LONGITUDINAL DYNAMICS

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Accelerator Technologies course
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Scope and Summary of the lectures:

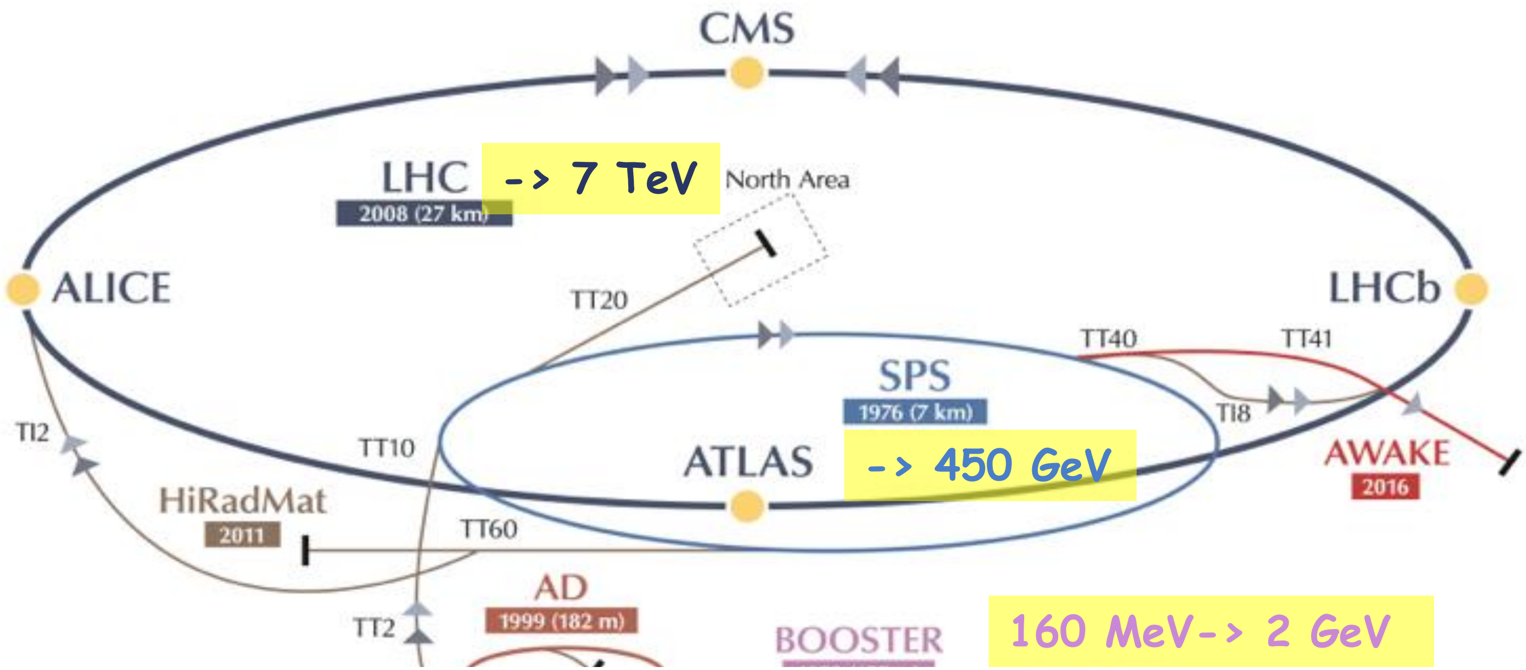
The goal of an accelerator is to provide a stable particle beam.

The particles nevertheless perform transverse betatron oscillations.
We will see that they also perform **oscillations** in the longitudinal plane and in energy.

We will look at the stability of these oscillations, and their dynamics.

- Acceleration methods
- Accelerating structures
- Linac: Phase Stability + Energy-Phase oscillations
- Circular accelerators: Cyclotron / Synchrotron
- Stability in a Synchrotron
- Longitudinal Phase Space Motion
- Bunch and Bucket
- Injection Matching + Filamentation
- RF manipulations in the PS

The CERN Accelerator Complex



- Linear accelerators scale in size and cost(!) ~linearly with the energy.
- Circular accelerators can each turn reuse
  - the **accelerating system**
  - the vacuum chamber 4
  - the bending/focusing magnets
  - beam instrumentation, …

-> economic solution to reach higher particle energies

But each accelerator has a limited energy range.

Particle types and acceleration

The accelerating system will depend upon the evolution of the particle velocity:

- electrons reach a constant velocity (~speed of light) at relatively low energy
- heavy particles reach a constant velocity only at very high energy
  - -> we need different types of resonators, optimized for different velocities
  - -> the **revolution frequency will vary**, so the RF frequency will be changing
  - -> magnetic field needs to follow the momentum increase

Particle rest mass m_0:

electron	0.511 MeV
proton	938 MeV
^{239}U	~220000 MeV

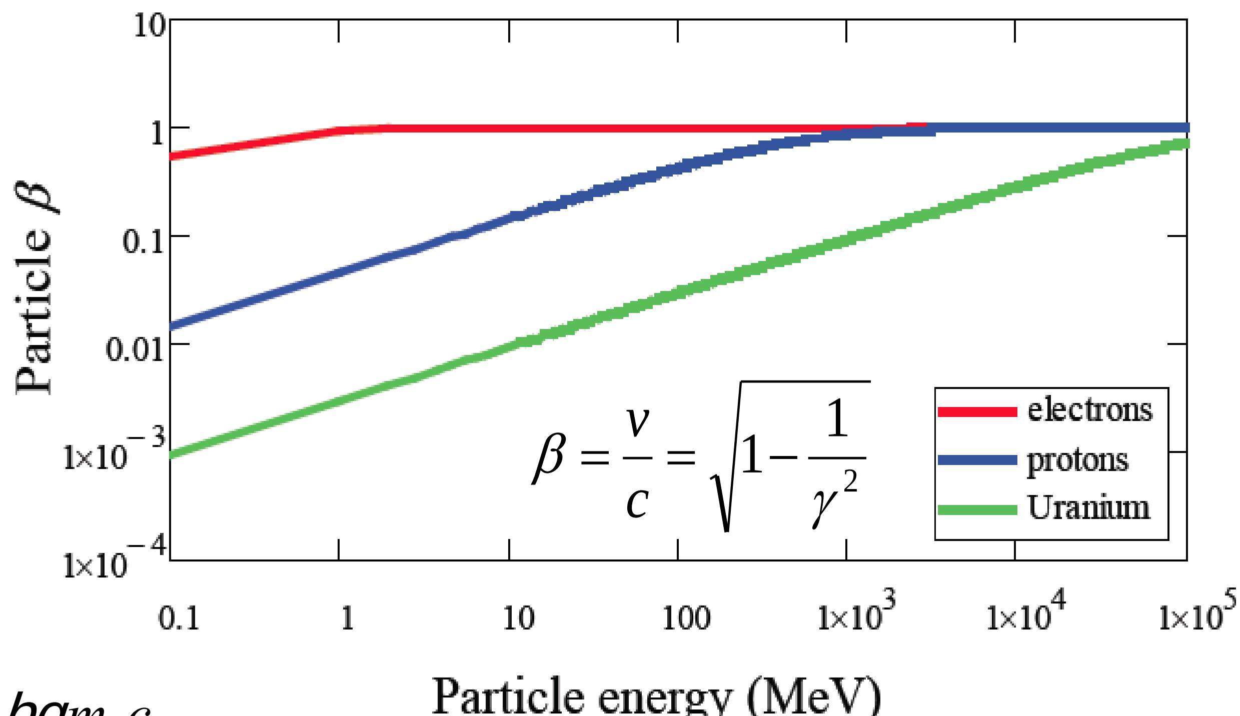
Total Energy: $E = \gamma m_0 c^2$

Relativistic gamma factor:

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$$

Momentum:

$$p = mv = \frac{E}{c^2}\beta c = \beta\frac{E}{c} = \beta\gamma m_0 c$$



Revolution frequency variation

The revolution and RF frequency will be changing during acceleration
Much more important for lower energies (values are kinetic energy - protons).

PS Booster: 50 MeV (β= 0.314) -> 1.4 GeV (β=0.915)
(pre LS2) 602 kHz -> 1746 kHz => **190% frequency increase**
(post LS2): 160 MeV (β= 0.520) -> 2 GeV (β=0.948) => **95% increase**

PS: 1.4 GeV (β=0.915) -> 25.4 GeV (β =0.9994)
437 KHz -> 477 kHz => **9% increase**
(post LS2): 2 GeV (β=0.948) -> 25.4 GeV (β =0.9994) => **5% increase**

SPS: 25.4 GeV -> 450 GeV (β=0.999998)
=> **0.06% frequency increase**

LHC: 450 GeV -> 7 TeV (β= 0.999999991)
=> only **$2\ 10^{-6}$ increase**

RF system needs more flexibility in lower energy accelerators.

Acceleration: May the force be with you

To accelerate, we need a **force in the direction of motion**!

Newton-Lorentz Force on a charged particle with charge e (electron, proton)

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{dt}} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

2nd term always perpendicular to motion => no acceleration

Hence, it is necessary to have an electric field E (preferably) along the direction of the initial momentum (z), which changes the momentum of the particle.

$$\frac{dp}{dt} = eE_z$$

The 2nd term - larger at high velocities – is used for:

- BENDING: generated by a magnetic field perpendicular to the plane of the particle trajectory. The bending radius ρ obeys to the relation :

$$\frac{p}{e} = B\rho$$

in practical units: $B\rho\,[\mathrm{Tm}] \approx \frac{p\,[\mathrm{GeV/c}]}{0.3}$

- FOCUSING: the bending effect is used to bring the particles trajectory closer to the axis, hence to increase the beam density.

Energy Gain

The acceleration increases the **momentum**, providing **kinetic energy** to the charged particles.

In relativistic dynamics, total energy E and momentum p are linked by

$$E^2 = E_0^2 + p^2c^2 \qquad (E = E_0 + W)$$

W kinetic energy
E_0 rest energy

Hence: $$dE = v\,dp \qquad \left(2E\,dE = 2c^2 p\,dp \Leftrightarrow dE = c^2 mv / E\,dp = v\,dp\right)$$

The rate of energy gain per unit length of acceleration (along z) is then:

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

and the kinetic energy gained from the field along the z path is:

$$dW = dE = eE_z\,dz \quad \rightarrow \quad W = e\int E_z\,dz = eV$$

where V is just a potential.

Unit of Energy

Today's accelerators and future projects work/aim at the TeV energy range.

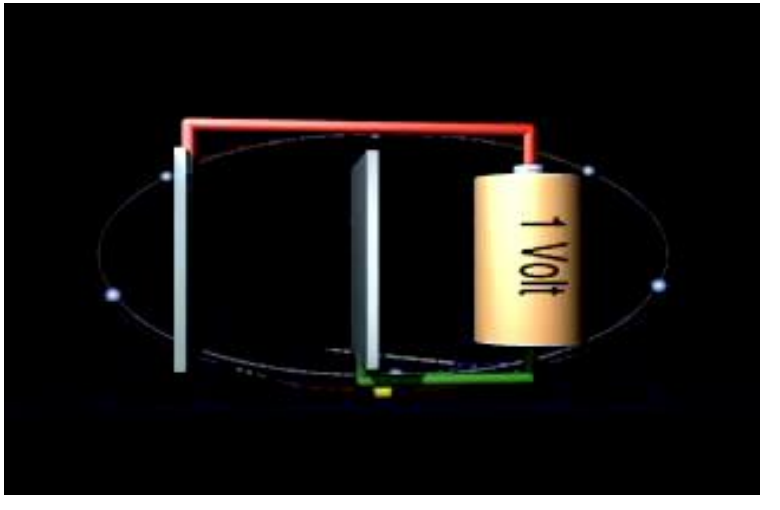
LHC:	7 TeV -> 14 TeV
CLIC:	3 TeV
FCC:	~100 TeV

In fact, this energy unit comes from acceleration:

> **1 eV (electron Volt)** is the energy that 1 elementary charge e (like one electron or proton) gains when it is accelerated in a potential (voltage) difference of 1 Volt.

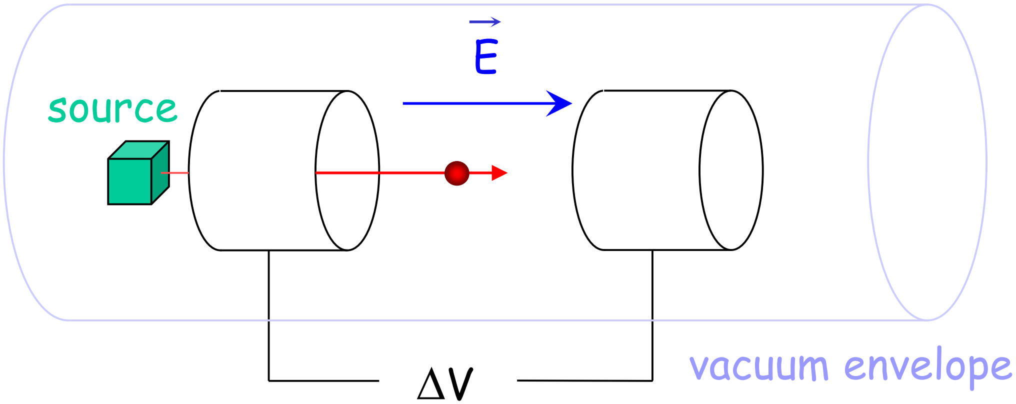
Basic Unit: **eV (electron Volt)**

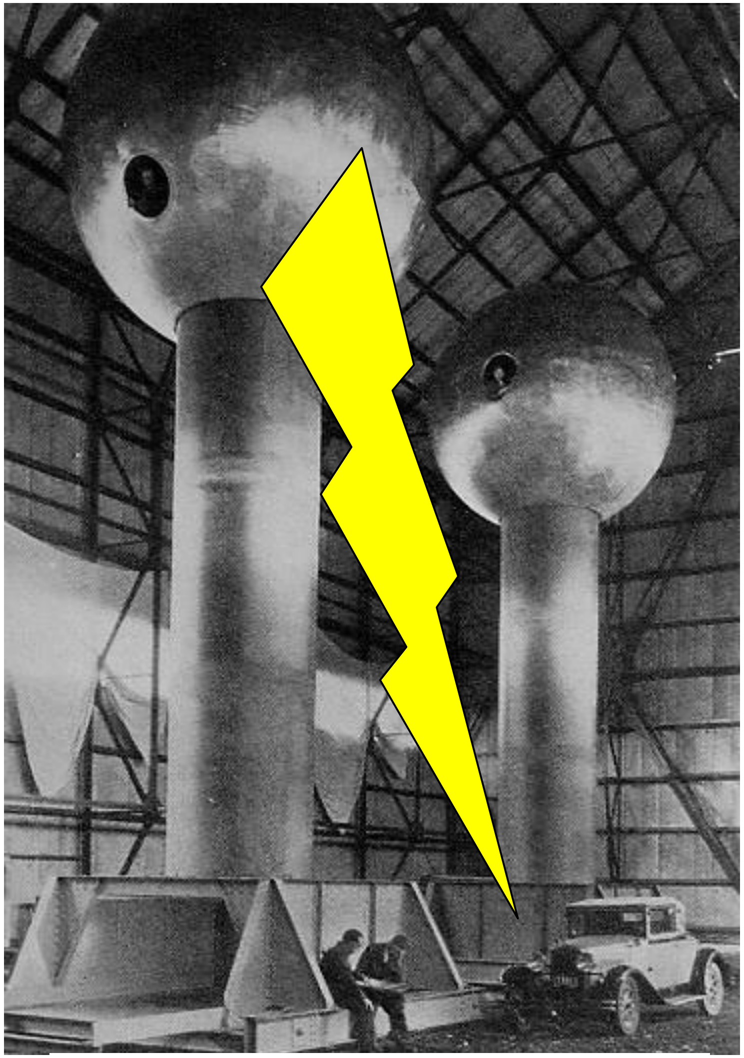
keV = 1000 eV = 10^3 eV
MeV = 10^6 eV
GeV = 10^9 eV
TeV = 10^{12} eV



LHC = ~450 Million km of batteries!!!
3x distance Earth-Sun

Electrostatic Acceleration





Van-de-Graaf generator at MIT

Electrostatic Field:

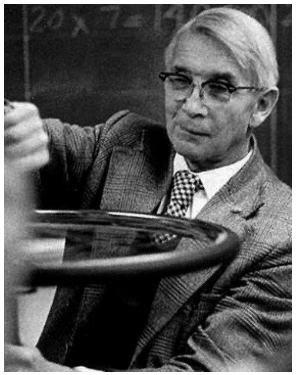
Force: $\vec{F} = \frac{d\vec{p}}{dt} = q\,\vec{E}$

Energy gain: $W = q\,\Delta V$

used for first stage of acceleration:
particle sources, electron guns,
x-ray tubes

Limitation: insulation problems
maximum high voltage (~ 10 MV)

Radio-Frequency (RF) Acceleration



G.Ising

Electrostatic acceleration limited by insulation possibilities => use time-varying fields

1924: Ising suggests drift-tubes with time-varying fields

1928: Widerøe builds first demonstration linac

R.Widerøe

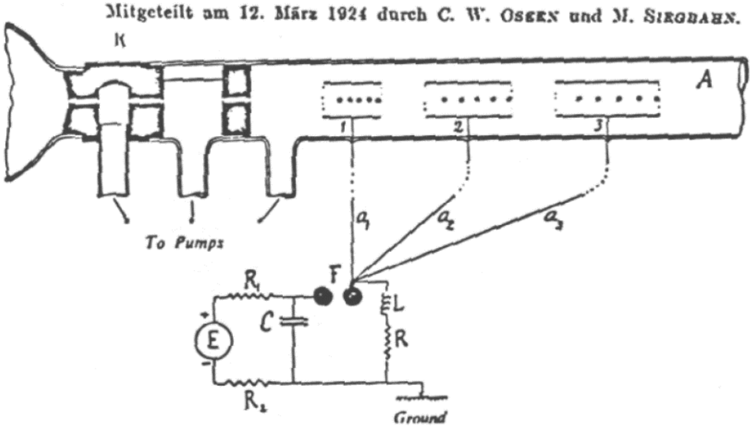
Prinzip einer Methode zur Herstellung von Kanalstrahlen hoher Voltzahl.

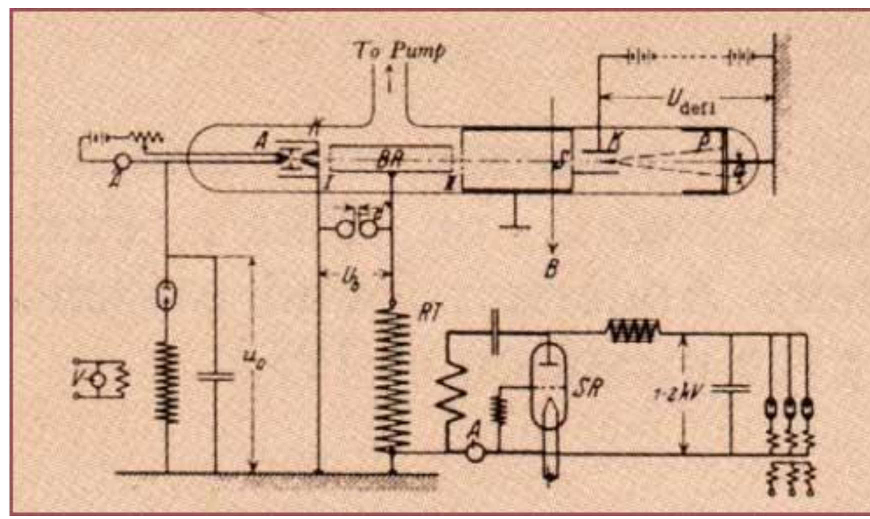
Von

GUSTAF ISING.

Mit 2 Figuren im Texte.

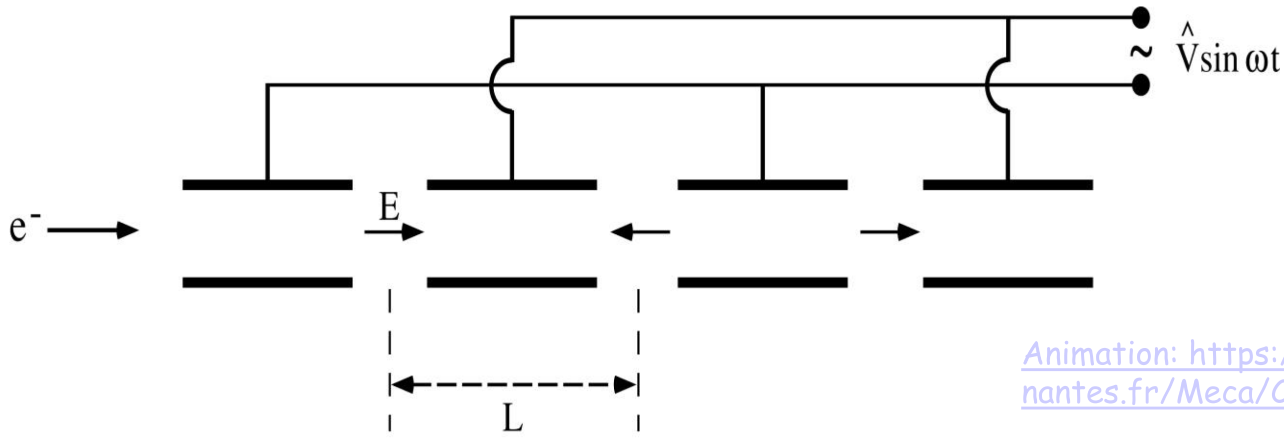
Mitgeteilt am 12. März 1924 durch C. W. Oseen und M. Siegbahn.





P.Lebrun

Radio-Frequency (RF) Acceleration



[Widerøe-type structure]

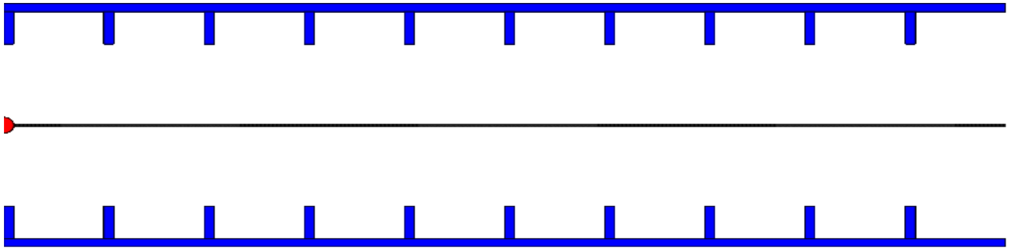
[Animation: https://phyanim.sciences.univ-nantes.fr/Meca/Charges/linac.php](https://phyanim.sciences.univ-nantes.fr/Meca/Charges/linac.php)

Cylindrical electrodes (drift tubes) separated by gaps and fed by a RF generator, as shown above, lead to an alternating electric field polarity

Synchronism condition → $L = v\,T/2$

v = particle velocity
T = RF period

Consequence: We can only accelerate bunched beam!

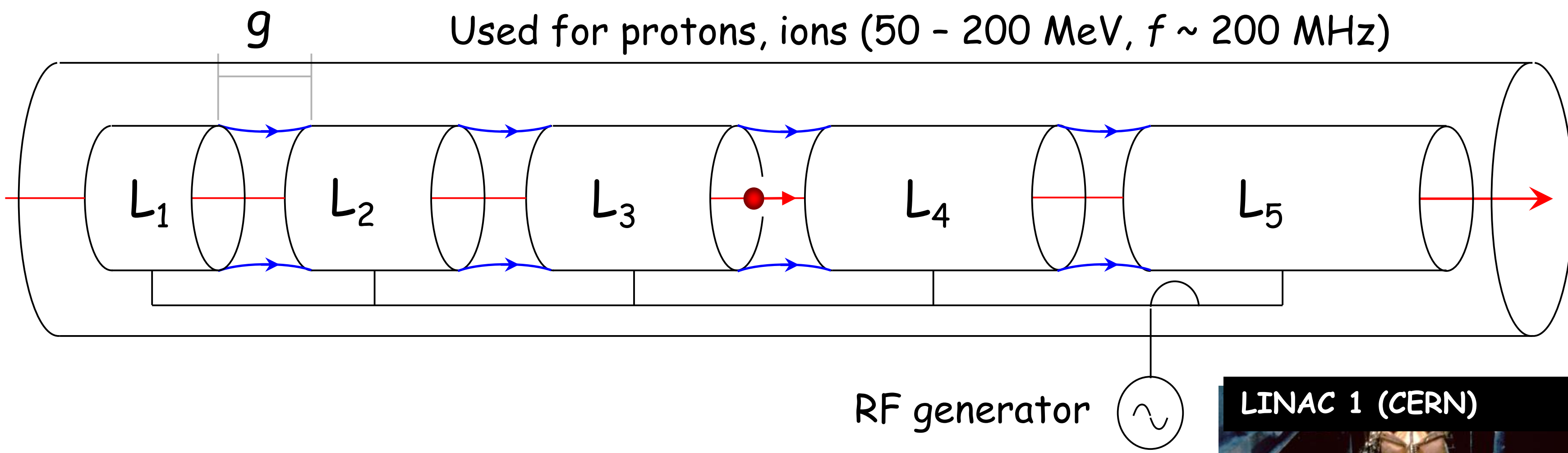


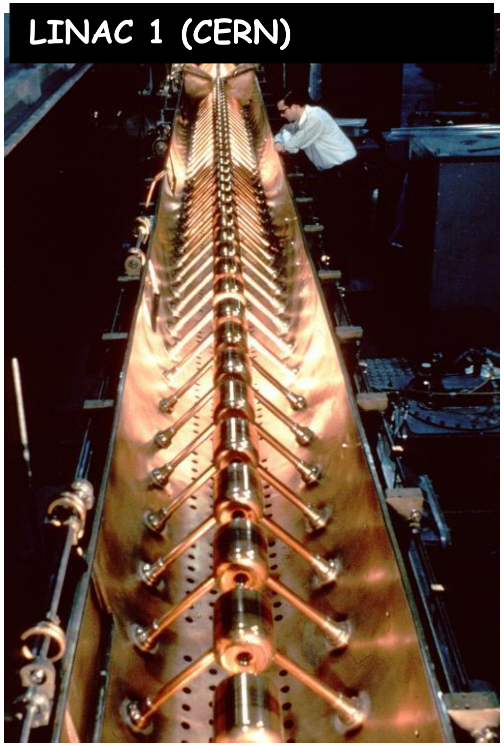
Similar for standing wave cavity as shown (with v≈c)

D.Schulte

RF acceleration: Alvarez Structure

Used for protons, ions (50 – 200 MeV, f ~ 200 MHz)





Synchronism condition $(g << L)$

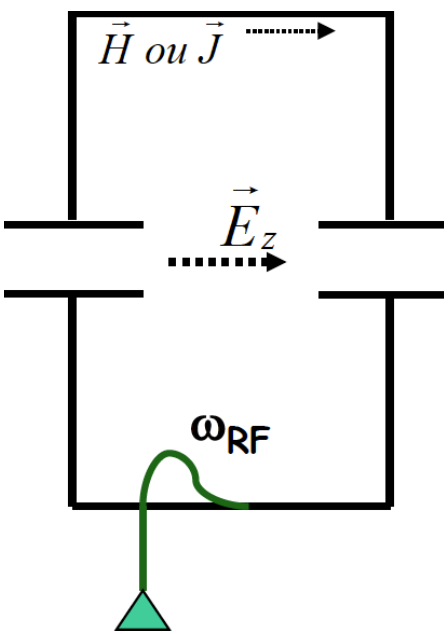
$$L = v_s T_{RF} = \beta_s \lambda_{RF}$$

$$\omega_{RF} = 2\pi f_{RF} = 2\pi \frac{v_s}{L}$$

Resonant RF Cavities

- Considering RF acceleration, it is obvious that when particles get high velocities the drift spaces get longer and one looses on the efficiency.
  => The solution consists of using a higher operating frequency.

