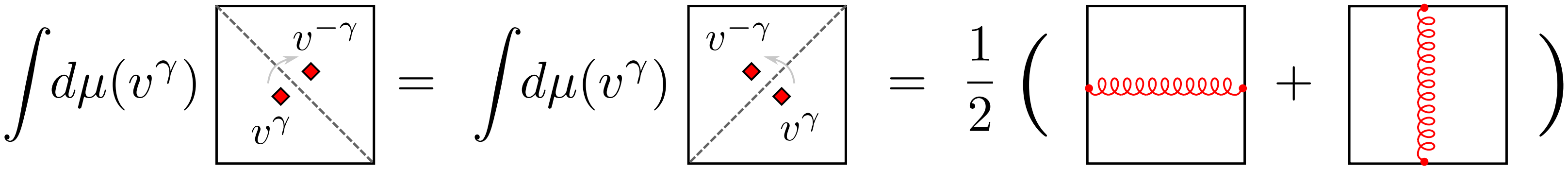
$$x_{13}^2 = y_{13}^2 = 1\,, \qquad x_{12}^2 = z\bar{z} \quad x_{23}^2 = (1-z)(1-\bar{z})\,, \qquad y_{12}^2 = \alpha\bar{\alpha}\,, \quad y_{23}^2 = (1-\alpha)(1-\bar{\alpha})\,.$$

Results for $\tilde{F}_{ij;kl}$:

$$\tilde{F}_{12} = z\bar{z}\left[\frac{1}{\alpha} + \frac{1}{\bar{\alpha}} - \frac{1}{z} - \frac{1}{\bar{z}}\right], \; \tilde{F}_{13} = \alpha + \bar{\alpha} - z - \bar{z}\,, \; \tilde{F}_{23} = -(1-z)(1-\bar{z})\left[\frac{1}{1-\alpha} + \frac{1}{1-\bar{\alpha}} - \frac{1}{1-z} - \frac{1}{1-\bar{z}}\right]$$

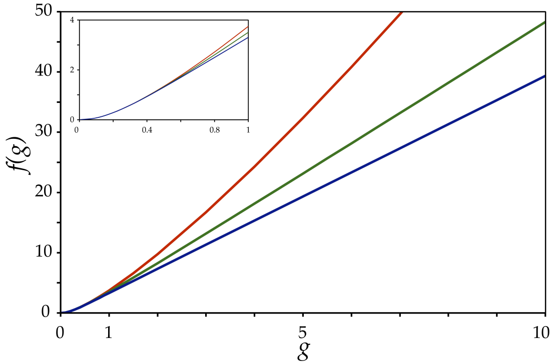
Square with corners **1243**: **field theory = integrability**, c.f. [Fleury, Komatsu (2016)]

$$\tilde{F}_{12;34} + \tilde{F}_{13;24} = -2(z+\bar{z}) + \left(\frac{1}{\alpha} + \frac{1}{\bar{\alpha}}\right)(\alpha\bar{\alpha} + z\bar{z})$$



Conclusions

- **Hexagon tesselations** compute **flavour** and **combinatorics** in tree-level $\mathcal{N} = 4$ correlators.
- Unlike in the leading N spectrum, **colour factors** are **needed**!
- Colour dressed tilings can address **non-planar corrections** and **multitrace admixtures**.
- An analysis of the underlying **Feynman graphs** is required to understand **integrability**.
- We can study **general gauge groups** $U(N)$, $SO(N)$, $USp(N)$.
- Spectrum: an **interpolation from weak to strong** coupling is possible: e.g. **BES**



- Superior technique: **Quantum Spectral Curve** incorporating **wrapping**
- Tessellations: for now **S**-matrix picture. How to compute **Lüscher terms**?
- Resummation of **residues** akin to **MB method**

By **crossing**: [Basso, Komatsu, Vieira (2015)]

$$\begin{aligned}
\mathfrak{h}(\{\bar{Y}_1\},\{\},\{Y_3\}),\ \mathfrak{h}(\{Y_1\},\{\},\{\bar{Y}_3\}) &\to -1\,,\\
\mathfrak{h}(\{\bar{Y}_1,\bar{Y}_2\},\{\},\{\}),\ \mathfrak{h}(\{Y_1,Y_2\},\{\},\{\}) &\to \frac{u_1}{u_1+\frac{i}{2}}\,,\\
\mathfrak{h}(\{\bar{Y}_1,\bar{Y}_2\},\{\},\{Y_3\}),\ -\mathfrak{h}(\{Y_1,Y_2\},\{\},\{\bar{Y}_3\}) &\to \frac{u_1}{u_1+\frac{i}{2}}\,,\\
-\mathfrak{h}(\{\bar{Y}_1\},\{\},\{Y_3,Y_4\}),\ \mathfrak{h}(\{Y_1\},\{\},\{\bar{Y}_3,\bar{Y}_4\}) &\to \frac{u_3}{u_3+\frac{i}{2}}\,,\\
\mathfrak{h}(\{\bar{Y}_1,\bar{Y}_2\},\{\},\{Y_3,Y_4\}),\ \mathfrak{h}(\{Y_1,Y_2\},\{\},\{\bar{Y}_3,\bar{Y}_4\}) &\to \frac{u_1u_3}{(u_1+\frac{i}{2})(u_3+\frac{i}{2})}\,,\\
\mathfrak{h}(\{\bar{Y}_1\},\{\},\{\bar{Y}_3\}),\ \mathfrak{h}(\{Y_1\},\{\},\{Y_3\}) &\to \frac{u_1-u_3+i}{u_1-u_3}\,,\\
-\mathfrak{h}(\{\bar{Y}_1,\bar{Y}_2\},\{\},\{\bar{Y}_3\}),\ \mathfrak{h}(\{Y_1,Y_2\},\{\},\{Y_3\}) &\to \frac{u_1(u_1-u_3+i)(u_1+u_3-i)}{(u_1+\frac{i}{2})(u_1-u_3)(u_1+u_3)}\,,\\
-\mathfrak{h}(\{\bar{Y}_1\},\{\},\{\bar{Y}_3,\bar{Y}_4\}),\ \mathfrak{h}(\{Y_1\},\{\},\{Y_3,Y_4\}) &\to \frac{u_3(u_1-u_3+i)(u_1+u_3+i)}{(u_3+\frac{i}{2})(u_1-u_3)(u_1+u_3)}\,,\\
\mathfrak{h}(\{\bar{Y}_1,\bar{Y}_2\},\{\},\{\bar{Y}_3,\bar{Y}_4\}),\ \mathfrak{h}(\{Y_1,Y_2\},\{\},\{Y_3,Y_4\}) &\to \frac{u_1u_3\,[(u_1-u_3)^2+1]\,[(u_1+u_3)^2+1]}{(u_1+\frac{i}{2})(u_3+\frac{i}{2})(u_1-u_3)^2(u_1+u_3)^2}\,.\\
\mathfrak{h}(\{X_1\},\{\},\{\bar{X}_3\}),\ \mathfrak{h}(\{\bar{X}_1\},\{\},\{X_3\}) &\to \frac{i}{u_1-u_3},\\
\mathfrak{h}(\{X_1,X_2\},\{\},\{\bar{X}_3,\bar{X}_4\}),\ \mathfrak{h}(\{\bar{X}_1,\bar{X}_2\},\{\},\{X_3,X_4\}) &\to \frac{u_1u_3(1+2\,u_1^2+2\,u_3^2)}{(u_1+\frac{i}{2})(u_3+\frac{i}{2})(u_1-u_3)^2(u_1+u_3)^2}\,.
\end{aligned}$$