- The power lost by radiation, due to circulating currents on the electrodes, is proportional to the RF frequency.
  => The solution consists of enclosing the system in a cavity which resonant frequency matches the RF generator frequency.



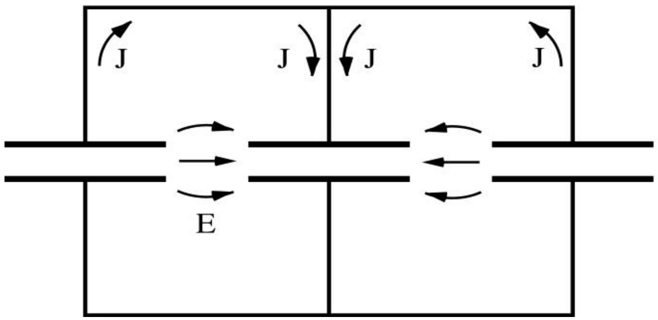
- The electromagnetic power is now constrained in the resonant volume

- Each such cavity can be independently powered from the RF generator

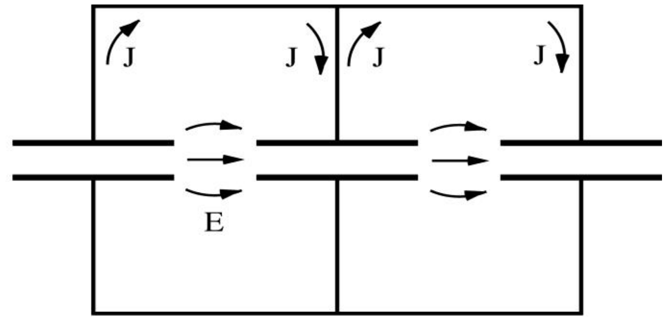
- Note however that joule losses will occur in the cavity walls (unless made of superconducting materials)

Some RF Cavity Examples

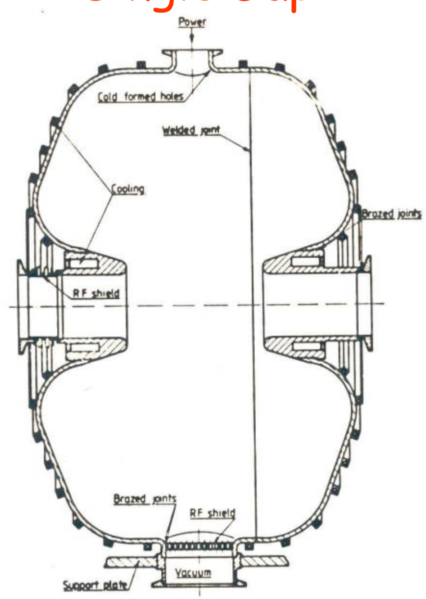
L = vT/2 (π mode)



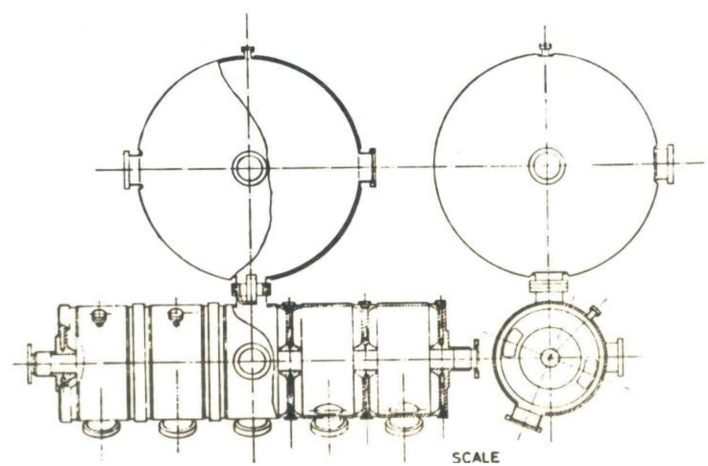
L = vT (2π mode)



Single Gap



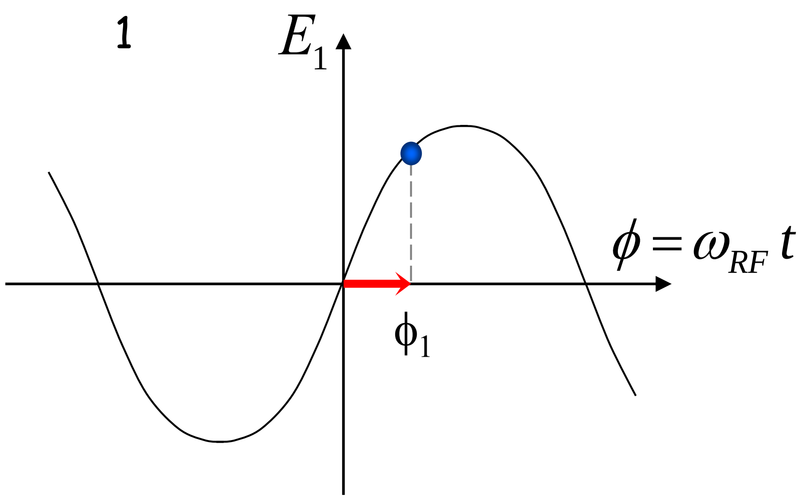
Multi-Gap



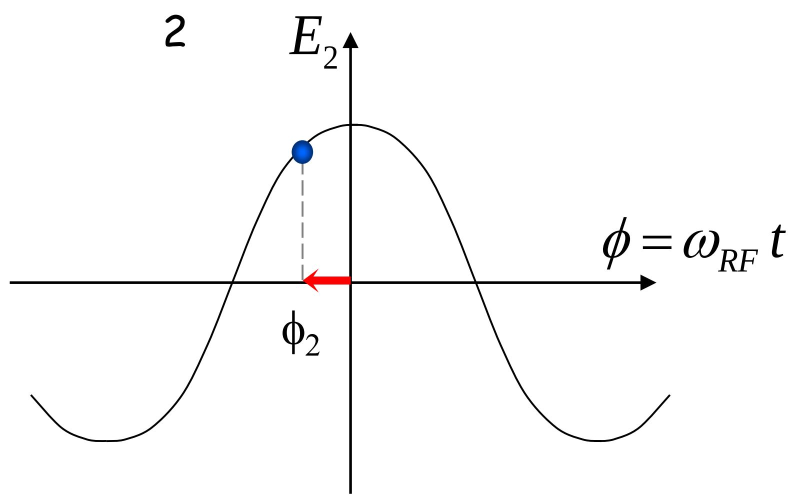
Common Phase Conventions

1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope
2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

Time t= 0 chosen such that:



$$E_1(t) = E_0 \sin\left(W_{RF}\, t\right)$$



$$E_2(t) = E_0 \cos\left(W_{RF}\, t\right)$$

3. I will stick to **convention 1** in the following to avoid confusion

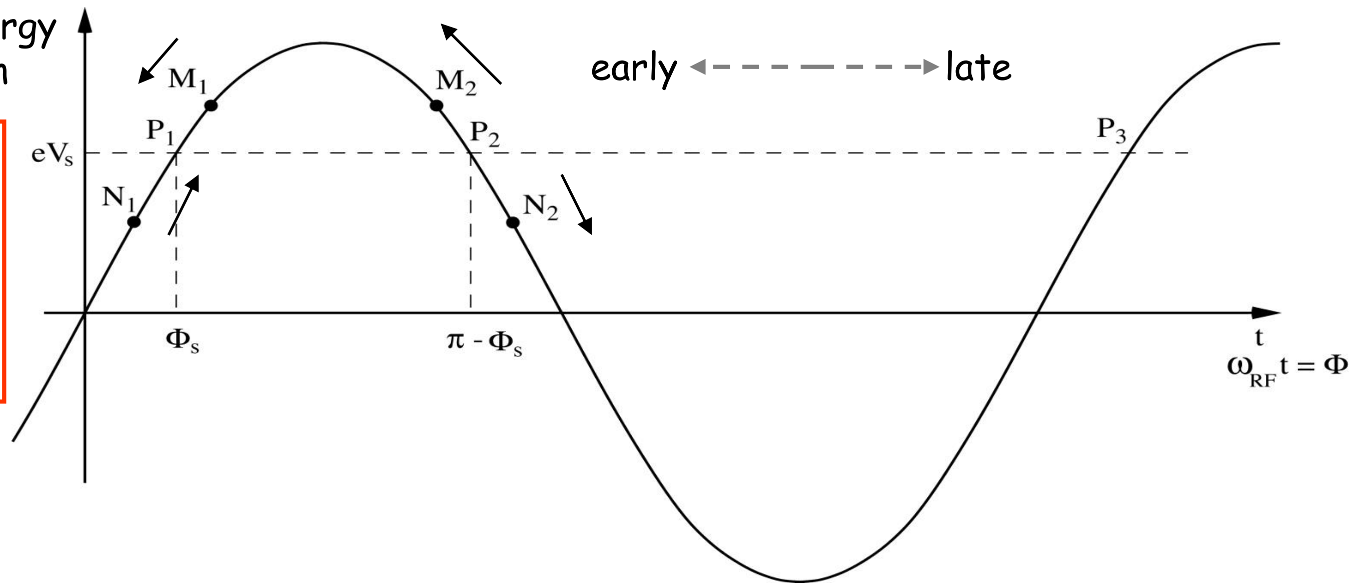
Principle of Phase Stability (Linac)

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

$$eV_s = e\hat{V}\sin\Phi_s$$

is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1 , P_2, …… are fixed points.

For a 2π mode, the electric field is the same in all gaps at any given time.



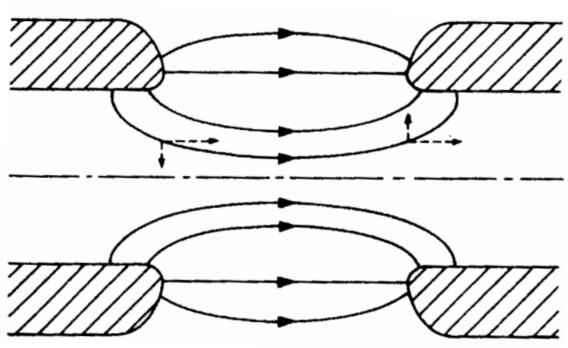
If an energy increase is transferred into a velocity increase =>

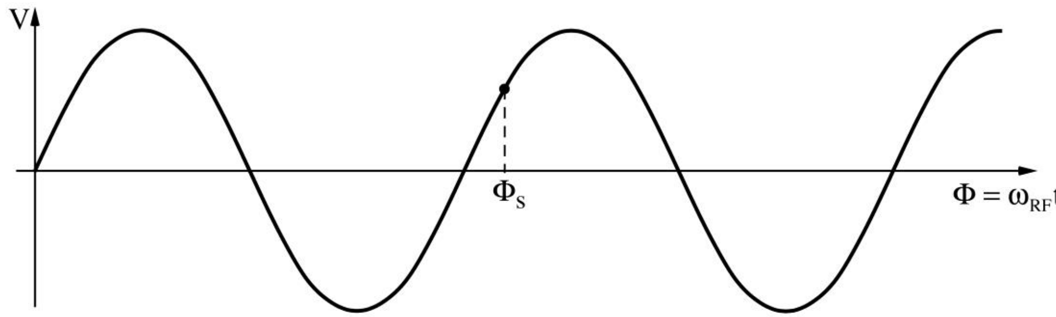
M_1 & N_1 will move towards P_1 => stable

M_2 & N_2 will go away from P_2 => unstable

(Highly relativistic particles have no significant velocity change)

A Consequence of Phase Stability





The divergence of the field is zero according to Maxwell :

$$\nabla \vec{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} = - \frac{\partial E_z}{\partial z}$$

Transverse fields

- focusing at the entrance and
- defocusing at the exit of the cavity.

Electrostatic case: Energy gain inside the cavity leads to focusing
RF case: Field increases during passage => transverse defocusing!

External focusing (solenoid, quadrupole) is then necessary

Energy-phase Oscillations (Small Amplitude) (1)

- Rate of energy gain for the synchronous particle:

$$\frac{dE_s}{dz} = \frac{dp_s}{dt} = eE_0 \sin f_s$$

- Use **reduced variables** with respect to synchronous particle

$$w = W - W_s = E - E_s \qquad \varphi = \phi - \phi_s$$

Energy gain: $$\frac{dw}{dz} = eE_0[\sin(\phi_s + \varphi) - \sin\phi_s] \approx eE_0 \cos\phi_s.\varphi \quad (small\ \varphi)$$

- Rate of **phase change** with respect to the synchronous one:

$$\frac{d\varphi}{dz} = \omega_{RF}\left(\frac{dt}{dz} - \left(\frac{dt}{dz}\right)_s\right) = \omega_{RF}\left(\frac{1}{v} - \frac{1}{v_s}\right) \cong -\frac{\omega_{RF}}{v_s^2}(v - v_s)$$

Leads finally to: $$\frac{d\varphi}{dz} = -\frac{\omega_{RF}}{m_0 v_s^3 \gamma_s^3} w$$

Energy-phase Oscillations (Small Amplitude) (2)

Combining the two 1st order equations into a 2nd order equation gives the equation of a **harmonic oscillator**:

$$\frac{d^2\varphi}{dz^2} + \Omega_s^2 \varphi = 0$$

with

$$\Omega_s^2 = \frac{eE_0 \omega_{RF} \cos\phi_s}{m_0 v_s^3 \gamma_s^3}$$

Slower for higher energy!

Stable harmonic oscillations imply: $W_s^2 > 0 \quad \text{and real}$

hence: $\cos\phi_s > 0$

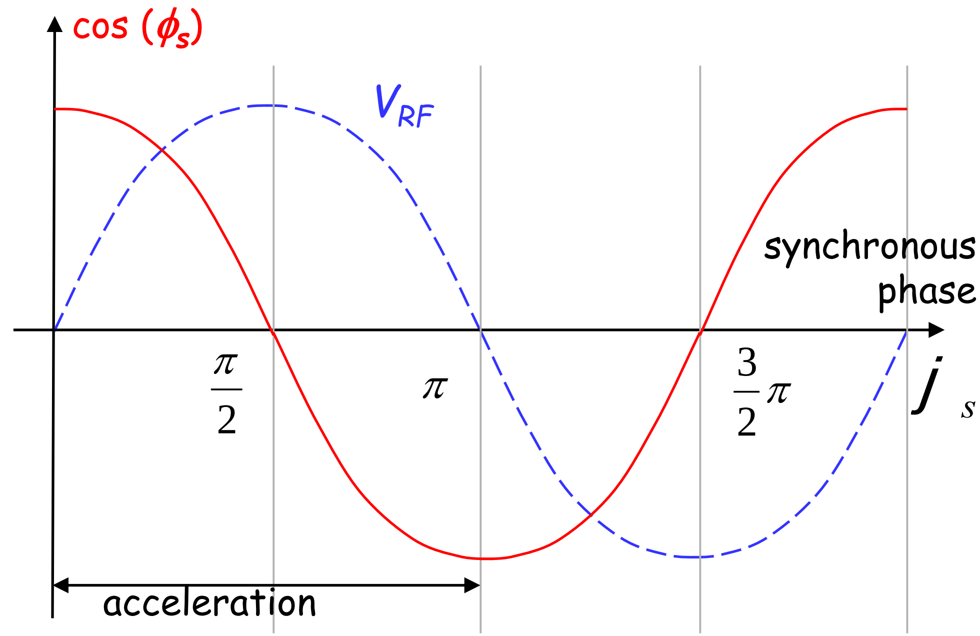
And since acceleration also means:

$$\sin\phi_s > 0$$

You finally get the result for the stable phase range:

$$0 < \phi_s < \frac{\pi}{2}$$

Positive rising RF slope!



Summary up to here…

- Acceleration by electric fields, static fields limited
  => time-varying fields
- Synchronous condition needs to be fulfilled for acceleration
- Particles perform oscillation around synchronous phase
- Stable acceleration on the rising slope in a linac.

- Electrons are quickly relativistic, speed does not change
- Protons and ions need changing structure geometry and certain RF frequency range

Circular accelerators

Betatron

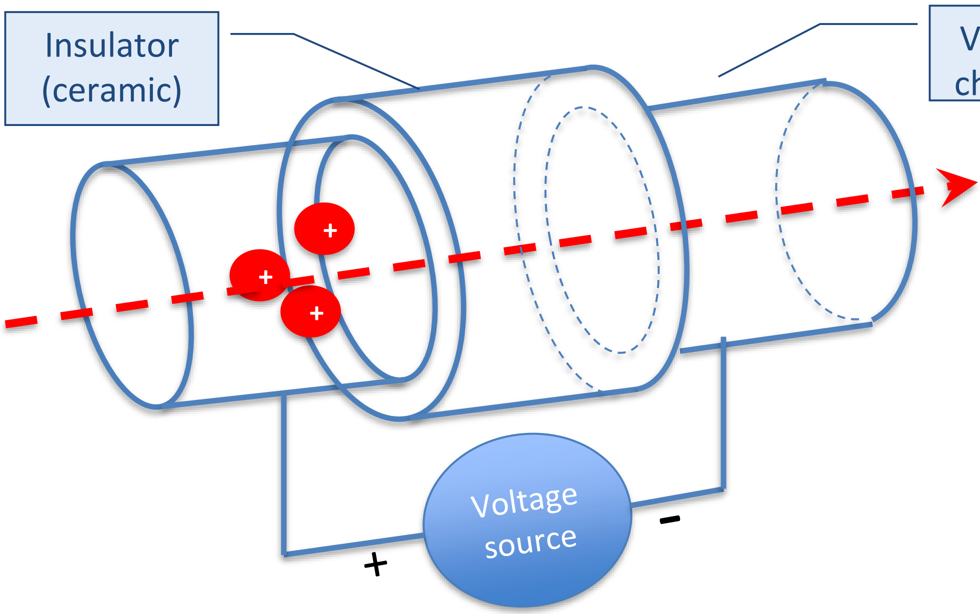
Cyclotron

Synchrotron

Methods of Acceleration in circular accelerators

Electrostatic field limited by insulation, magnetic field doesn't accelerate at all.

Circular machine: DC acceleration impossible since $\oint \vec{E} \cdot \mathrm{d}\vec{s} = 0$



~~First attracted~~

~~**Acceleration**~~

~~Then again attracted~~

~~**Deceleration**~~

➡ **no Acceleration**

The electric field is derived from a scalar potential φ and a vector potential A
The time variation of the magnetic field H generates an electric field E

The solution: => time varying electric fields
- Induction
- RF frequency fields

$$\oint \vec{E} \cdot \mathrm{d}\vec{s} = -\iint \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{A}$$

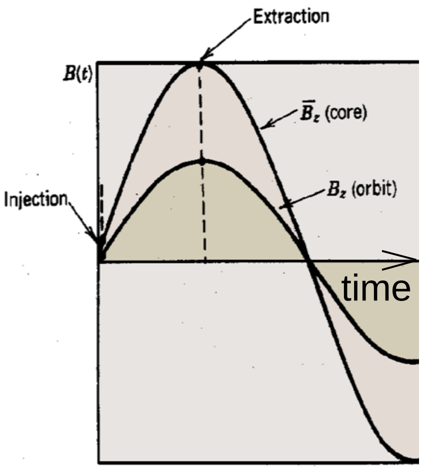
Acceleration by Induction: The Betatron

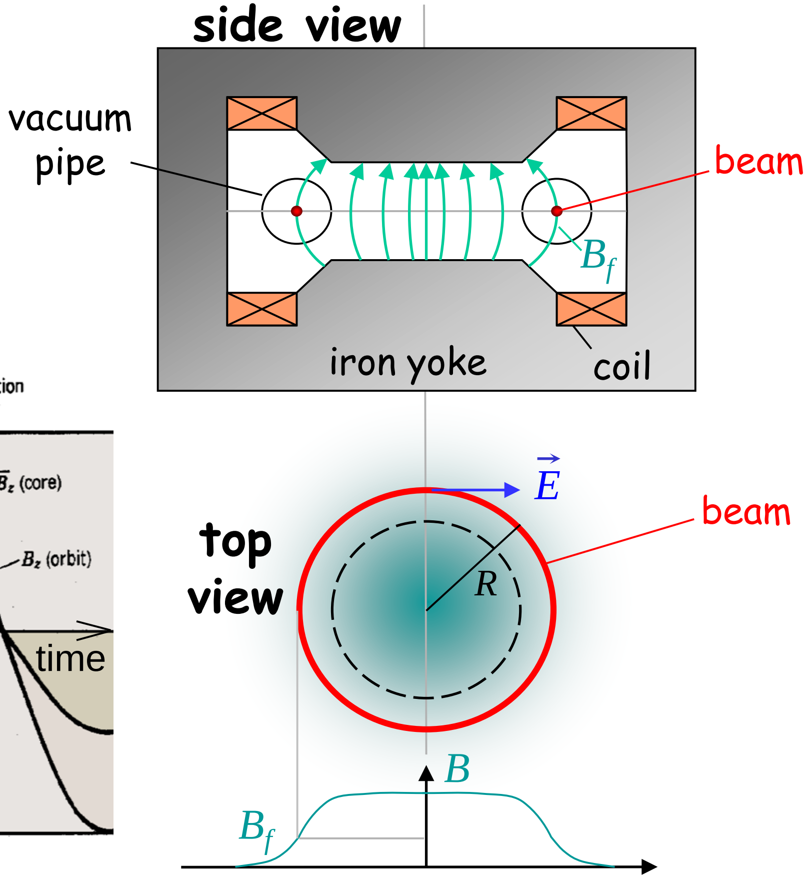
A ramping magnetic field
- Guides particles on a circular trajectory and
- Creates a tangential electric field that accelerates the particles

Limited by saturation in iron (~300 MeV e-)

Used in industry and medicine, as they are compact accelerators for electrons

Donald Kerst with the first betatron, invented at the University of Illinois in 1940

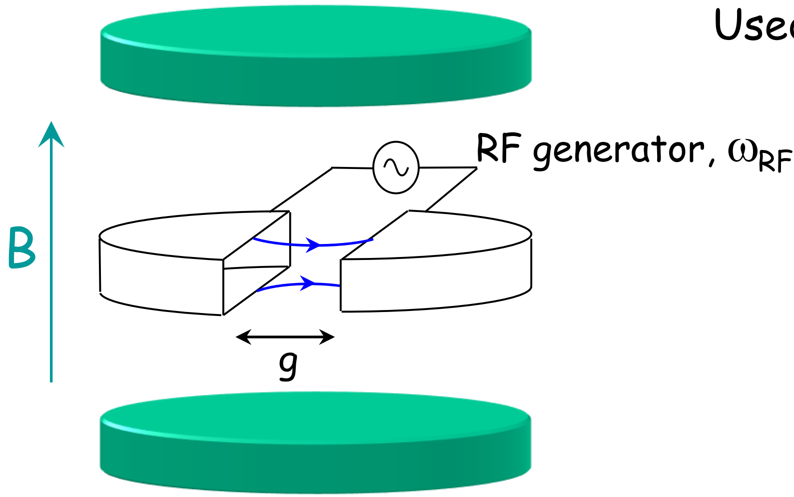




Circular accelerators: Cyclotron

Courtesy: EdukiteLearning, https://youtu.be/cNnNM2ZqIsc

Circular accelerators: Cyclotron



Used for protons, ions

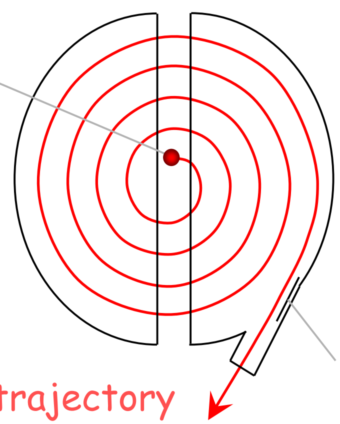
B = constant

ω_{RF} = constant

Synchronism condition

$$\omega_s = \omega_{RF}$$

$$2\pi\ \rho = v_s\ T_{RF}$$



Cyclotron frequency $\omega = \dfrac{q\ B}{m_0\ \gamma}$

1. γ increases with the energy
   ⇒ no exact synchronism
2. if **v << c** ⇒ $\gamma \cong 1$

Cyclotron Animation

Animation: https://phyanim.sciences.univ-nantes.fr/Meca/Charges/cyclotron.php

Cyclotron / Synchrocyclotron

TRIUMF 520 MeV cyclotron Vancouver - Canada

CERN 600 MeV synchrocyclotron

Synchrocyclotron: Same as cyclotron, except a modulation of ω_{RF}

B = constant

$\gamma\ \omega_{RF}$ = constant

ω_{RF} decreases with time

The condition:

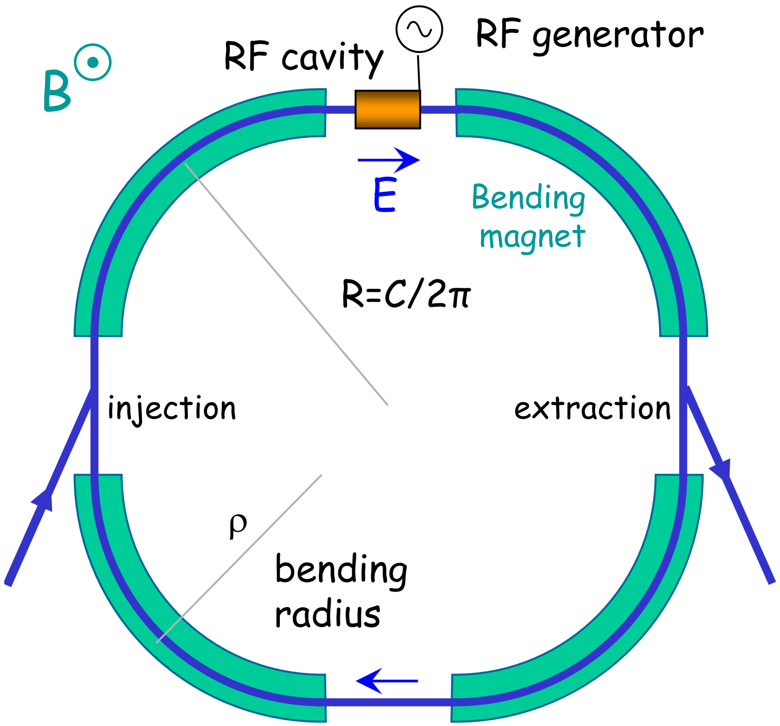
$$\omega_s(t) = \omega_{RF}(t) = \frac{q\ B}{m_0\ \gamma(t)}$$

Allows to go beyond the non-relativistic energies

Circular accelerators: Cyclotron

Courtesy Berkeley Lab,
https://www.youtube.com/watch?v=cutKuFxeXmQ

Circular accelerators: The Synchrotron



1. Constant orbit during acceleration
2. To keep particles on the closed orbit, B should increase with time
3. ω and ω_{RF} increase with energy

RF frequency can be multiple of revolution frequency

$$\omega_{RF} = h\omega$$

Synchronism condition ➡

$$T_s = h\,T_{RF}$$

$$\frac{2\pi\,R}{v_s} = h\,T_{RF}$$

h integer,
harmonic number: number of RF cycles per revolution

h is the maximum number of bunches in the synchrotron.
Normally less bunches due to gaps for kickers, collision constraints,…

Circular accelerators: The Synchrotron

LEAR (CERN)
Low Energy Antiproton Ring

EPA (CERN)
Electron Positron Accumulator

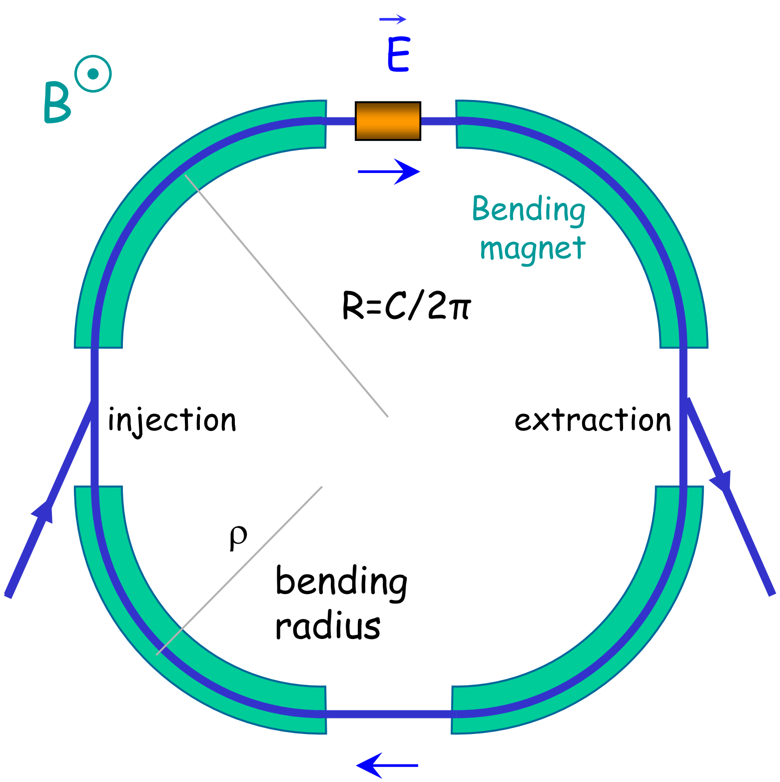
PS (CERN)
Proton Synchrotron

Examples of different proton and electron synchrotrons at CERN

+ LHC (of course!)

The Synchrotron

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:



$e\hat{V}\sin f$ ⟶ **Energy gain per turn**

$f = f_s = cte$ ⟶ **Synchronous particle**

$\omega_{RF} = h\omega$ ⟶ **RF synchronism (h - harmonic number)**

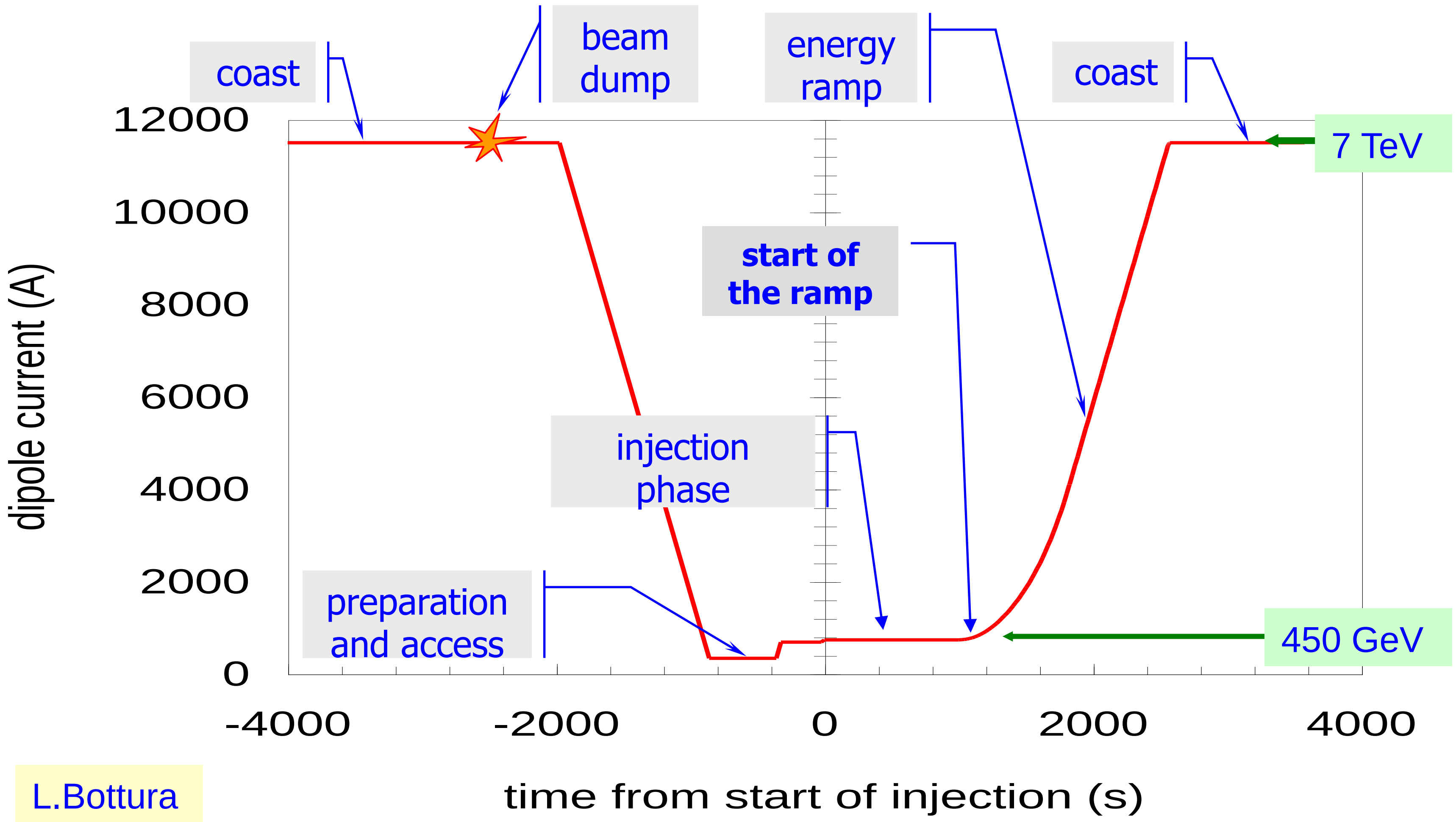
$r = cte \quad R = cte$ ⟶ **Constant orbit**

$Br = P/e$ Þ B ⟶ **Variable magnetic field**

If v≈c, ω hence ω_{RF} remain constant (ultra-relativistic e^-)

The Synchrotron – LHC Operation Cycle

The magnetic field (dipole current) is increased during the acceleration.



The Synchrotron – Energy ramping

Energy ramping by increasing the B field (frequency has to follow v):

$$p = eB\rho \underset{\rho\ \text{const.}}{\Rightarrow} \frac{dp}{dt} = e\rho\,\dot{B} \Rightarrow (\Delta p)_{turn} = e\rho\,\dot{B}T_r = \frac{2\pi e\rho\,R\dot{B}}{v}$$

With

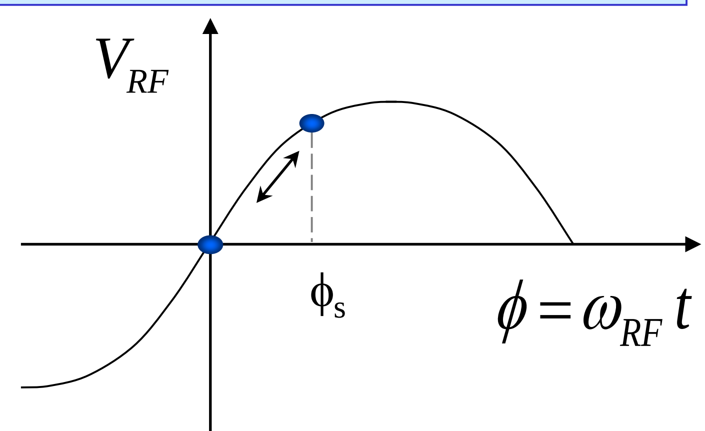
$$E^2 = E_0^2 + p^2c^2 \Rightarrow \Delta E = v\Delta p$$

$$(\Delta E)_{turn} = (\Delta W)_s = 2\pi e\rho\,R\dot{B} = e\hat{V}\sin\phi_s$$

Synchronous phase φ_s changes during energy ramping

$$\sin\phi_s = 2\pi\,\rho\,R\,\frac{\dot{B}}{\hat{V}_{RF}} \quad \Rightarrow \quad \phi_s = \arcsin\left(2\pi\,\rho\,R\,\frac{\dot{B}}{\hat{V}_{RF}}\right)$$

- **The synchronous phase depends on**
  - **the change of the magnetic field**
  - **and the RF voltage**



The Synchrotron – Frequency change

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency :

$$\omega = \frac{\omega_{RF}}{h} = \omega(B, R_s)$$

Hence: $$\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi}\frac{ec^2}{E_s(t)}\frac{\rho}{R_s}B(t)$$ (using $p(t) = eB(t)\rho, \quad E = mc^2$)

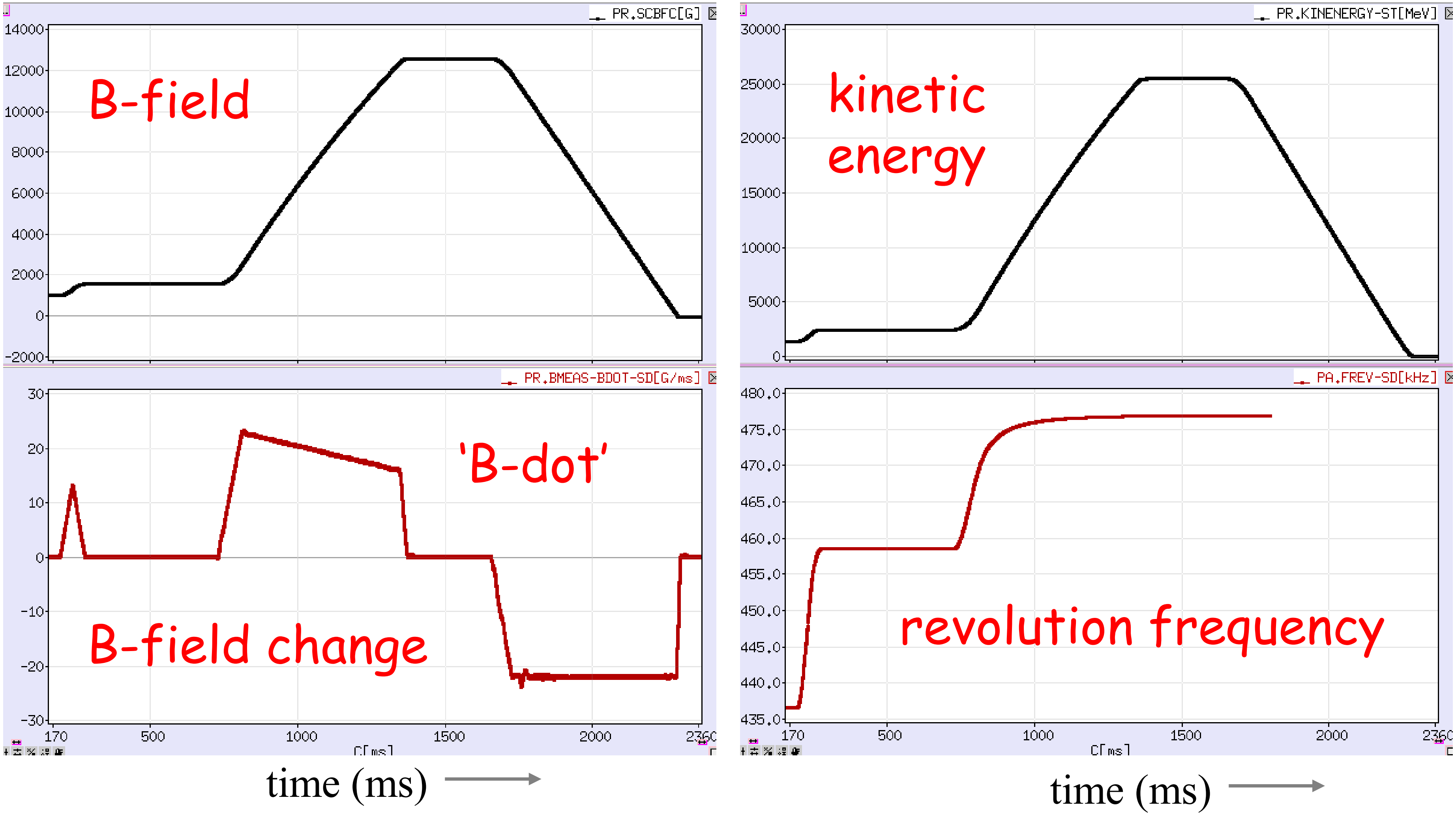
Since $E^2 = (m_0c^2)^2 + p^2c^2$ the RF frequency must follow the variation of the B field with the law

$$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s}\left\{\frac{B(t)^2}{(m_0c^2/ec\rho)^2 + B(t)^2}\right\}^{1/2}$$

RF frequency program during acceleration determined by B-field !

Example: PS - Field / Frequency change

During the energy ramping, the B-field and the revolution frequency increase



Overtaking in a roundabout

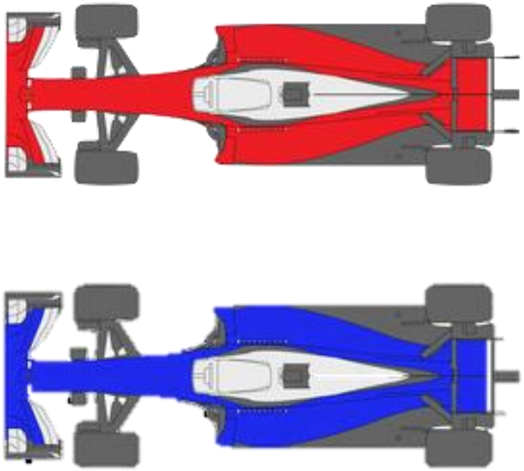
Finally a real-life problem: what is the fastest way through a roundabout?

Most CERN people encounter this near the French entrance to CERN.

Optimize the roundabout!

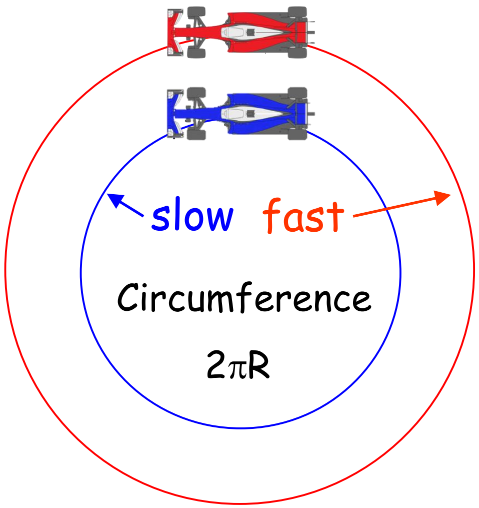
The magic roundabout in Swindon, UK!
Video: https://www.youtube.com/watch?v=6OGvj7GZSIo

Overtaking in a Formula 1 Race



Overtaking in a Formula 1 Race

Overtaking in a Formula 1 Race



A F1 car wants to overtake another car! It will have a

- a different track length due to a 'dispersion orbit'
- and a different velocity.

$$T = \frac{L}{v} = \frac{2\pi R}{v} \quad \text{and} \quad f_r = \frac{1}{T} = \frac{v}{2\pi R}$$

$$\Rightarrow \frac{\Delta T}{T} = \frac{\Delta R}{R} - \frac{\Delta v}{v}$$

The winner depends on the relative change in speed compared to the relative change in track length!

v=speed of the car

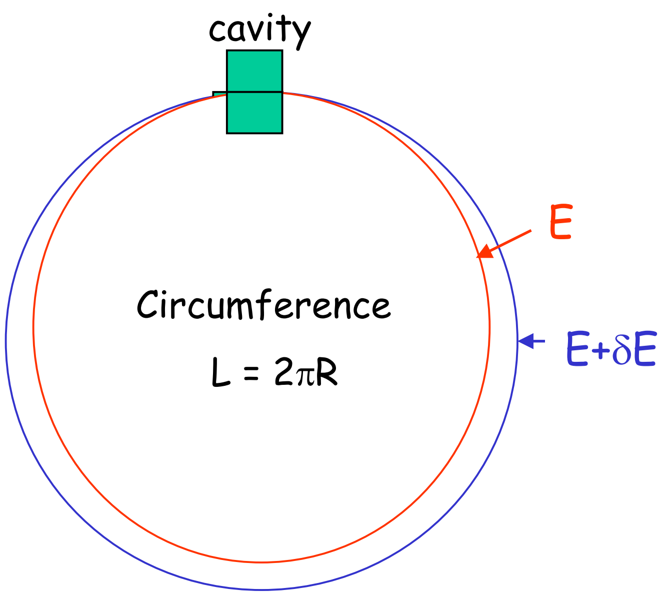
R=track physical radius

T=revolution period

f_r=revolution frequency

If the relative change in speed is larger than the relative change in track length => the red car will win!

Overtaking in a Synchrotron



A particle slightly shifted in momentum will have a

- dispersion orbit and a different orbit length
- a different velocity.

As a result of both effects the revolution period T changes with a "slip factor" η :

$$\eta = \frac{dT/T}{dp/p}$$

Note: you also find η defined with a minus sign!

p=particle momentum

R=synchrotron physical radius

T=revolution period

The "momentum compaction factor" is defined as relative orbit length change with momentum:

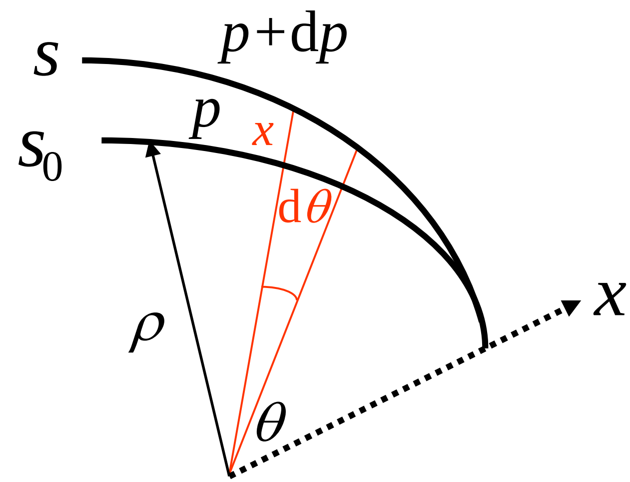
$$\alpha_c = \frac{dL/_L}{dp/_p} \qquad \alpha_c = \frac{p}{L}\frac{dL}{dp}$$

Momentum Compaction Factor

$$\alpha_c = \frac{p}{L}\frac{dL}{dp}$$

$$ds_0 = \rho\, d\theta$$

$$ds = (\rho + x)\, d\theta$$



The elementary path difference from the two orbits is:

definition of dispersion D_x

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D_x}{\rho}\frac{dp}{p}$$

$$x = x_0 + D_x \frac{\Delta p}{p}$$

leading to the total change in the circumference:

$$dL = \oint_C dl = \oint \frac{x}{\rho}\, ds_0 = \oint \frac{D_x}{\rho}\frac{dp}{p}\, ds_0$$

$$\alpha_c = \frac{1}{L}\int_C \frac{D_x(s)}{\rho(s)}\, ds_0$$

With ρ=∞ in straight sections we get:

$$\alpha_c = \frac{\langle D_x \rangle_m}{R}$$

$\langle\ \rangle_m$ means that the average is considered over the bending magnet only

Property of the **transverse** beam optics!

Dispersion Effects – Revolution Period

The two effects of the orbit length and the particle velocity change the revolution period as:

$$T = \frac{L}{\beta c} \qquad \Rightarrow \qquad \frac{dT}{T} = \frac{dL}{L} - \frac{d\beta}{\beta} = \alpha_c \frac{dp}{p} - \frac{d\beta}{\beta}$$

definition of momentum compaction factor

$$\frac{dT}{T} = \left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{dp}{p}$$

$$p = mv = \beta\gamma \frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-\frac{1}{2}}}{\left(1-\beta^2\right)^{-\frac{1}{2}}} = \underbrace{\left(1-\beta^2\right)^{-1}}_{\gamma^2}\frac{d\beta}{\beta}$$

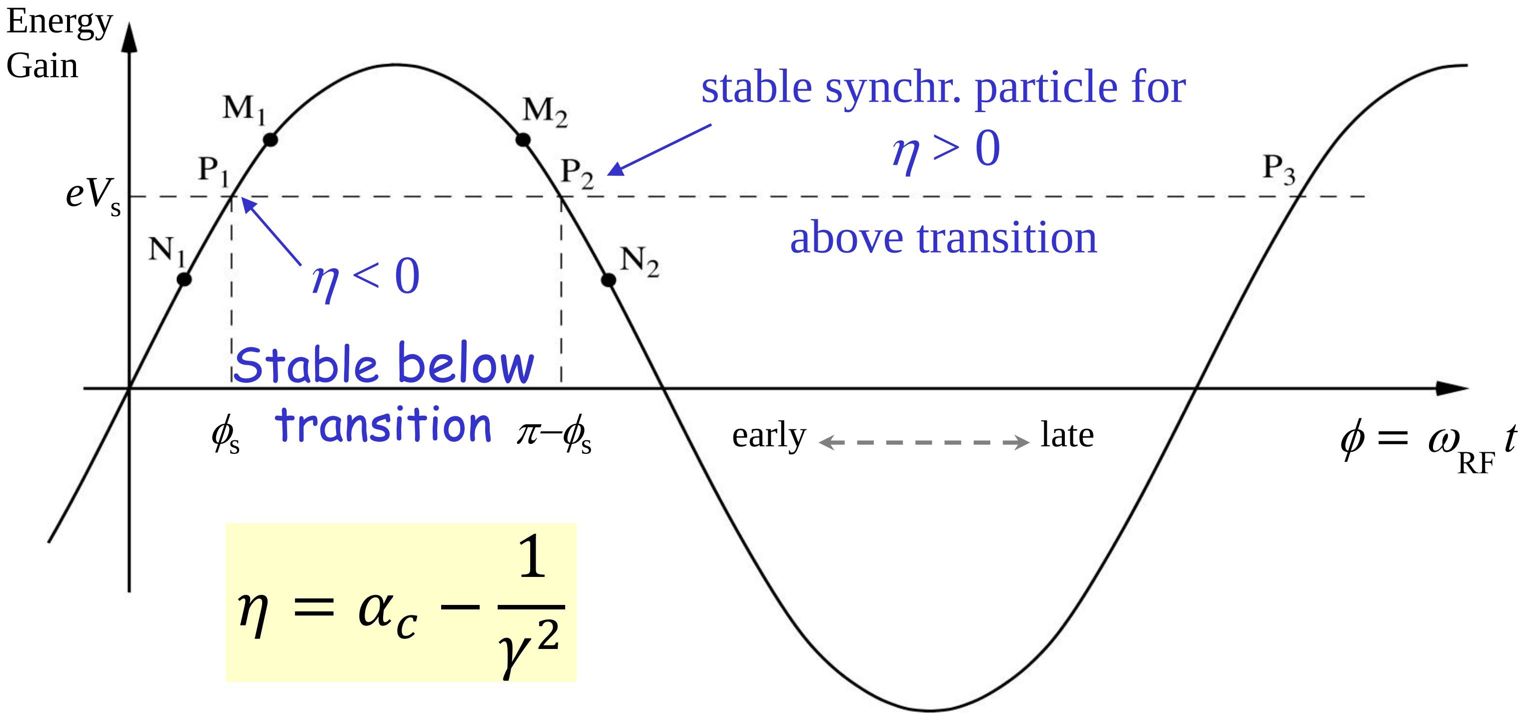
Slip factor: $\eta = \alpha_c - \frac{1}{\gamma^2}$ or $\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$ with $\gamma_t = \frac{1}{\sqrt{\alpha_c}}$

Note: you also find η defined with a minus sign!

At transition energy, $\eta = 0$, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Phase Stability in a Synchrotron

From the definition of η it is clear that an increase in momentum gives
- below transition ($\eta < 0$) a higher revolution frequency (increase in velocity dominates) while
- above transition ($\eta > 0$) a lower revolution frequency ($v \approx c$ and longer path) where the momentum compaction (generally > 0) dominates.

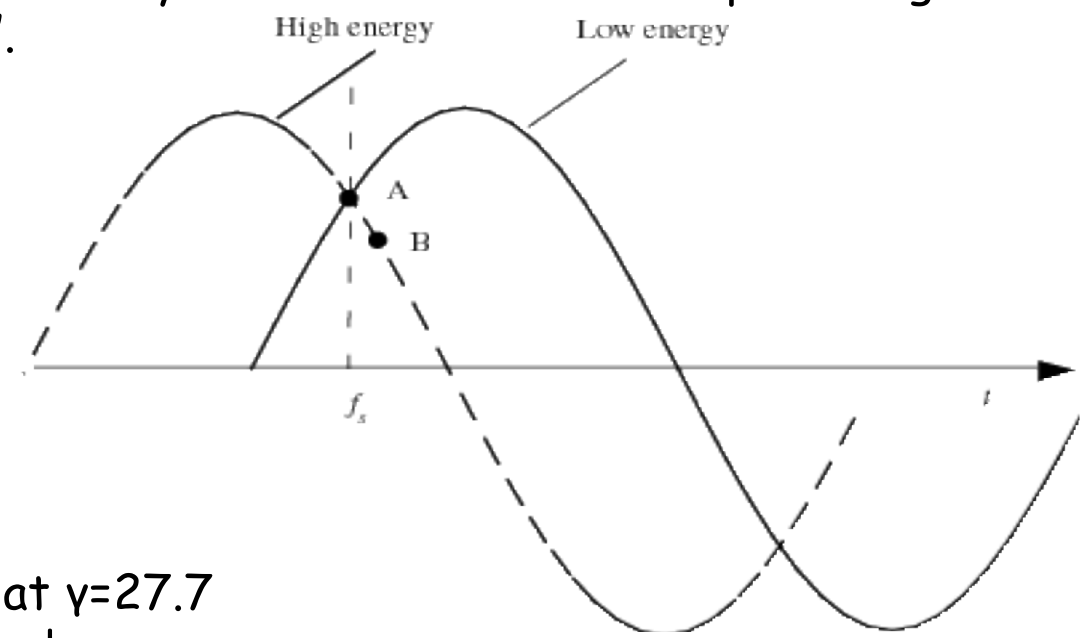


$$\eta = \alpha_c - \frac{1}{\gamma^2}$$

Crossing Transition

At transition, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a rapid change of the RF phase, a 'phase jump'.



$$\alpha_c \sim \frac{1}{Q_x^2} \qquad \gamma_t = \frac{1}{\sqrt{\alpha_c}} \sim Q_x$$

In the PS: γ_t is at ~6 GeV
In the SPS: γ_t= 22.8, injection at γ=27.7
=> no transition crossing!
In the LHC: γ_t is at ~55 GeV, also far below injection energy

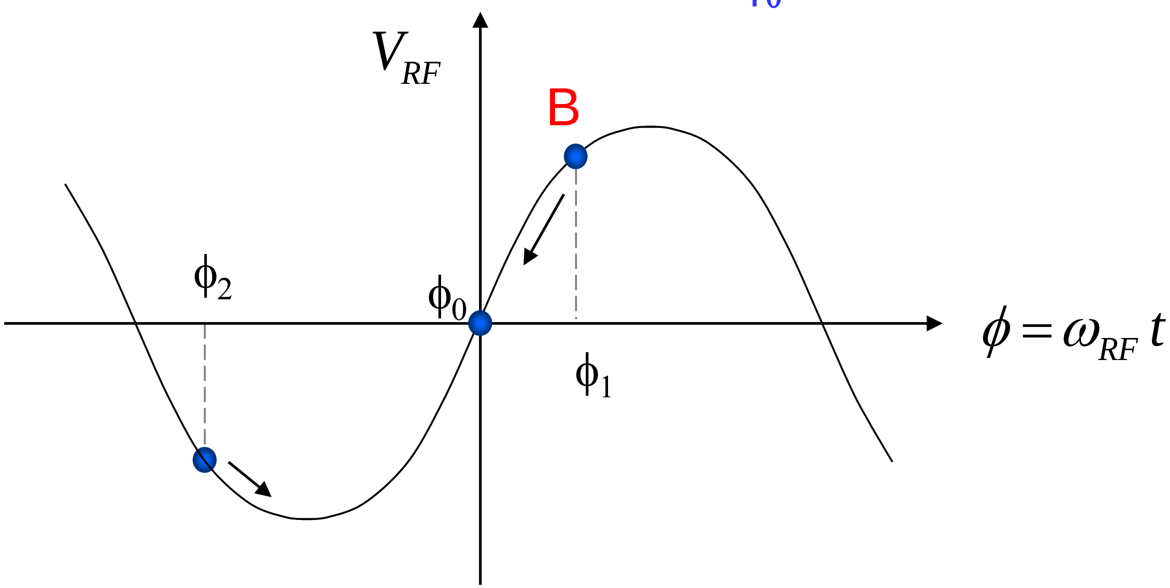
Transition crossing not needed in leptons machines, why?

Dynamics: Synchrotron oscillations

Simple case (no accel.): B = const., below transition $\gamma < \gamma_t$

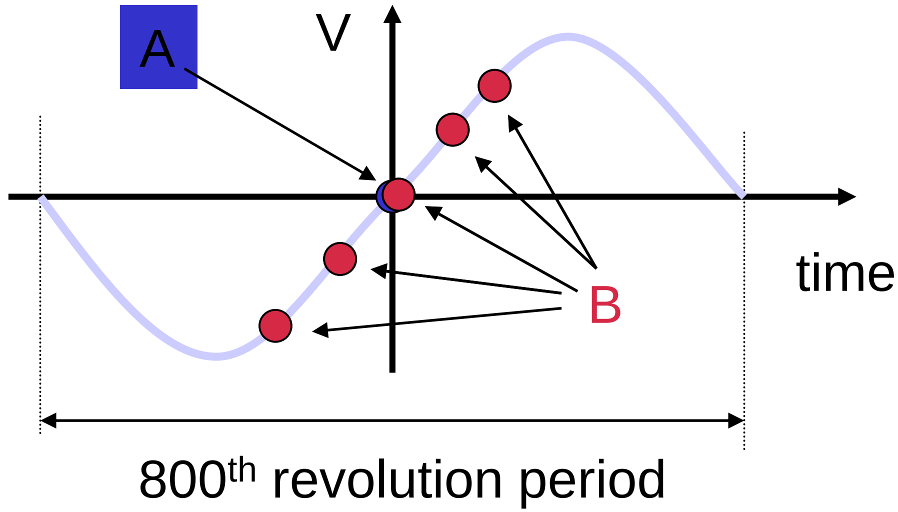
The phase of the synchronous particle must therefore be $\phi_0 = 0$.

Φ_1 - The particle B is accelerated
- Below transition, an energy increase means an increase in revolution frequency
- The particle arrives earlier – tends toward ϕ_0



ϕ_2 - The particle is decelerated
- decrease in energy - decrease in revolution frequency
- The particle arrives later – tends toward ϕ_0

Synchrotron oscillations



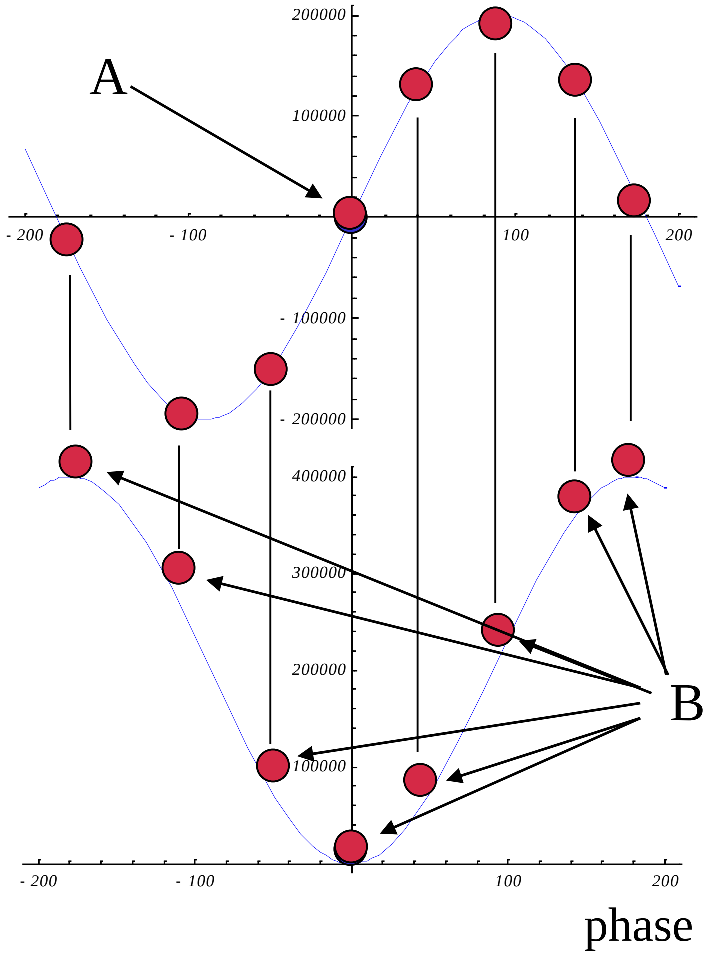
Particle B performs Synchrotron Oscillations around synchronous particle A.

The amplitude depends on the initial phase and energy.

The oscillation frequency is much slower than in the transverse plane.
It takes a large number of revolutions for one complete oscillation.
The restoring electric force is smaller than the magnetic force.

- proton synchrotrons of the order of 1000 turns
- electron storage rings of the order of ~10 turns

The Potential Well

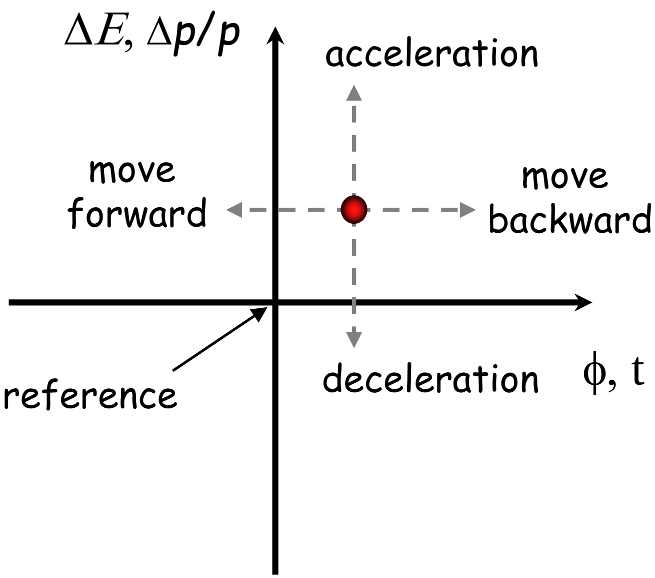


Cavity voltage

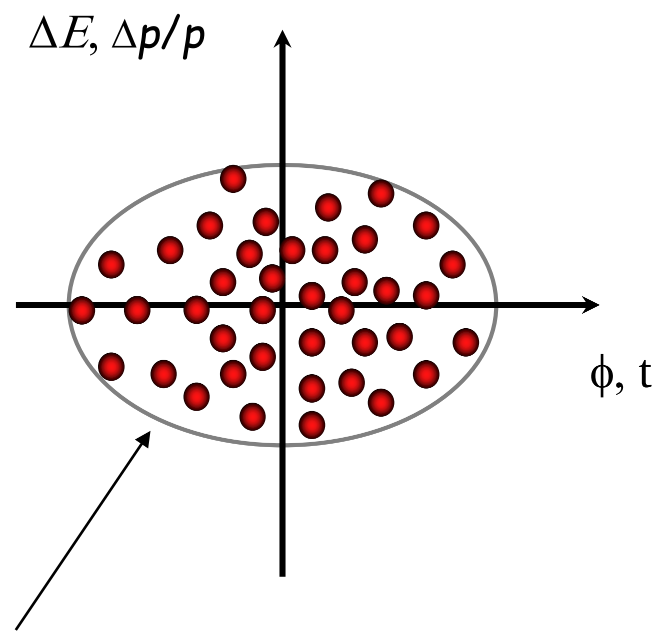
Potential well

Longitudinal phase space

The energy – phase oscillations can be drawn in phase space.
Similar to transverse, but here it's TIME and ENERGY!



The particle trajectory in the phase space ($\Delta p/p$, ϕ) describes its longitudinal motion.



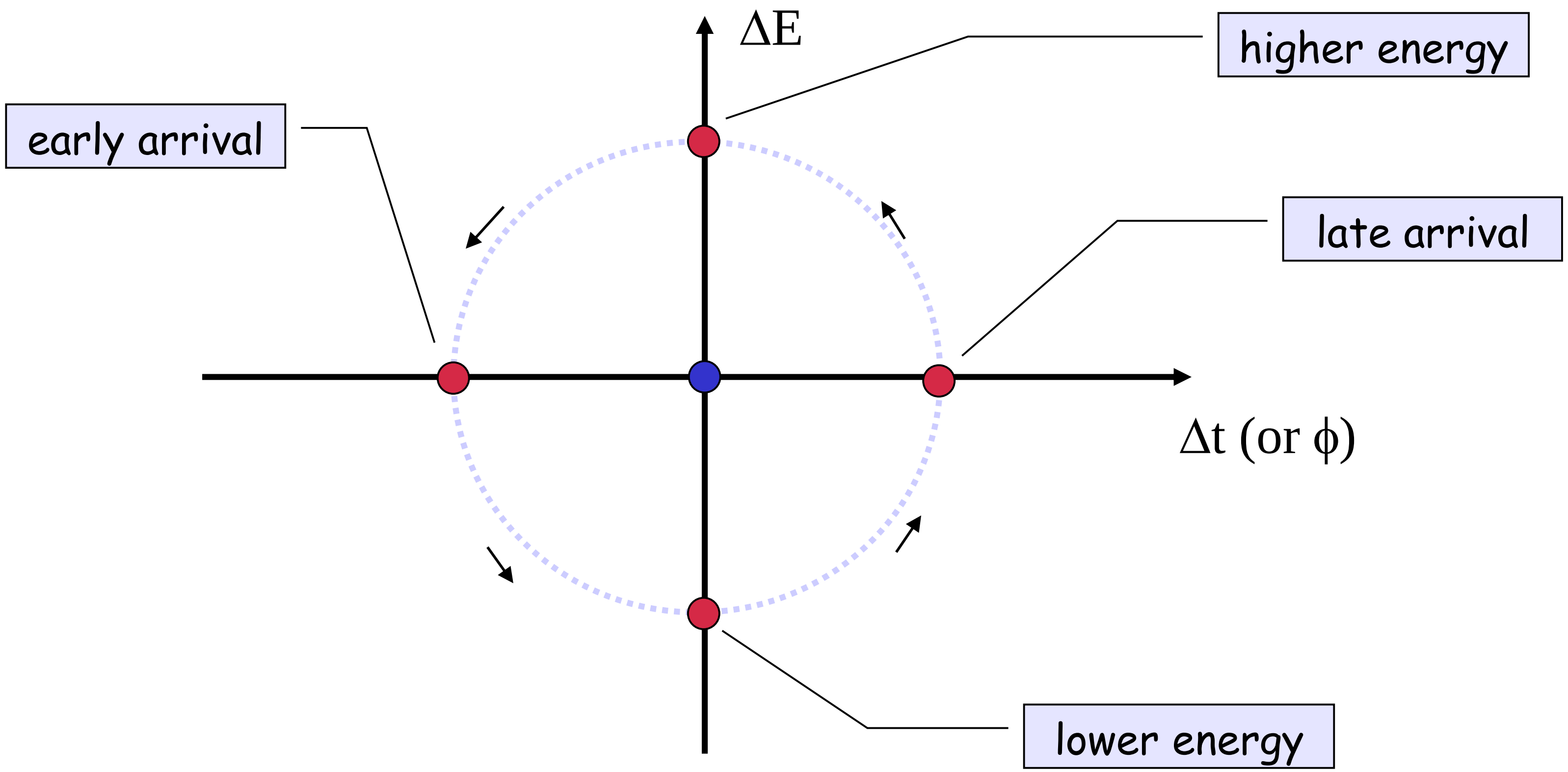
Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)

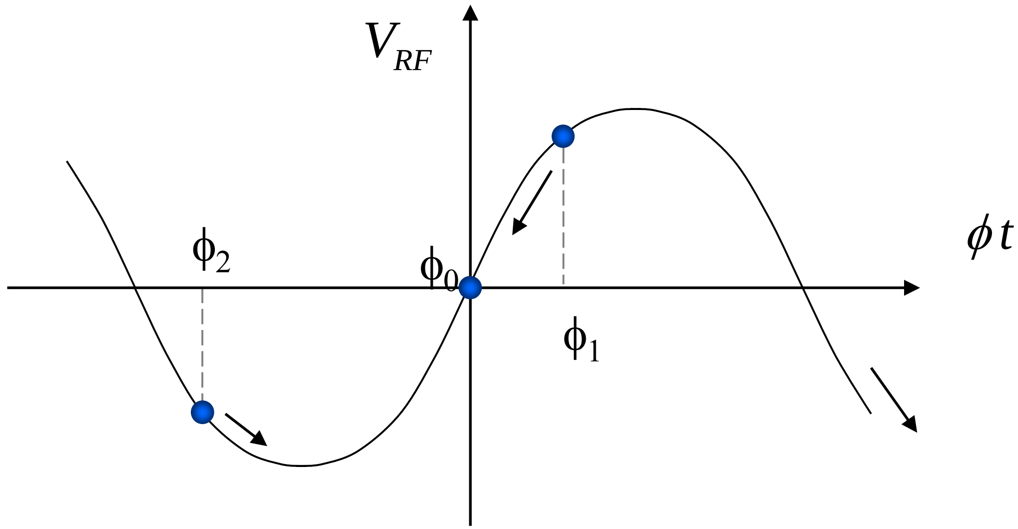
Longitudinal Phase Space Motion

Particle B oscillates around particle A in a synchrotron oscillation.

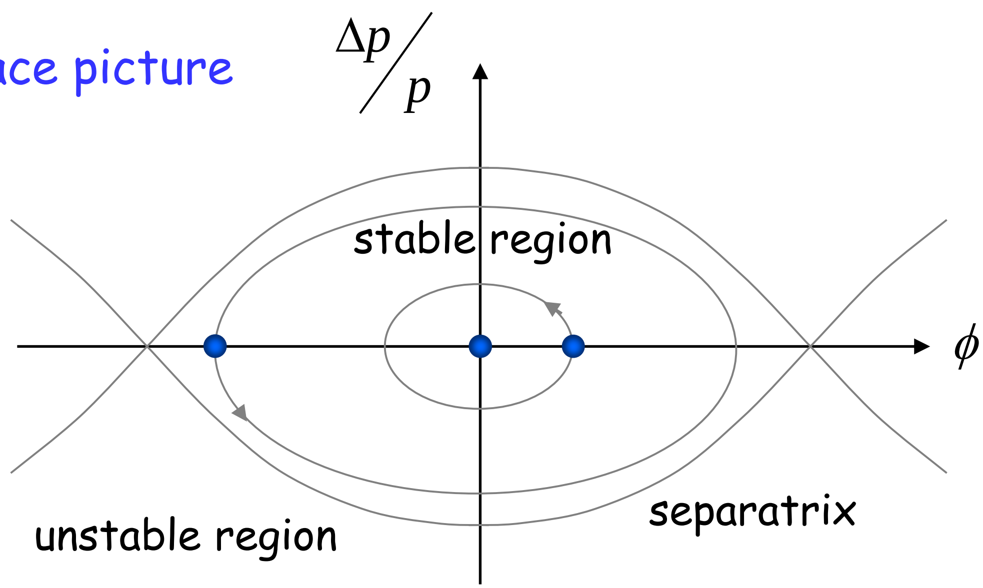
Plotting this motion in longitudinal phase space (time, energy) gives:



Synchrotron oscillations – No acceleration

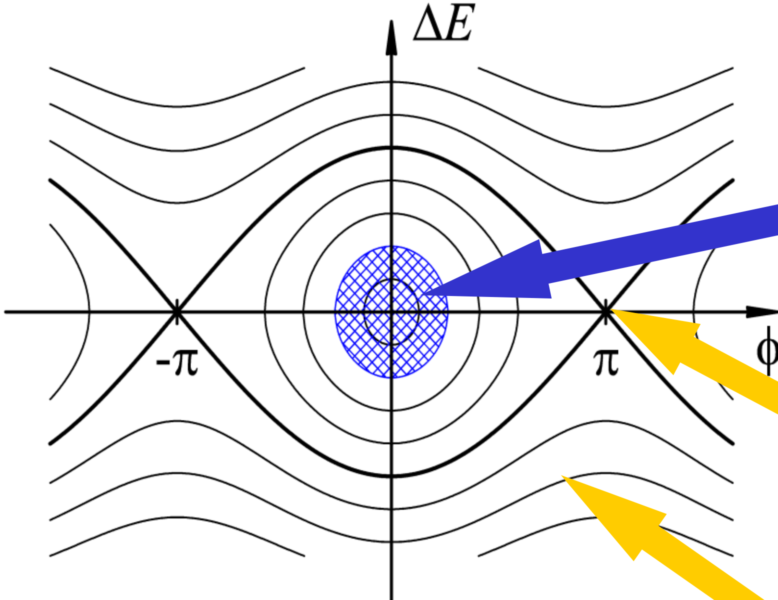


Phase space picture



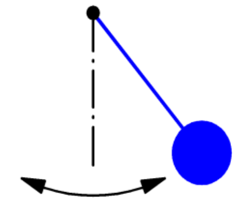
Synchrotron motion in phase space

ΔE-ϕ **phase space** of a stationary bucket
(when there is no acceleration)



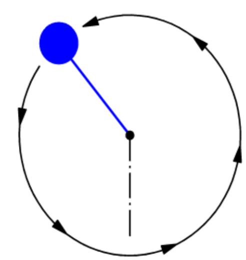
Dynamics of a particle
Non-linear, conservative oscillator → e.g. pendulum

Particle **inside the separatrix:**



Particle at the **unstable fix-point**

Particle **outside the separatrix:**

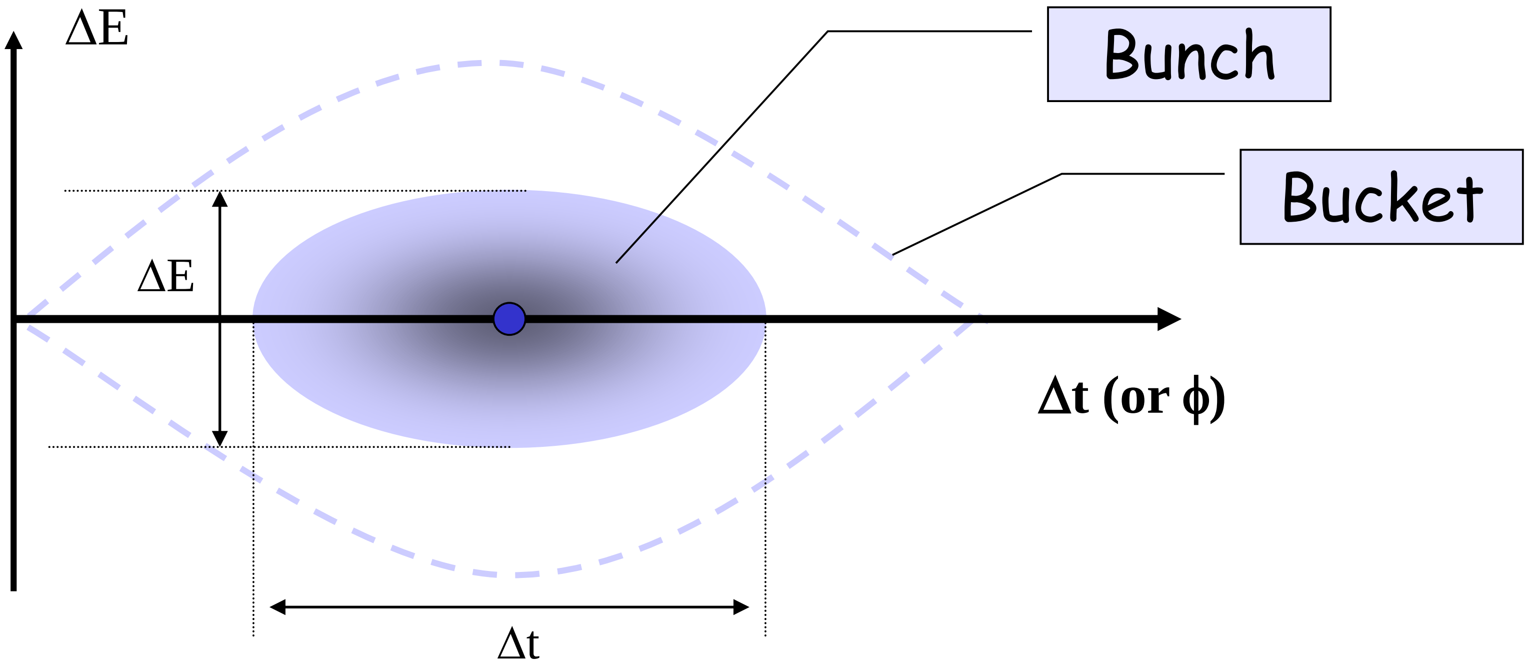


Bucket area: area enclosed by the separatrix

The area covered by particles is the longitudinal emittance

(Stationary) Bunch & Bucket

The bunches of the beam fill usually a part of the bucket area.



Bucket area = **longitudinal Acceptance** [eVs]

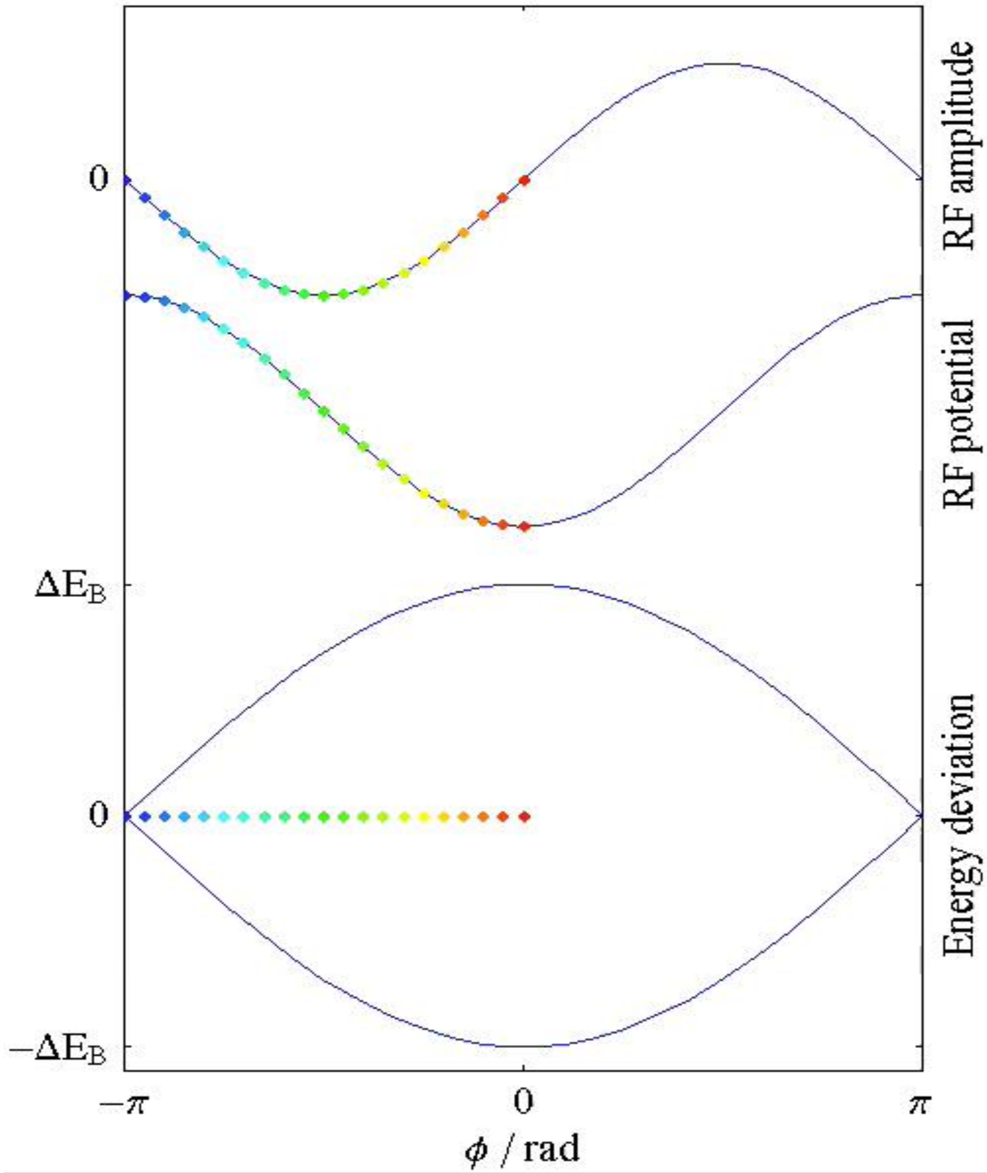
Bunch area = **longitudinal beam emittance** = $4\pi\, \sigma_E\, \sigma_t$ **[eVs]**

Attention: Different definitions are used!

Synchrotron motion in phase space

The restoring force is non-linear.
⇒ speed of motion depends on position in phase-space

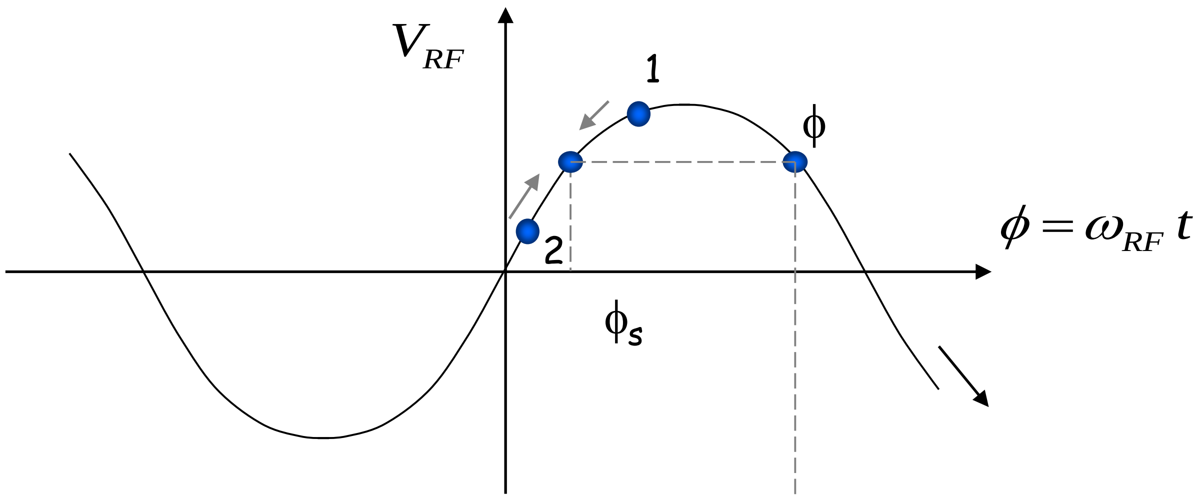
(here shown for a stationary bucket)



Synchrotron oscillations (with acceleration)

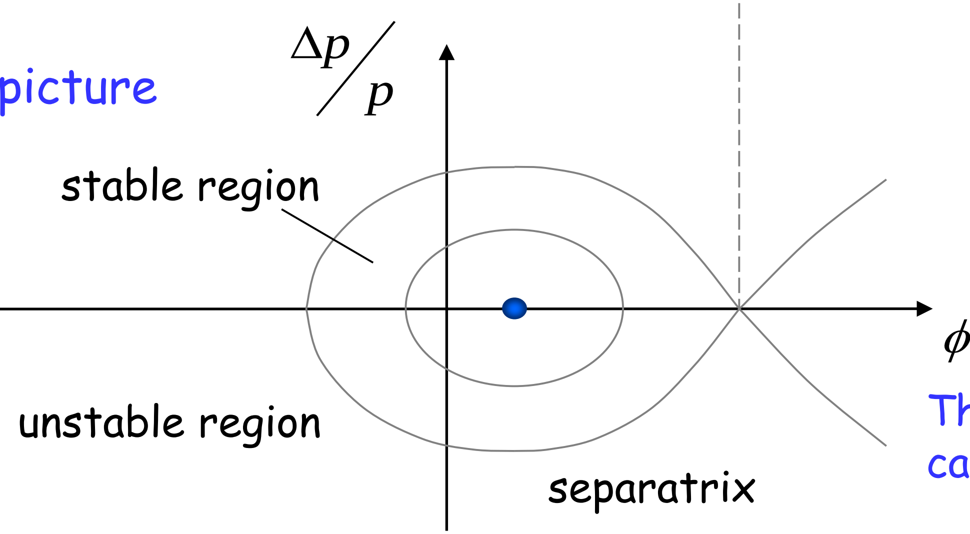
Case with acceleration *B increasing*

$\gamma < \gamma_t$



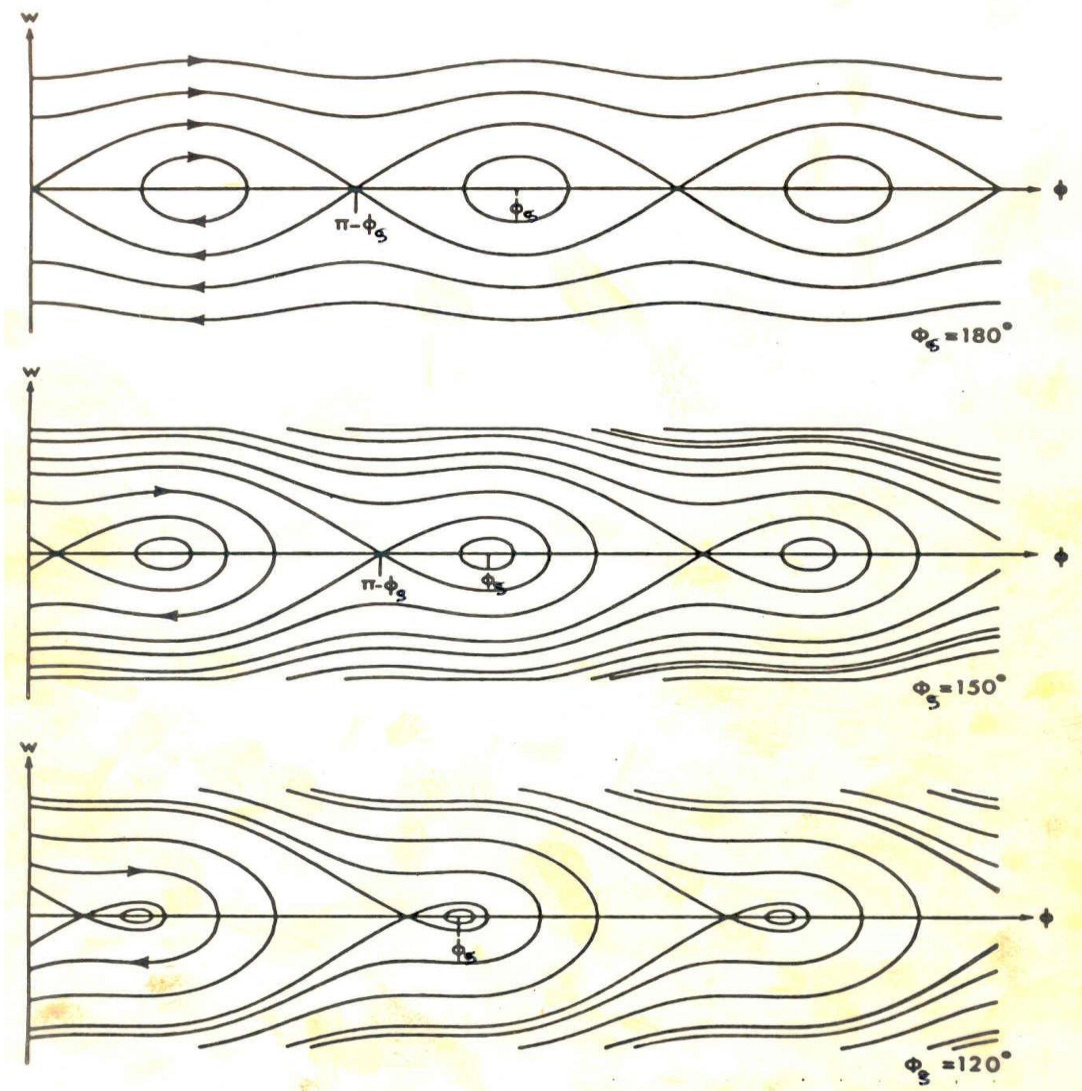
$$\phi_s < \phi < \pi - \phi_s$$

Phase space picture



The symmetry of the case B = const. is lost

RF Acceptance versus Synchronous Phase



The areas of stable motion (closed trajectories) are called "BUCKET". The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which means no acceleration.

During acceleration, the buckets get smaller, both in length and energy acceptance.

=> Injection preferably without acceleration.

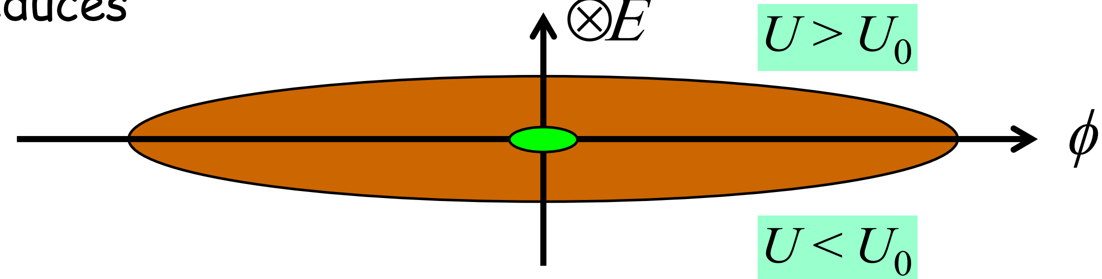
Longitudinal Motion with Synchrotron Radiation

Synchrotron radiation energy-loss energy dependant:

$$U_0 = \frac{4}{3} \square \frac{r_{e,p}}{(m_0 c^2)^3} \frac{E^4}{\rho}$$

During one period of synchrotron oscillation:

- when the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces



- when the particle is in the lower half-plane, it loses less energy per turn, but receives U_0 on the average, so its energy deviation gradually reduces

The phase space trajectory spirals towards the origin (limited by quantum excitations)

=> The synchrotron motion is damped toward an equilibrium bunch length and energy spread.

$$\sigma_\tau = \frac{\alpha}{\Omega_s}\left(\frac{\sigma_\varepsilon}{E}\right)$$

More details in the lectures on *Electron Beam Dynamics*

Longitudinal Dynamics in Synchrotrons

Now we will look more quantitatively at the "synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle.
Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient to follow other particles with respect to that particle.
So let's introduce the following **reduced variables**:

revolution frequency	:	$\Delta f_r = f_r - f_{rs}$
particle RF phase	:	$\Delta\phi = \phi - \phi_s$
particle momentum	:	$\Delta p = p - p_s$
particle energy	:	$\Delta E = E - E_s$
azimuth angle	:	$\Delta\theta = \theta - \theta_s$

Look at difference from synchronous particle

Equations of Longitudinal Motion

$$\frac{\Delta E}{\omega_0} = \frac{p_0 R}{h\eta\omega_0}\dot{\phi}$$

$$2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right) = e\,\hat{V}(\sin\phi - \sin\phi_s)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{-p_0 R}{h\eta\omega_0}\frac{d\phi}{dt}\right] + \frac{e\,\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.

We will study some cases in the following…

Small Amplitude Oscillations

Let's assume constant parameters R, p_0, ω_0 and η:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

with

$$\Omega_s^2 = \frac{-q\hat{V}_{RF}\eta h \omega_0}{2\pi R p_0}\cos\phi_s$$

Consider now small phase deviations from the reference particle:

$$\sin\phi - \sin\phi_s = \sin\left(\phi_s + \Delta\phi\right) - \sin\phi_s \cong \cos\phi_s \Delta\phi$$

(for small $\Delta\phi$)

and the corresponding linearized motion reduces to a harmonic oscillation:

$$\ddot{\phi} + \Omega_s^2 \Delta\phi = 0$$

where Ω_s is the synchrotron angular frequency.

The **synchrotron tune ν_s** is the number of synchrotron oscillations per revolution:

$$\nu_s = \Omega_s / \omega_0$$

Typical values are <<1, as it takes several 10 - 1000 turns per oscillation.

- proton synchrotrons of the order 10^{-3}
- electron storage rings of the order 10^{-1}

Stability condition for ϕ_s

$$\Omega_s^2 = \frac{-q\hat{V}_{RF}\eta h\omega_0}{2\pi R p_0}\cos\phi_s \Leftrightarrow \Omega_s^2 = \omega_0^2 \frac{-q\hat{V}_{RF}\eta h}{2\pi\beta^2 E}\cos\phi_s$$

with $Rp = \frac{\beta^2 E}{\omega}$

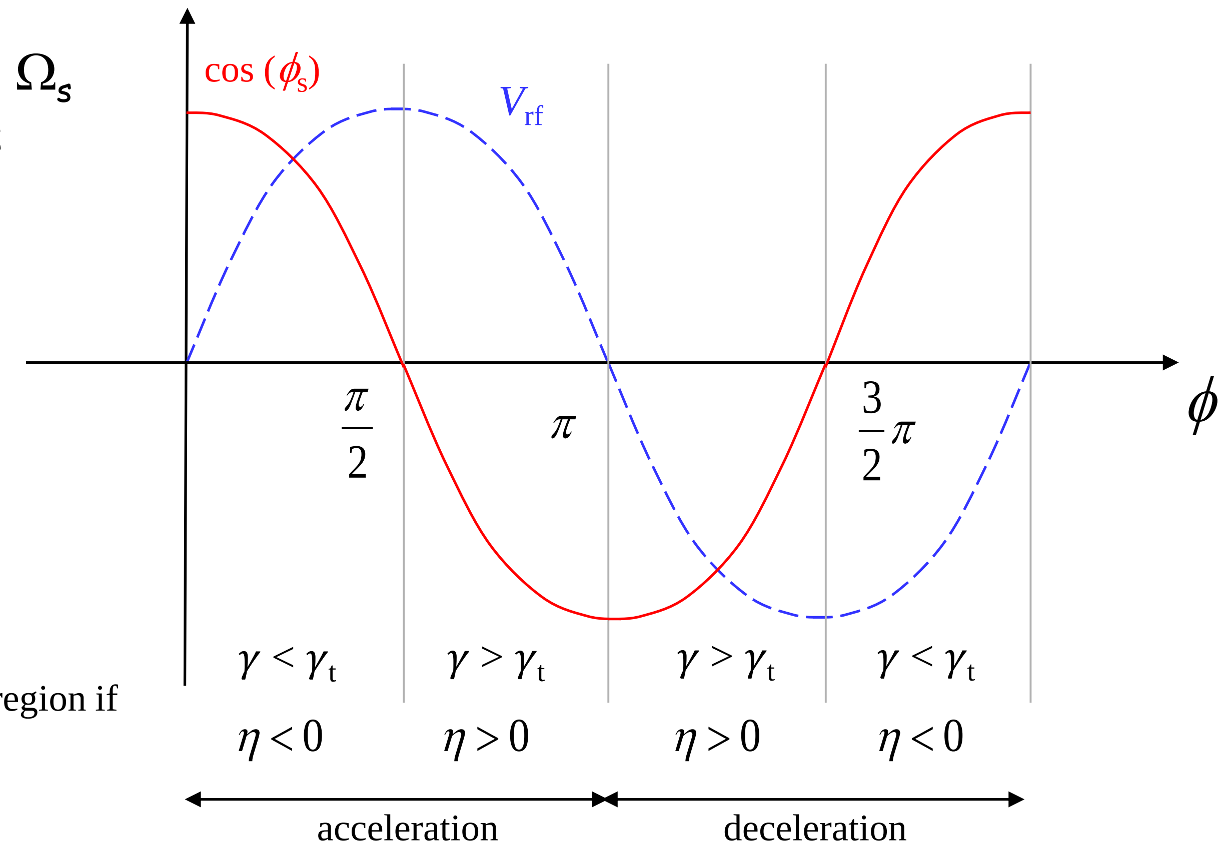
Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\Omega_s^2 > 0$$

$$\Updownarrow$$

$$\eta \cos\phi_s < 0$$

Stable in the region if

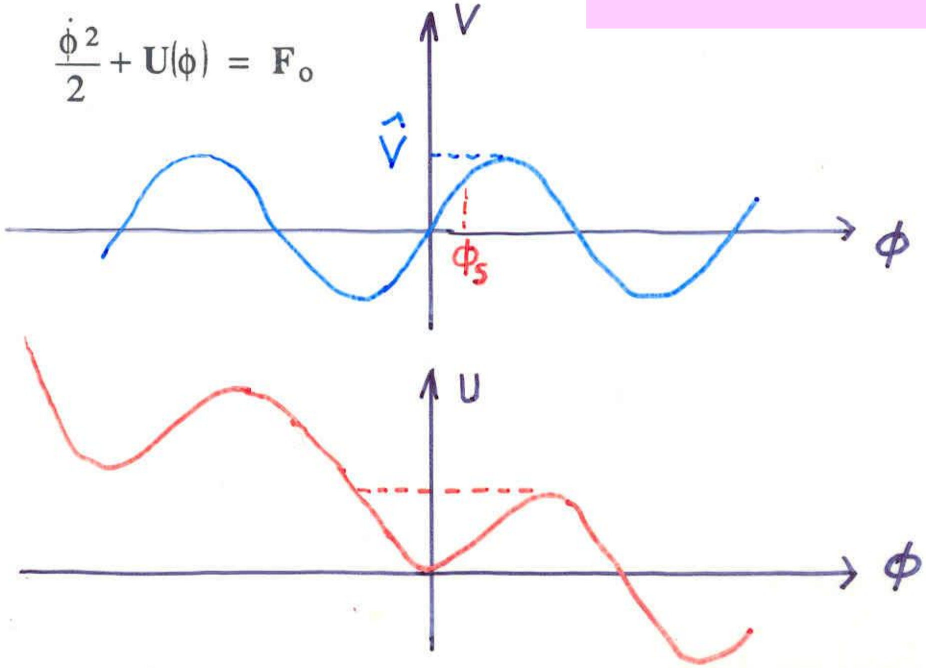


Potential Energy Function

The longitudinal motion is produced by a force that can be derived from a scalar potential:

$$\frac{d^2\phi}{dt^2} = F(\phi) \qquad F(\phi) = -\frac{\partial U}{\partial \phi}$$

$$U = -\int_0^\phi F(\phi)\,d\phi = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) - F_0$$



The sum of the potential energy and kinetic energy is constant and by analogy represents the total energy of a non-dissipative system.

Hamiltonian of Longitudinal Motion

Introducing a new convenient variable, W, leads to the 1st order equations:

$$W = \frac{\Delta E}{\omega_0} \longrightarrow \begin{aligned} \frac{d\phi}{dt} &= \frac{h\eta\omega_0}{p_0 R} W \\ \frac{dW}{dt} &= \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) \end{aligned}$$

The two variables ϕ,W are canonical since these equations of motion can be derived from a Hamiltonian H(ϕ,W,t):

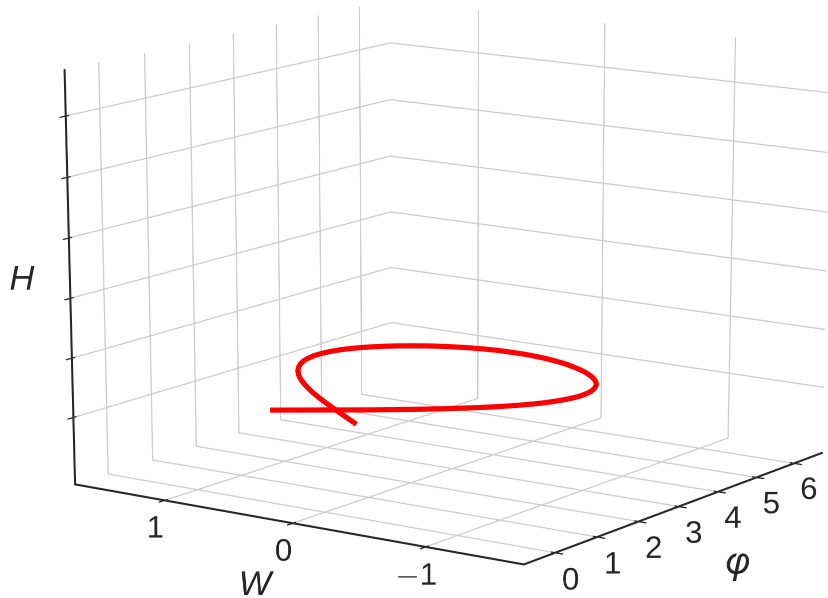
$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} \qquad \frac{dW}{dt} = -\frac{\partial H}{\partial \phi}$$

$$H(\phi, W) = \frac{1}{2}\frac{h\eta\omega_0}{p_0 R}W^2 + \frac{e\hat{V}}{2\pi}[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s]$$

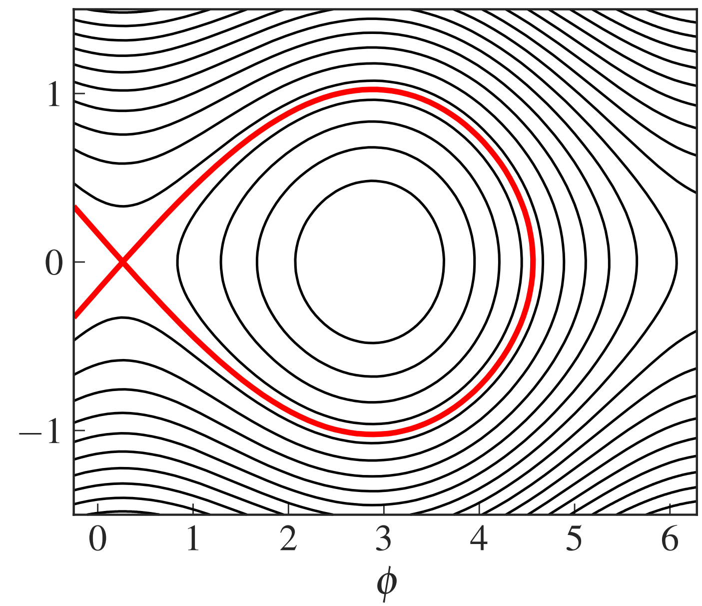
Hamiltonian of Longitudinal Motion

What does it represent? The total energy of the system!

Surface of H (φ,W)



Contours of H (φ,W)



Contours of constant H are particle trajectories in phase space!
(H is conserved)

Hamiltonian Mechanics can help us understand some fairly complicated dynamics (multiple harmonics, bunch splitting, …)

Energy Acceptance

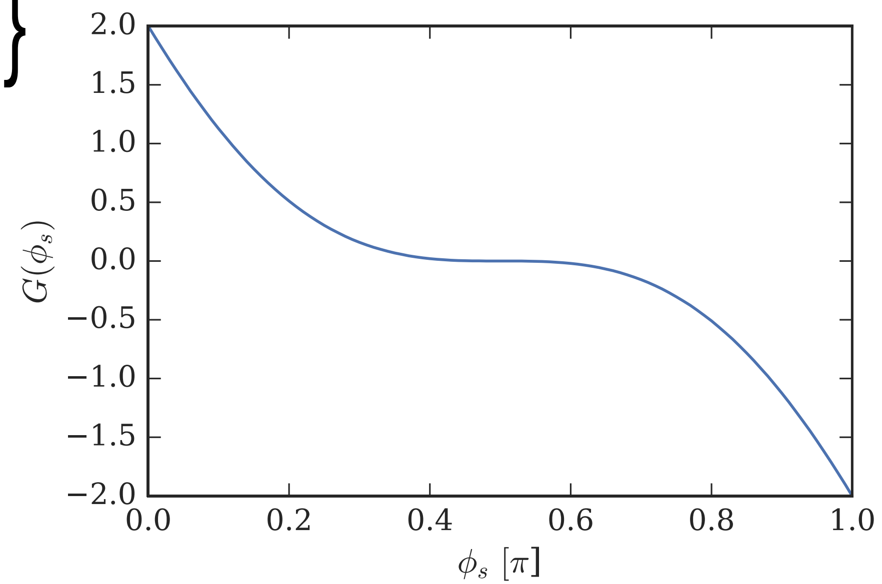
From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme at $\phi = \phi_s$.
Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{\max} = 2\Omega_s^2 \left\{ 2 + (2\phi_s - \pi)\tan\phi_s \right\}$$

That translates into an energy acceptance:

$$\left(\frac{\Delta E}{E_0}\right)_{\max} = \pm\beta \sqrt{\frac{-q\hat{V}}{\pi h \eta E_0} G(\phi_s)}$$

$$G(\phi_s) = \left[2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s \right]$$



This "RF acceptance" depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

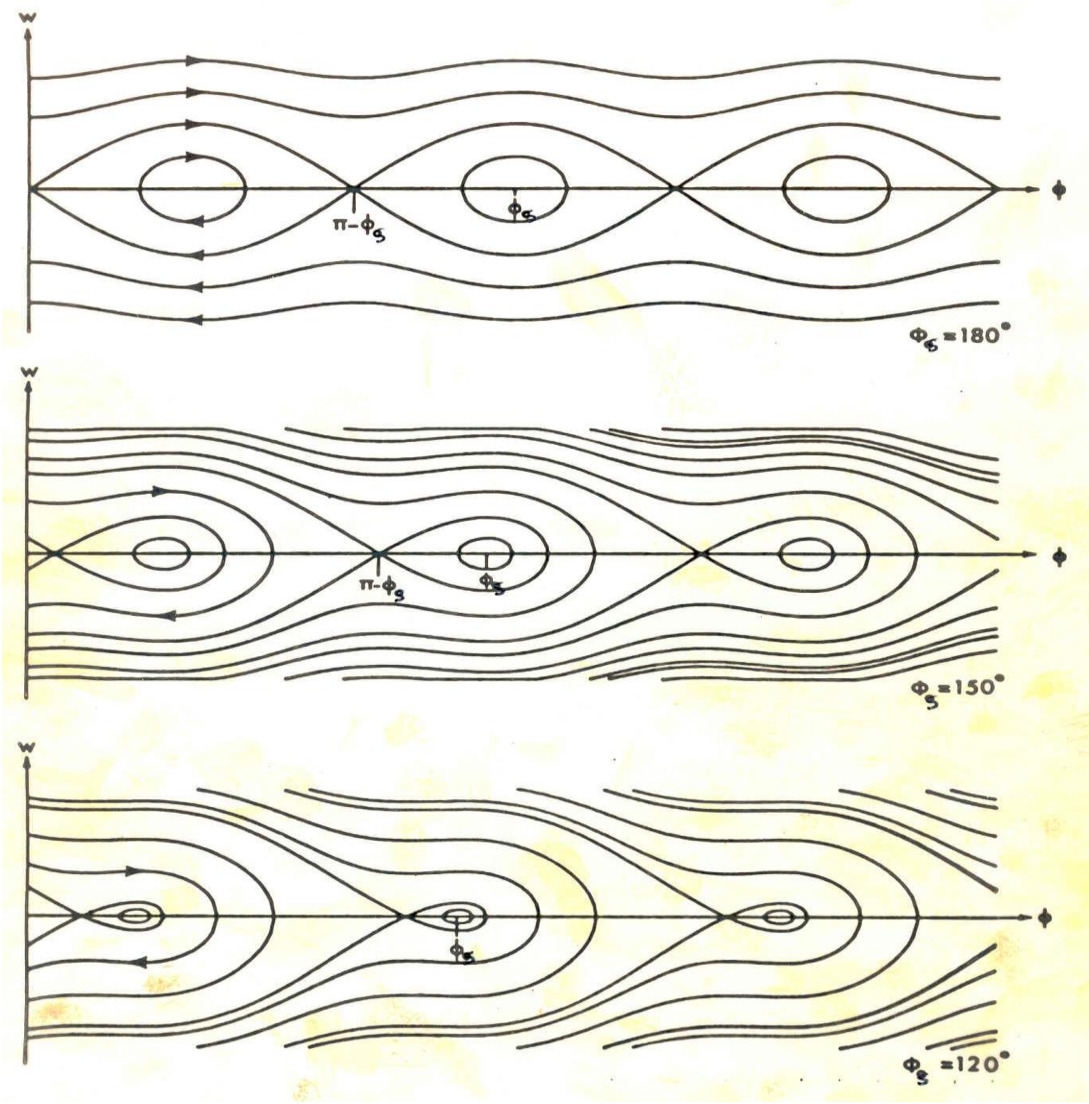
It's largest for ϕ_s=0 and ϕ_s=π (no acceleration, depending on η).

It becomes smaller during acceleration, when ϕ_s is changing

Need a higher RF voltage for higher acceptance.

For the same RF voltage it is smaller for higher harmonics h.

RF Acceptance versus Synchronous Phase



The areas of stable motion (closed trajectories) are called "BUCKET". The number of circulating buckets is equal to "h".

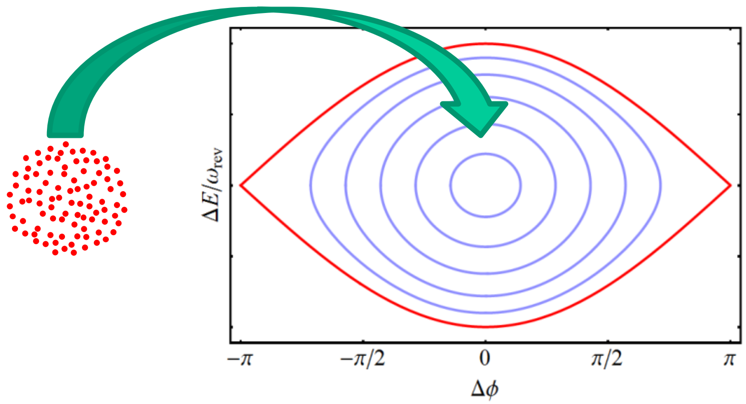
The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which means no acceleration.

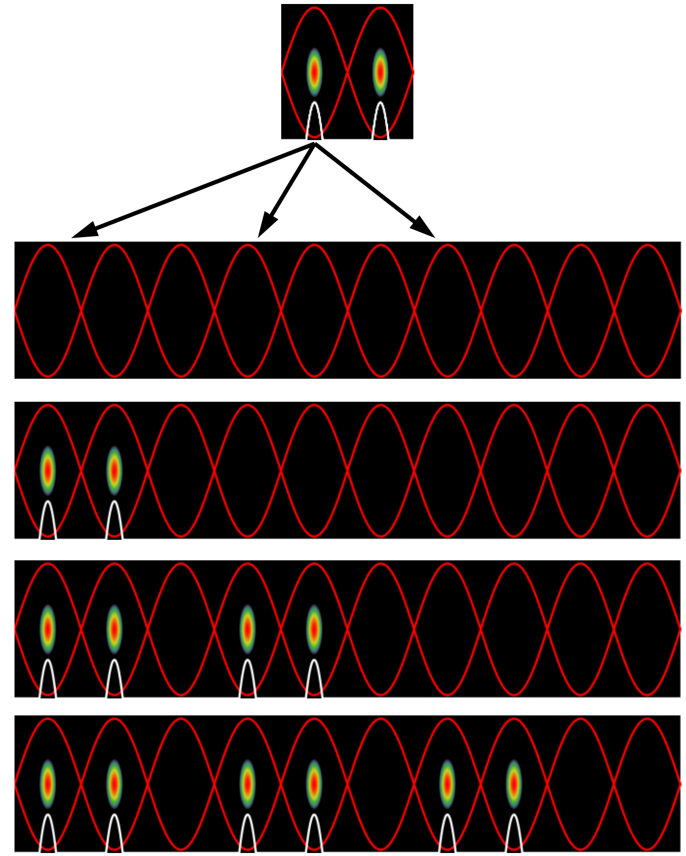
During acceleration, the buckets get smaller, both in length and energy acceptance.

=> Injection preferably without acceleration.

Injection: Bunch-to-bucket transfer

- Bunch from sending accelerator into the bucket of receiving





Advantages:

→ Particles always subject to longitudinal focusing

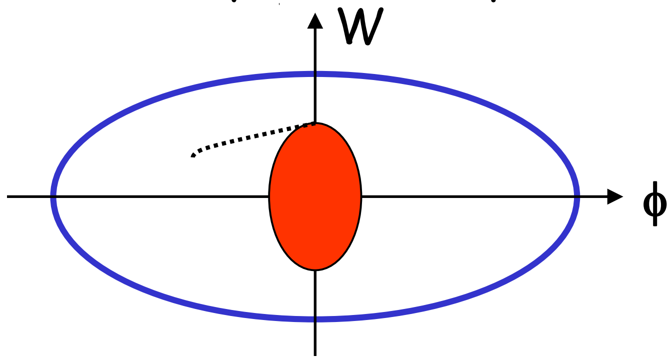
→ No need for RF capture of de-bunched beam in receiving accelerator

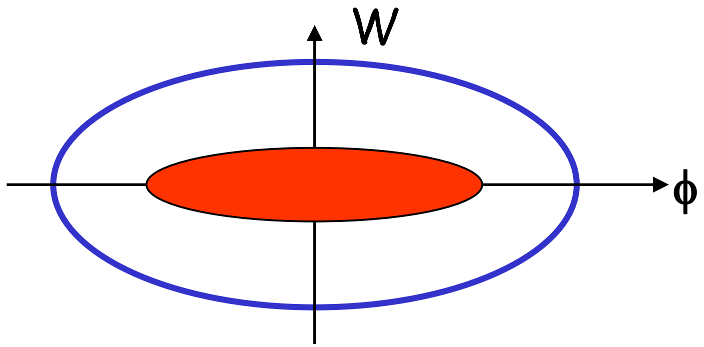
→ No particles at unstable fixed point

→ Time structure of beam preserved during transfer

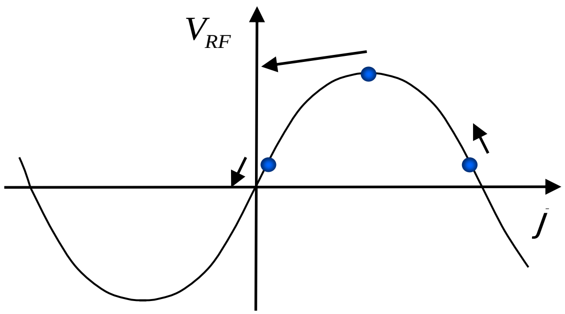
Effect of a Mismatch

Injected bunch: short length and large energy spread
after 1/4 synchrotron period: longer bunch with a smaller energy spread.

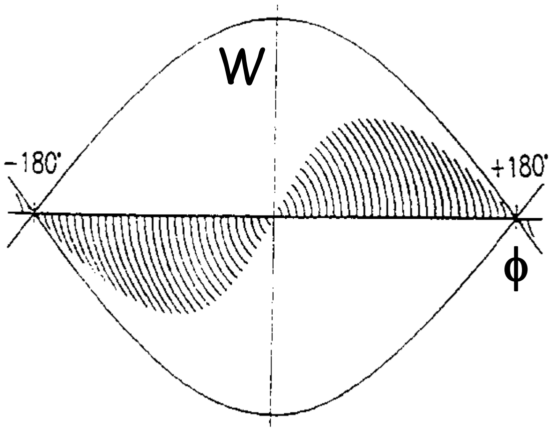




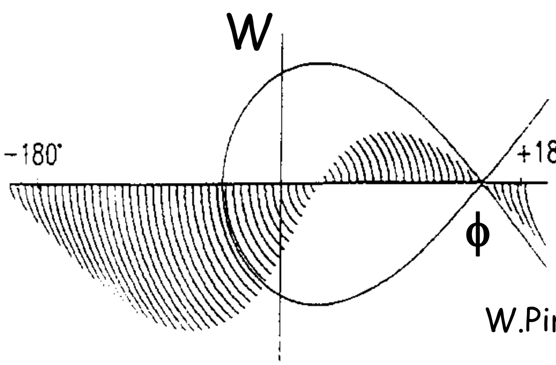
For larger amplitudes, the angular phase space motion is slower
(1/8 period shown below) => can lead to filamentation and emittance growth



restoring force is non-linear



stationary bucket



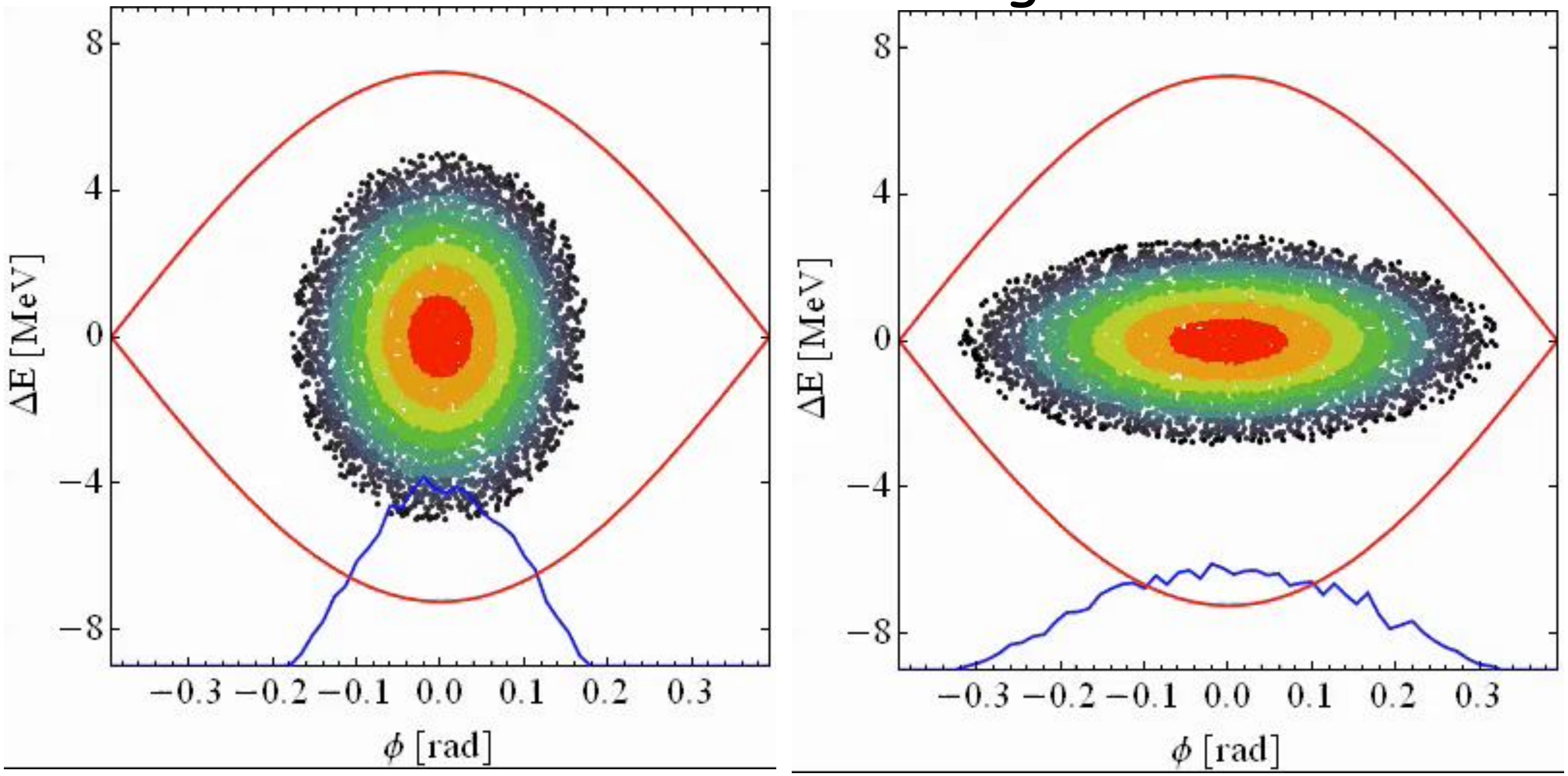
W.Pirkl

accelerating bucket

Effect of a Mismatch (2)

- Long. emittance is only preserved for correct RF voltage

Matched case | Longitudinal mismatch



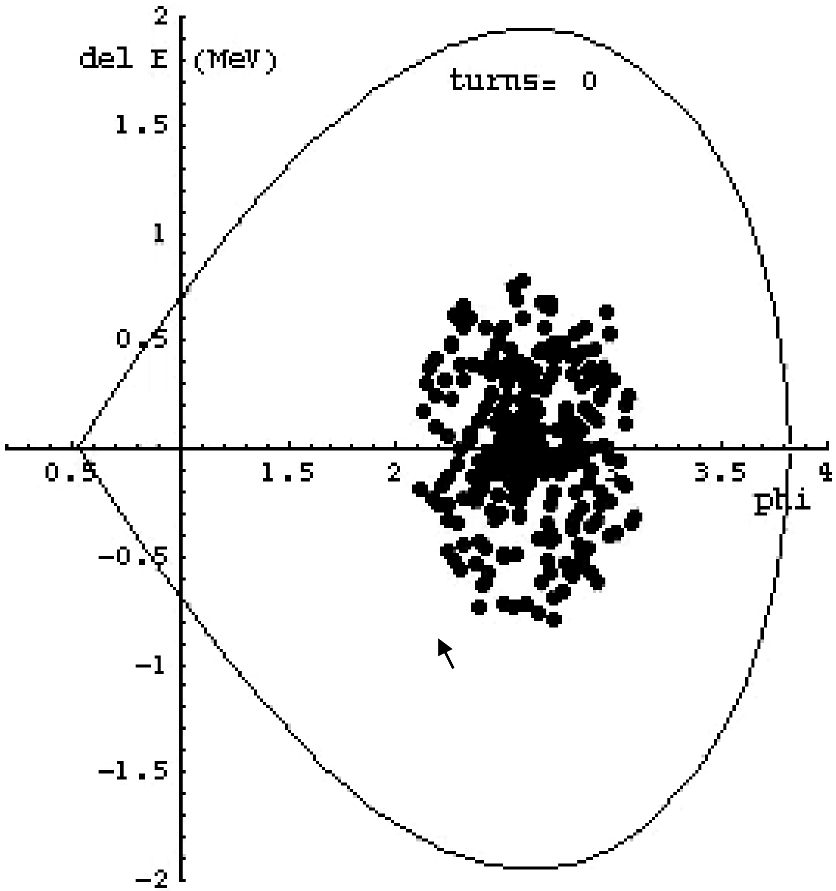
→ Bunch is fine, longitudinal emittance remains constant

→ Dilution of bunch results in increase of long. emittance

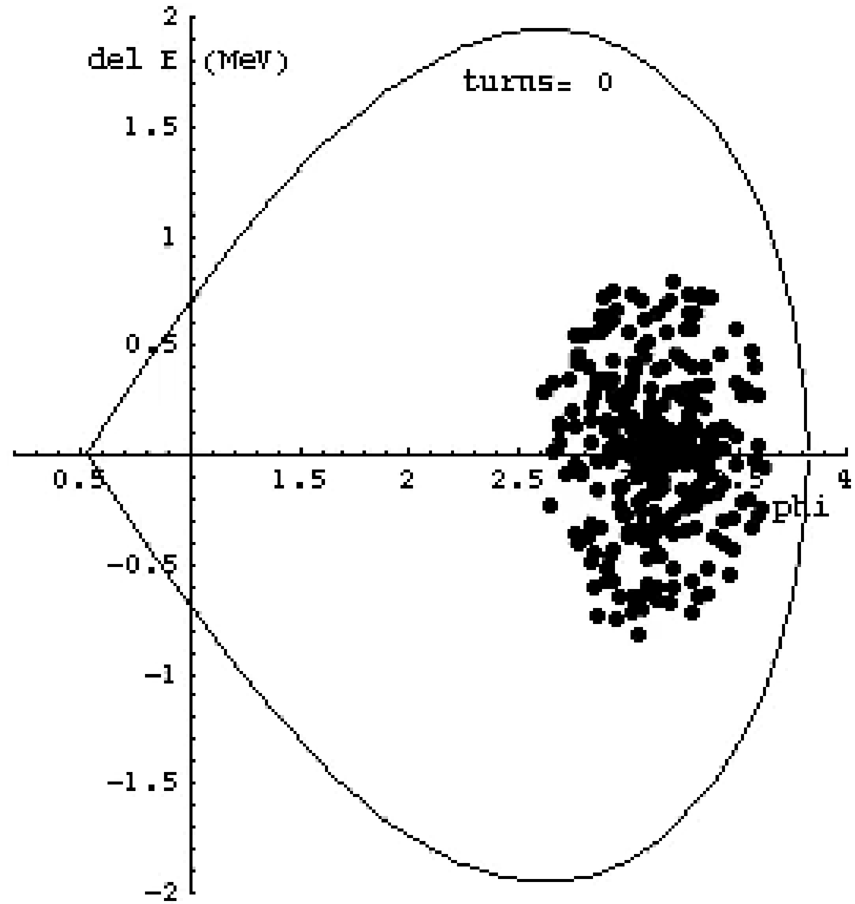
Effect of a Mismatch (3)

Evolution of an injected beam for the first 100 turns.

For a mismatched transfer, the emittance increases (right).



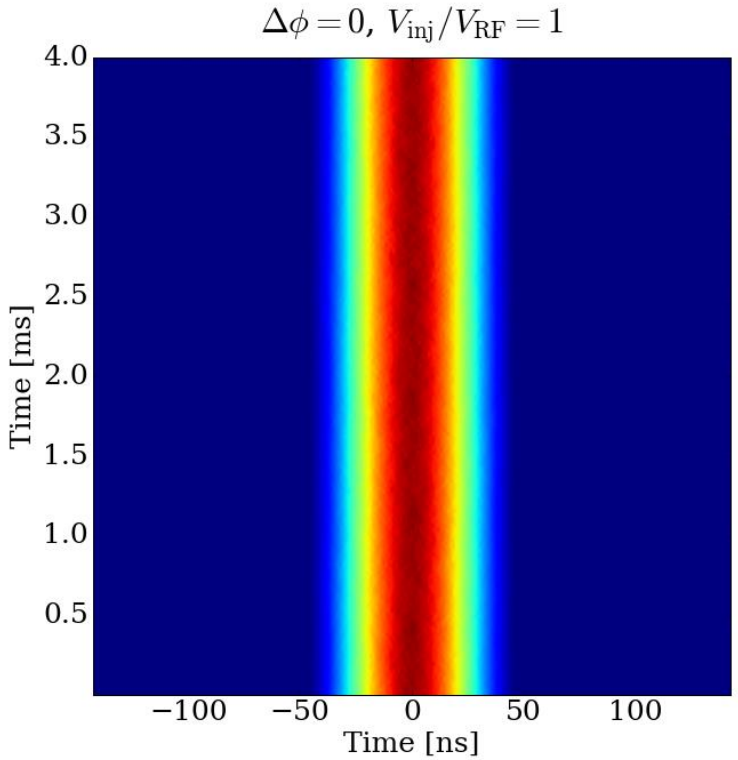
matched beam



mismatched beam – phase error

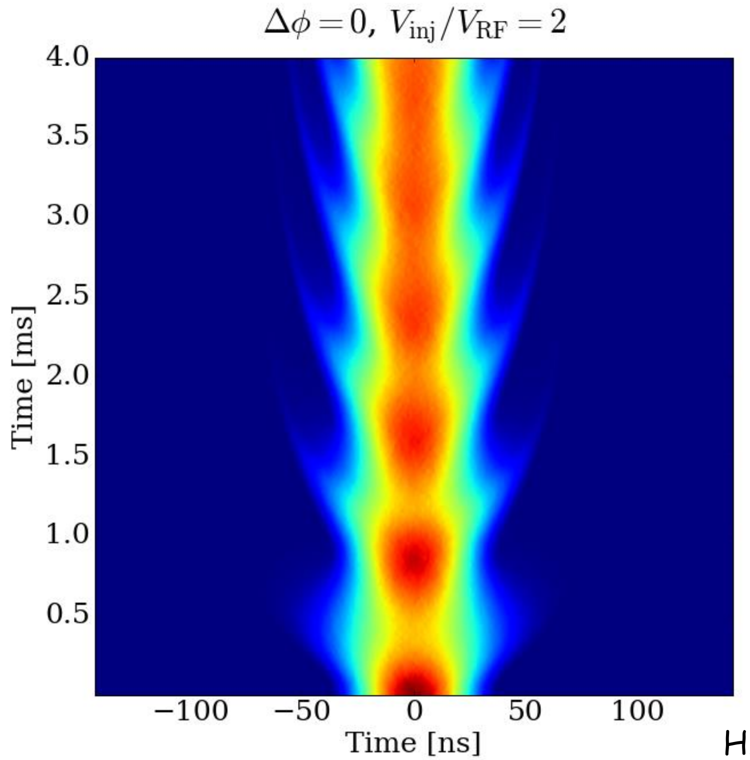
Longitudinal matching – Beam profile

Matched case



→ Bunch is fine, longitudinal emittance remains constant

Longitudinal mismatch

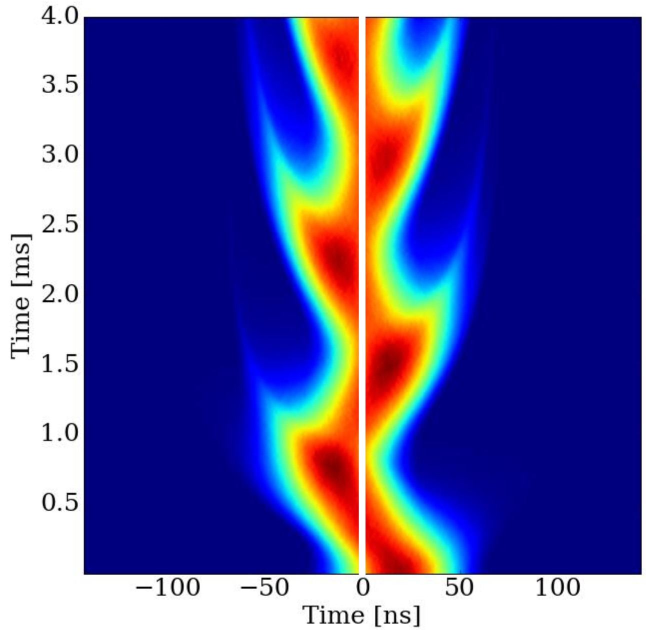


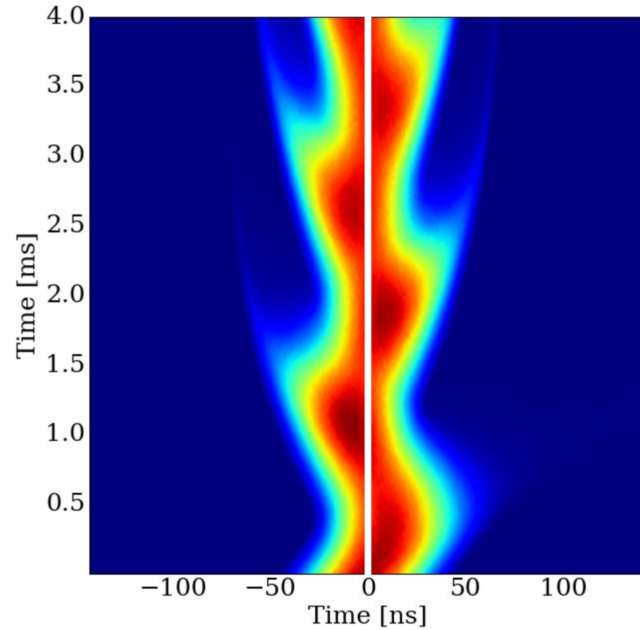
H.Damerau

→ Dilution of bunch results in increase of long. emittance

Matching quiz!

- Find the difference!





H.Damerau

→ -45° phase error at injection

→ Can be easily corrected by bucket phase

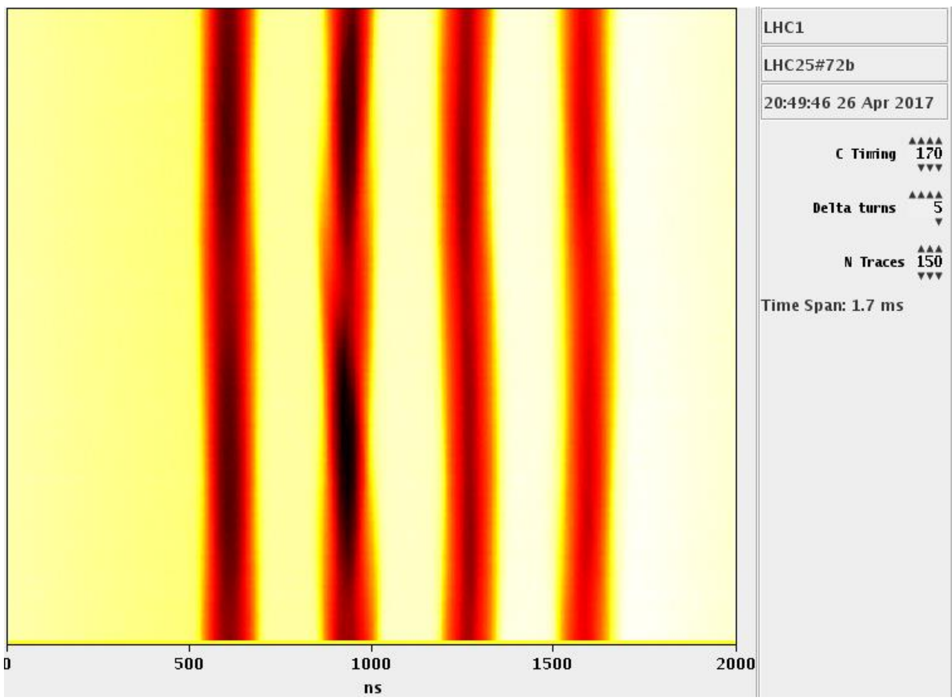
→ Equivalent energy error

→ Phase does not help: requires beam energy change

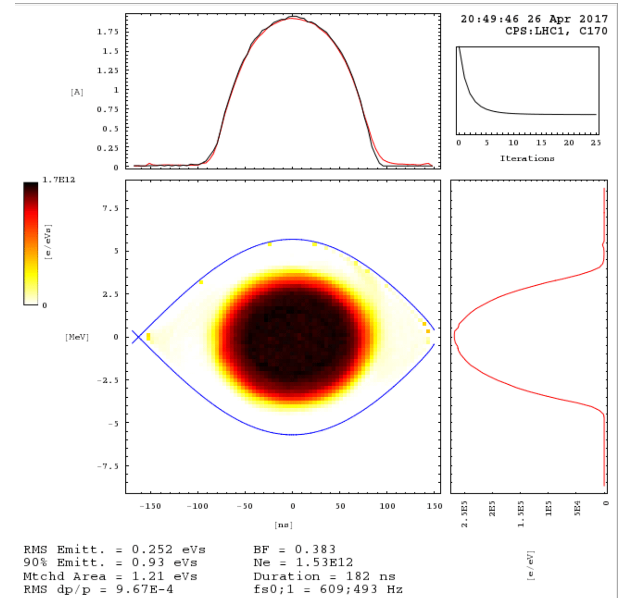
Phase Space Tomography

We can reconstruct the phase space distribution of the beam.

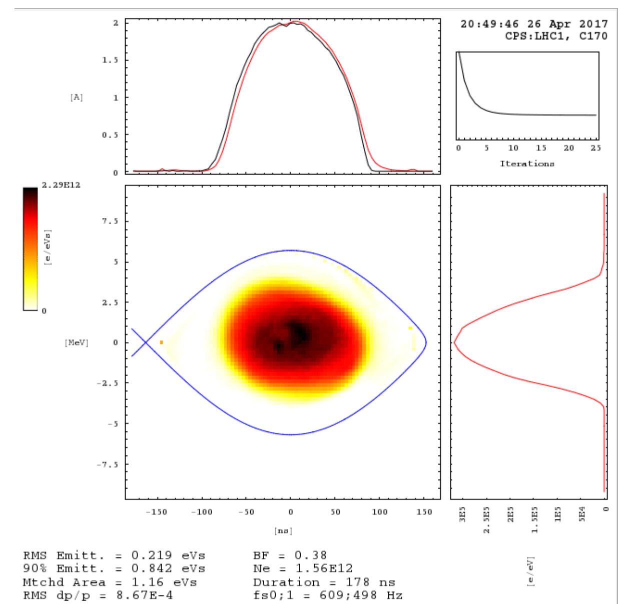
- Longitudinal bunch profiles over a number of turns
- Parameters determining Ω_s



1st bunch



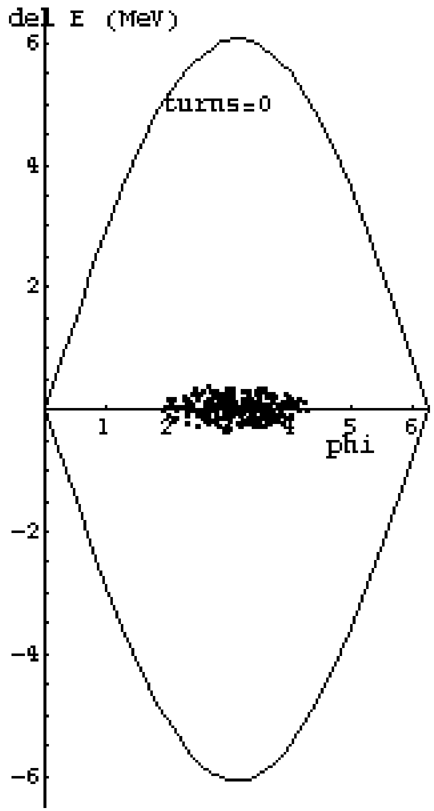
2nd bunch

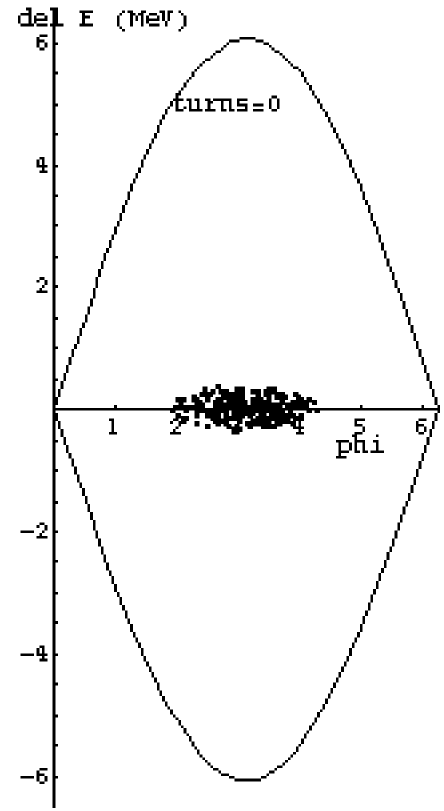


Bunch Rotation

Phase space motion can be used to make short bunches.

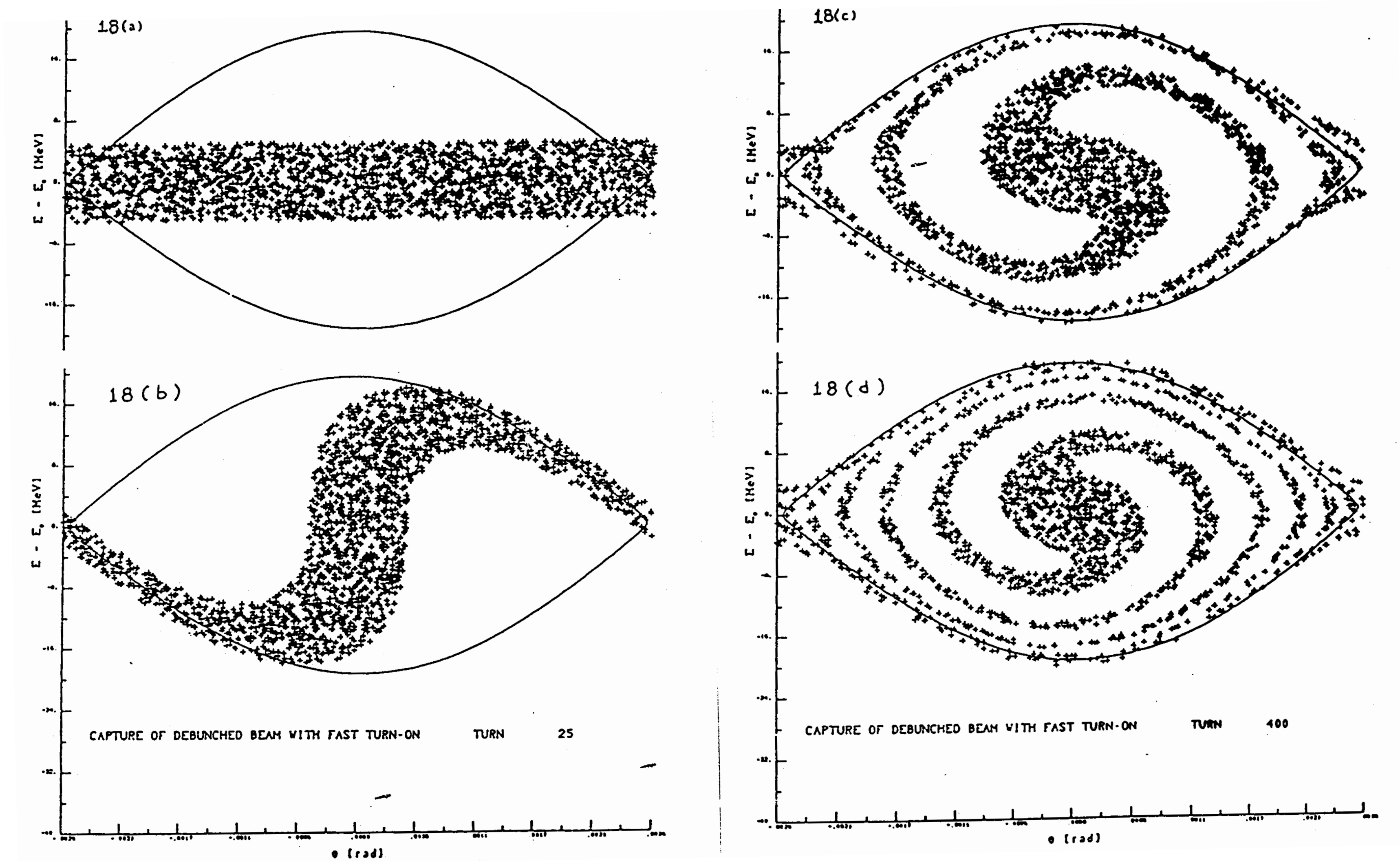
Start with a long bunch and extract or recapture when it's short.



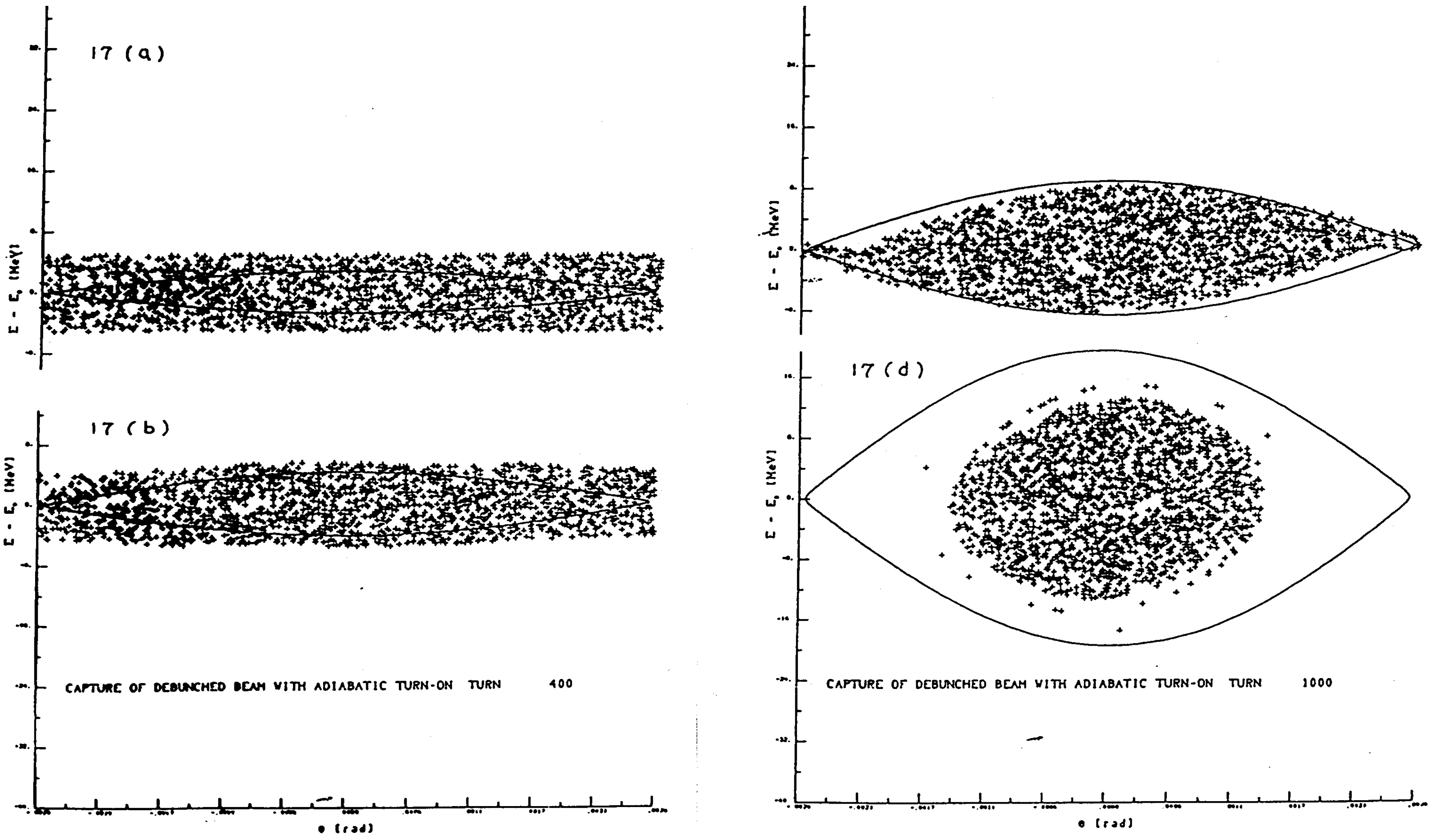


initial beam

Capture of a Debunched Beam with Fast Turn-On

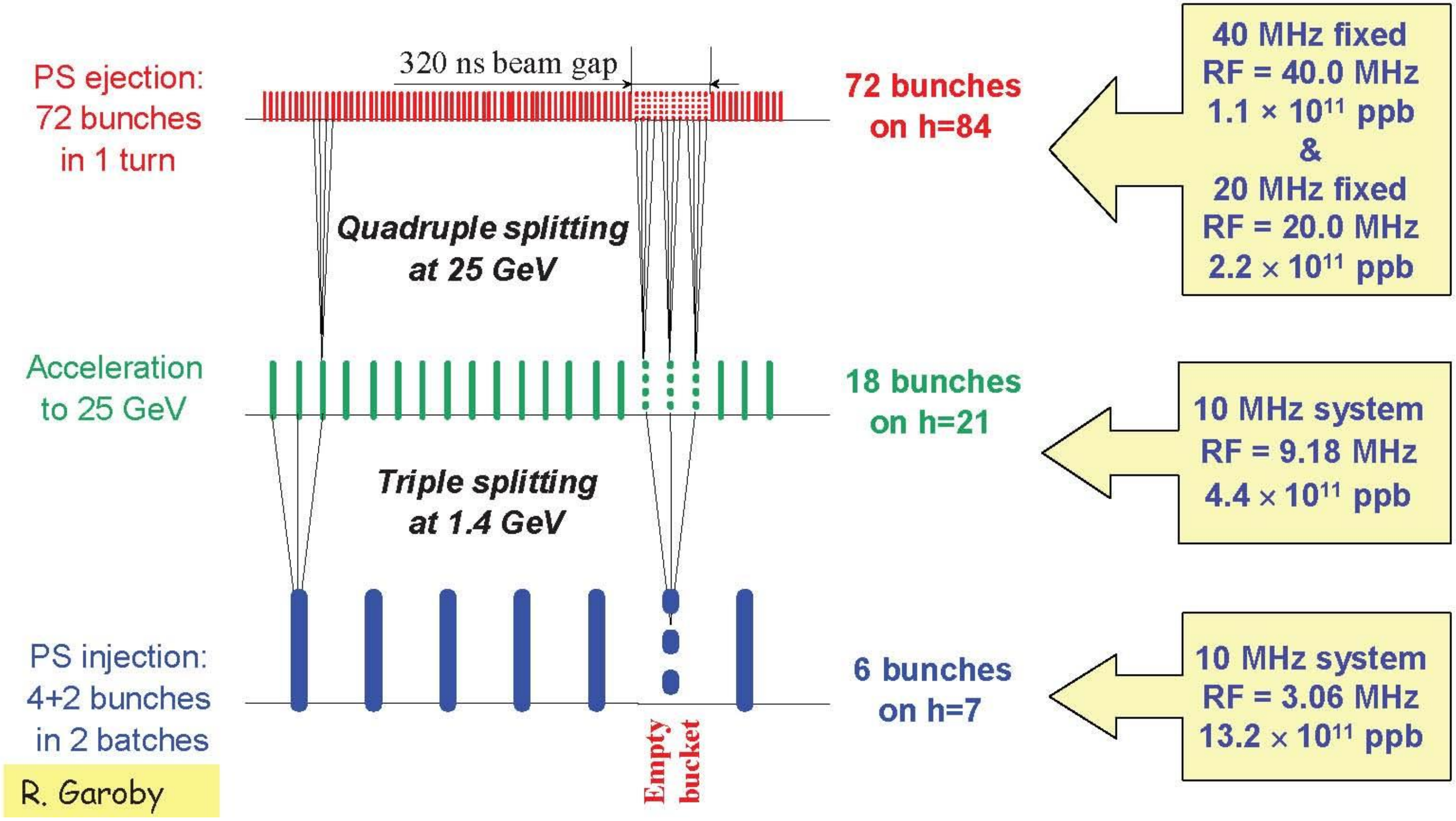


Capture of a Debunched Beam with Adiabatic Turn-On



Generating a 25ns Bunch Train in the PS

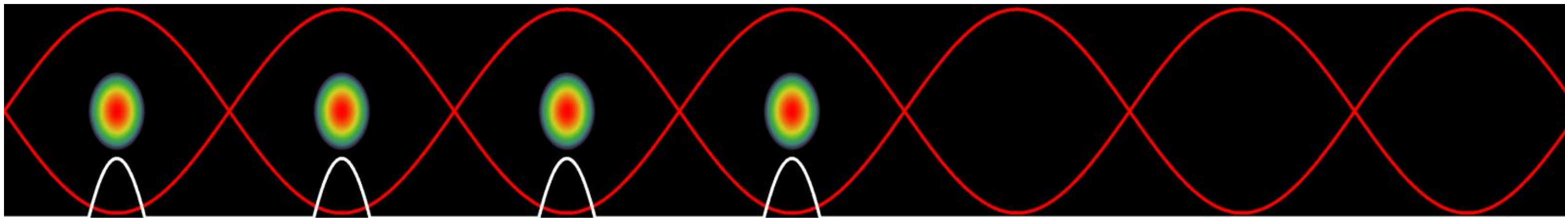
- **Longitudinal bunch splitting (basic principle)**
  - Reduce voltage on principal RF harmonic and simultaneously rise voltage on multiple harmonics (adiabatically with correct phase, etc.)



Use double splitting at 25 GeV to generate 50ns bunch trains instead

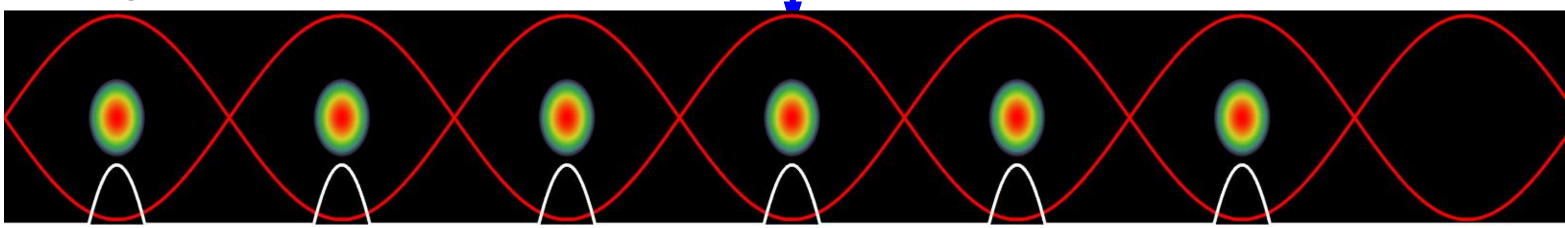
Production of the LHC 25 ns beam

1. Inject four bunches **~ 180 ns, 1.3 eVs**

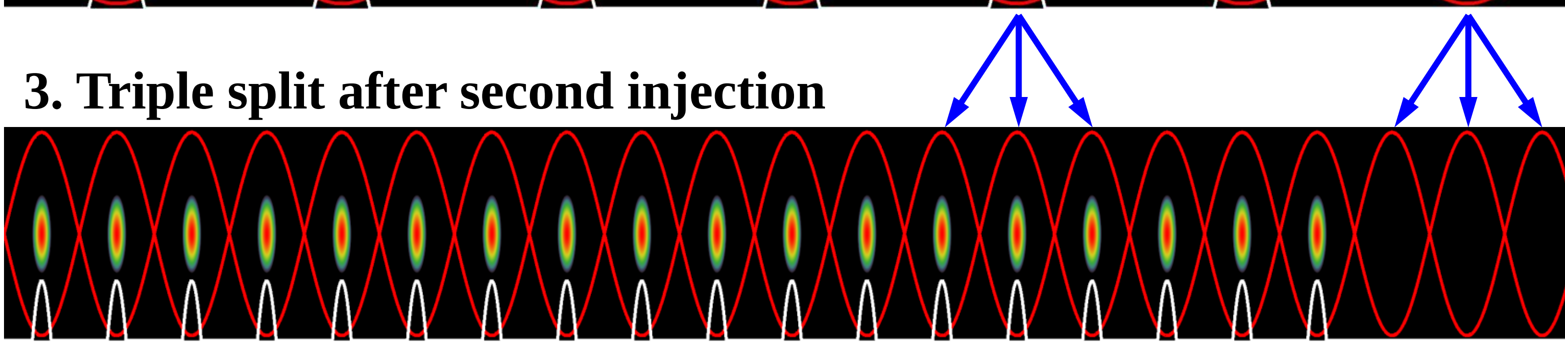


Wait 1.2 s for second injection

2. Inject two bunches



3. Triple split after second injection

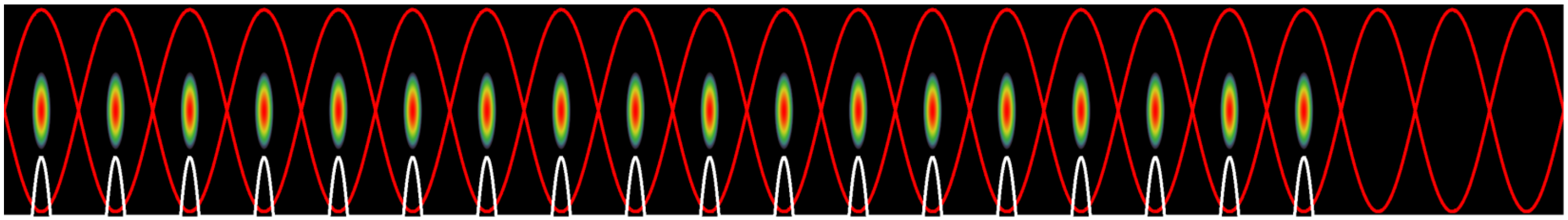


~ 0.7 eVs

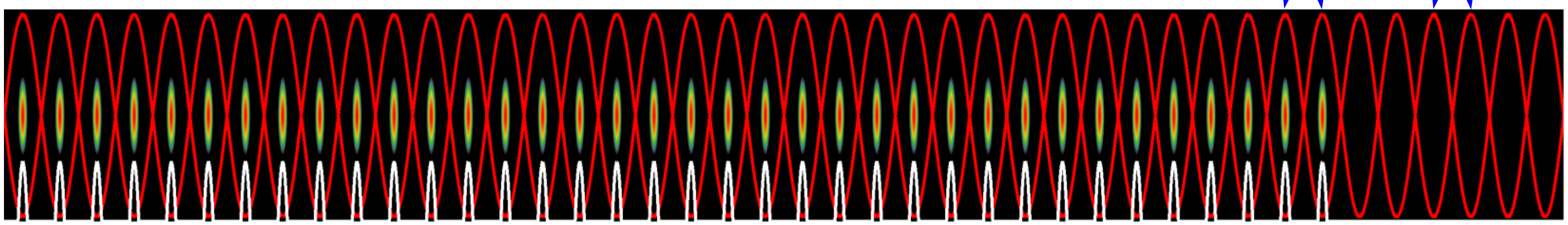
4. Accelerate from 1.4 GeV (E_{kin}) to 26 GeV

Production of the LHC 25 ns beam

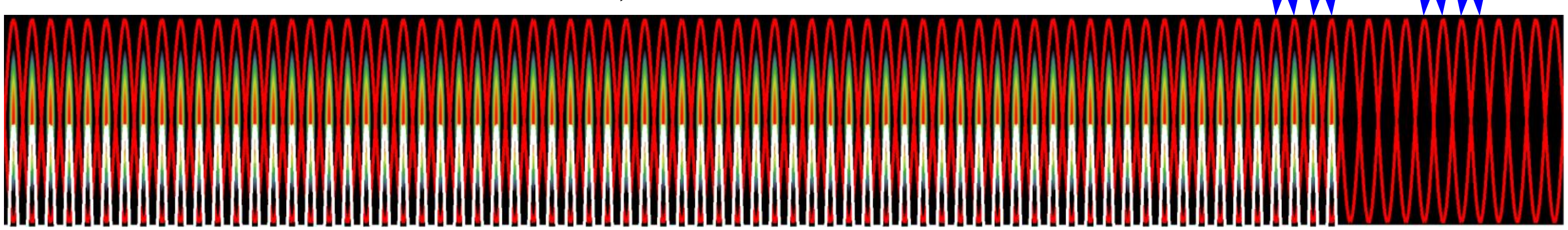
5. During acceleration: longitudinal emittance blow-up: 0.7 – 1.3 eVs



6. Double split (*h*21 → *h*42)

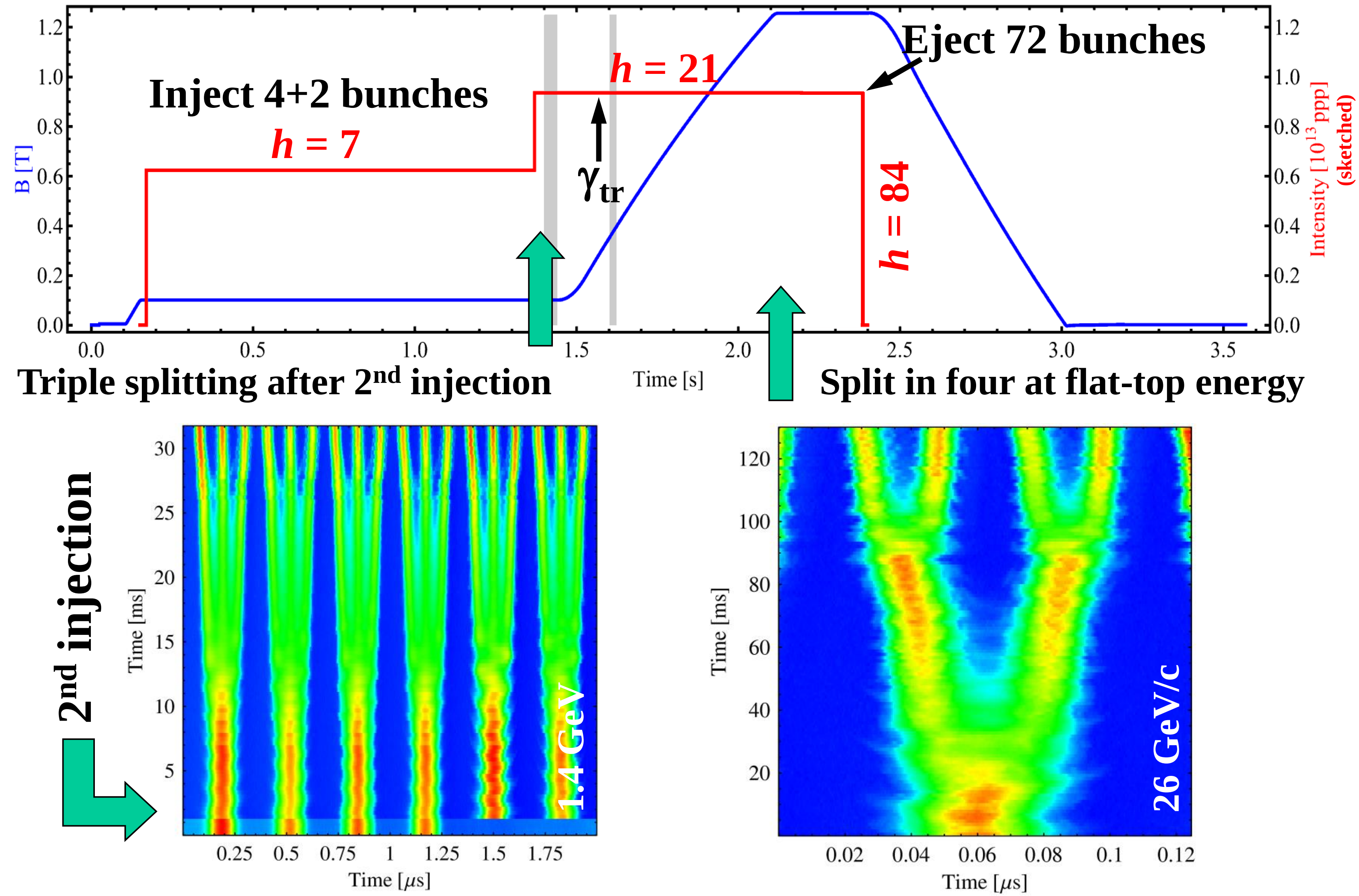


7. Double split (*h*42 → *h*84) **~ 0.35 eVs, 4 ns**



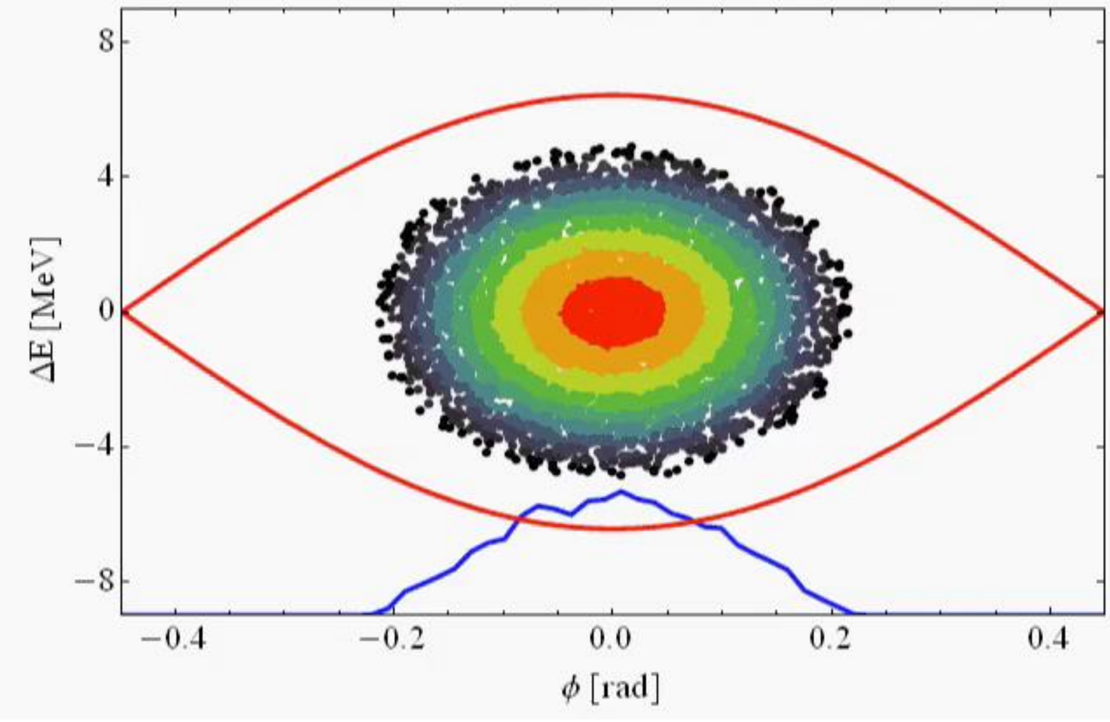
10. Fine synchronization, bunch rotation → Extraction!

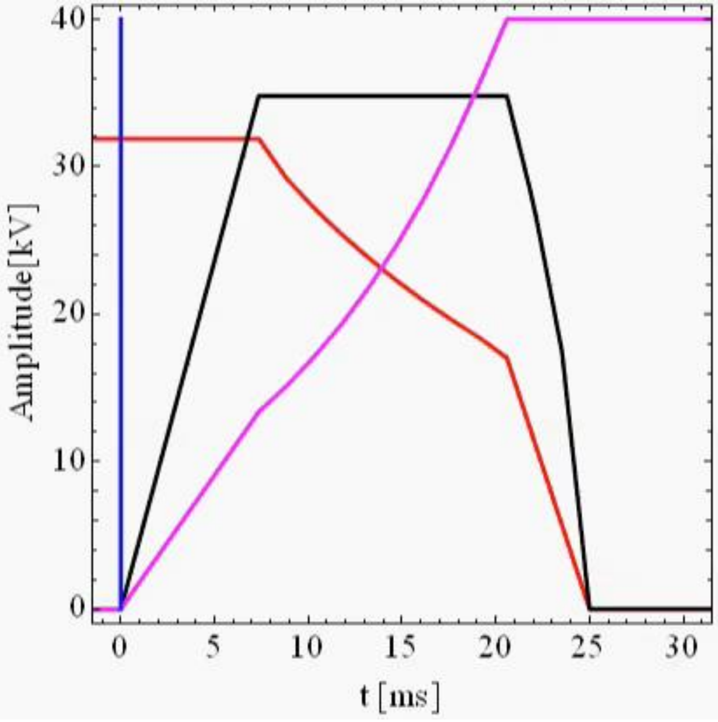
The LHC25 (ns) cycle in the PS



→ Each bunch from the Booster divided by 12 → 6 × 3 × 2 × 2 = 72

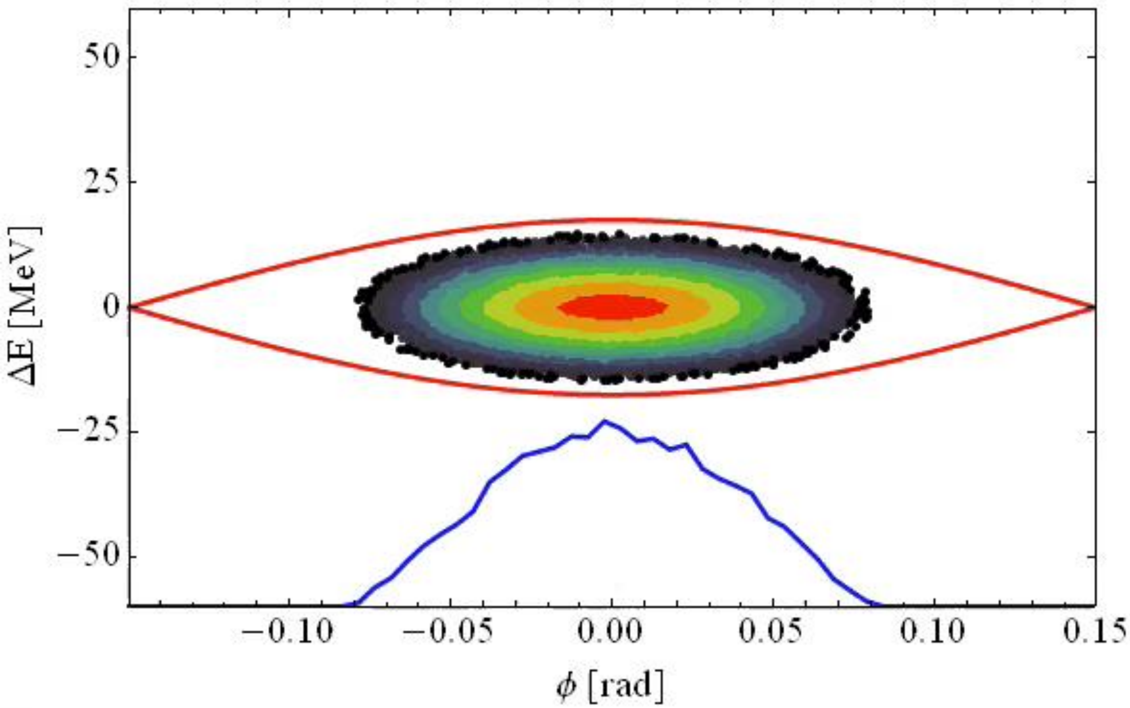
Triple splitting in the PS

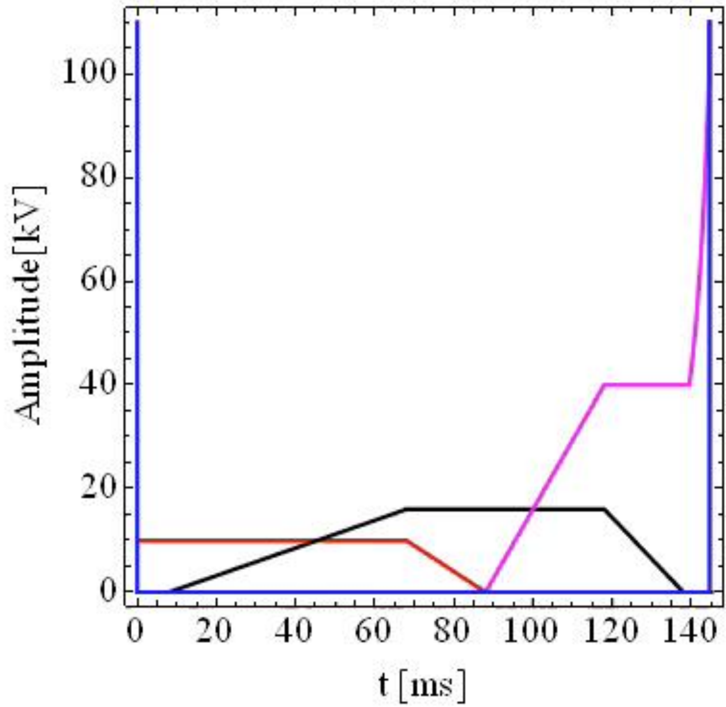




Two times double splitting in the PS

Two times double splitting and bunch rotation:





- Bunch is divided twice using RF systems at
  h = 21/42 (10/20 MHz) and *h* = 42/84 (20/40 MHz)

- Rotation: first part *h*84 only + *h*168 (80 MHz) for final part

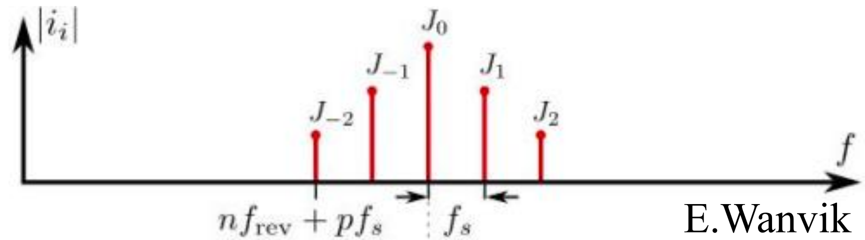
Synchrotron tune measurement

Reminder: Non-linear force => Synchrotron tune depends on amplitude

Principle A: The synchrotron oscillation modulates the arrival time of a bunch.

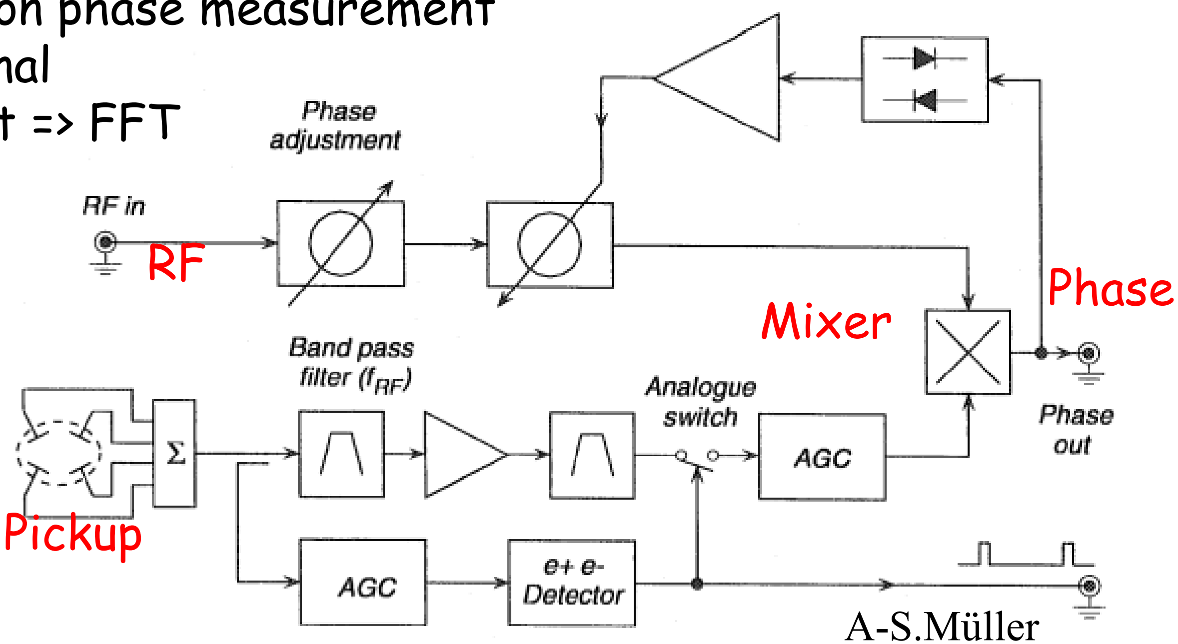
Use pick-up intensity signal and perform an FFT

⇒ The synchrotron tune will appear as sideband of revolution harmonics



Practical approach: Synchrotron phase measurement
Mix the signal with the RF signal
=> proportional to phase offset => FFT

Problem for proton machines since the synchrotron tune is very small.
The revolution harmonic lines are huge compared to the synchrotron lines,
so a very good and narrow bandwidth filter is needed to separate them

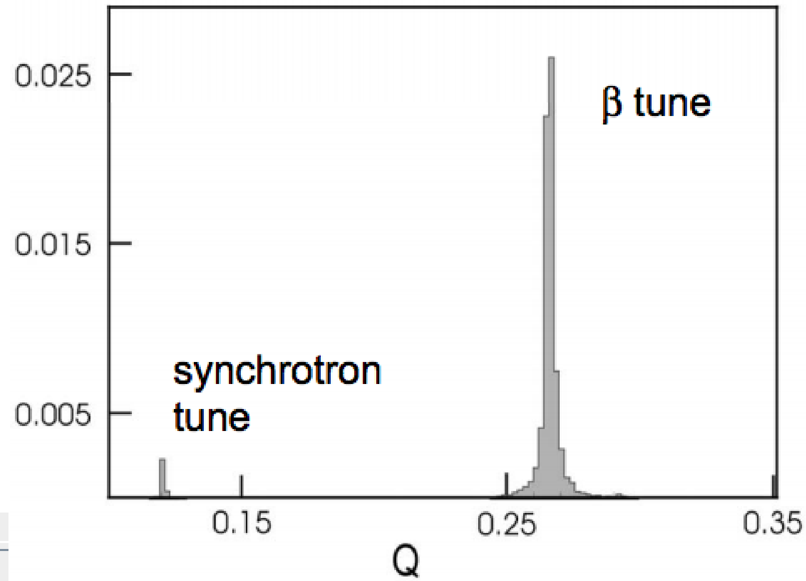


Synchrotron tune measurement - cont.

Principle B: The transverse beam position is modulated through dispersion:

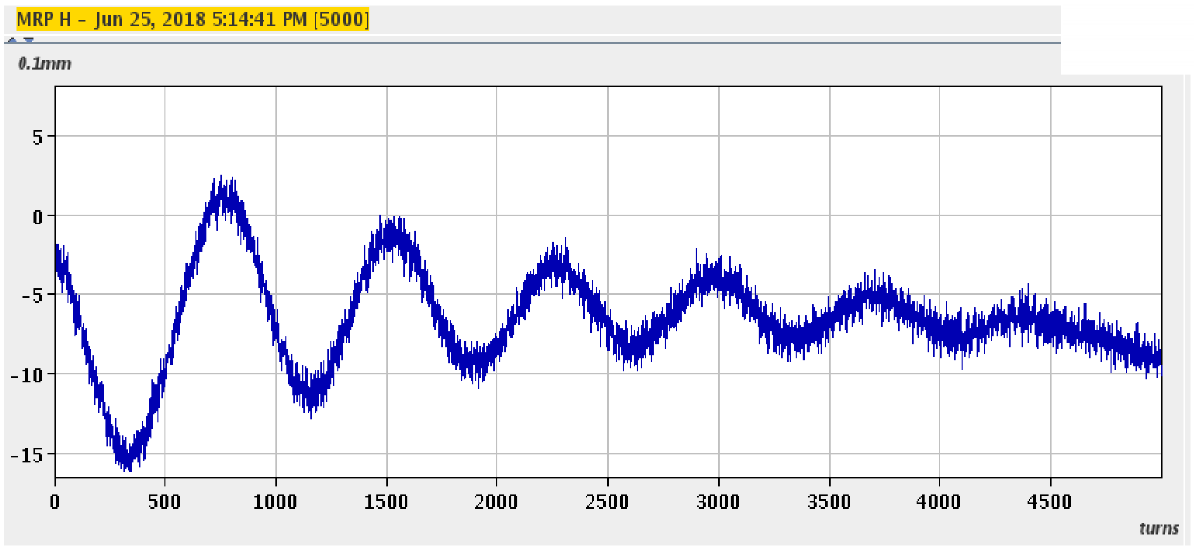
$$x = x_0 + D\,\frac{\Delta p}{p}$$

Use horizontal position signal from a BPM in dispersive region + perform FFT



A-S.Müller

Radial beam position after injection with phase/energy offset (at the PS)



Synchrotron tune measurement - cont.

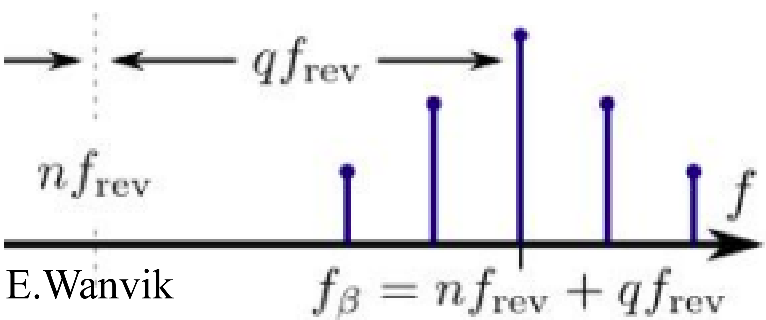
Principle C: The transverse tune is modulated through chromaticity:

$$Q = Q_0 + \xi \frac{\Delta p}{p}$$

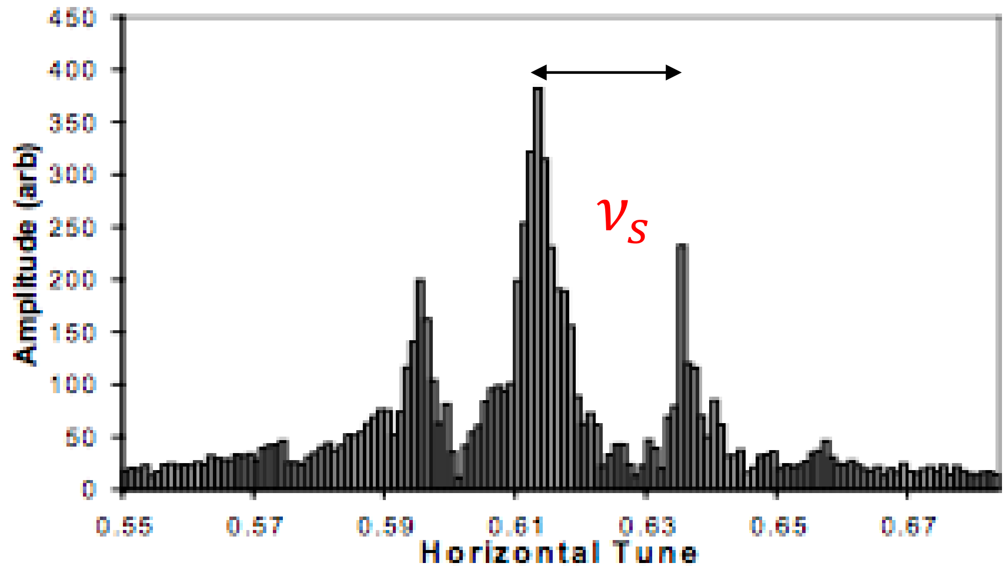
Frequency modulation (FM) of the betatron tunes.

Use horizontal position signal from a BPM + perform FFT

The synchrotron tune will appear as sidebands of the betatron tune.



Tune measurement for positrons (at the SPS)



A Boudzko (EPAC'98)

Summary


End of our
crash course in
longitudinal dynamics
www.formula1.com

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And CERN Accelerator Schools (CAS) Proceedings
In particular: CERN-2014-009
Advanced Accelerator Physics - CAS

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- Luca Bottura
- Berkeley Lab
- Edukite Learning

Appendix

- Summary Relativity and Energy Gain
- Velocity, Energy, and Momentum
- Momentum compaction factor
- Synchrotron energy-phase oscillations
- Stability condition
- Separatrix stationary bucket
- Large amplitude oscillations
- Bunch matching into stationary bucket

Appendix: Relativity + Energy Gain

Newton-Lorentz Force $\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$

2nd term always perpendicular to motion => no acceleration

Relativistics Dynamics

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \qquad \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$$

$$p = mv = \frac{E}{c^2}\beta c = \beta\frac{E}{c} = \beta\gamma m_0 c$$

$$E^2 = E_0^2 + p^2c^2 \longrightarrow dE = v\,dp$$

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

$$dE = dW = eE_z\,dz \rightarrow W = e\int E_z\,dz$$

RF Acceleration

$$E_z = \hat{E}_z \sin\omega_{RF}t = \hat{E}_z \sin\phi(t)$$

$$\int \hat{E}_z\,dz = \hat{V}$$

$$W = e\hat{V}\sin\phi$$

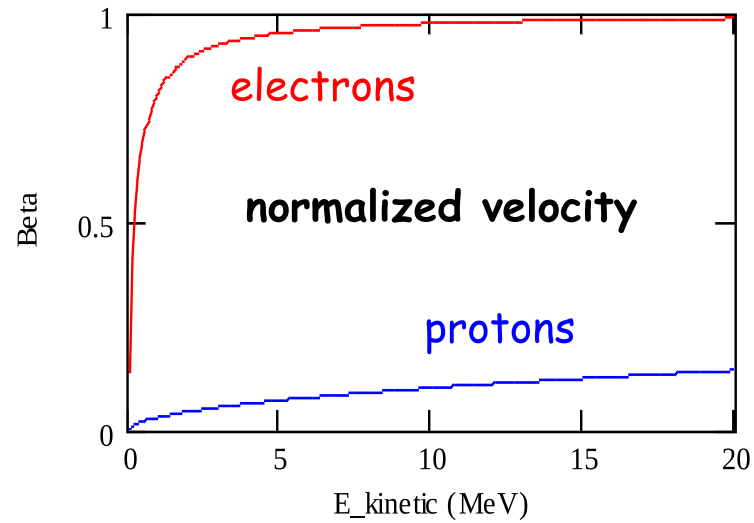
(neglecting transit time factor)

The field will change during the passage of the particle through the cavity
=> effective energy gain is lower

Appendix: Velocity, Energy and Momentum

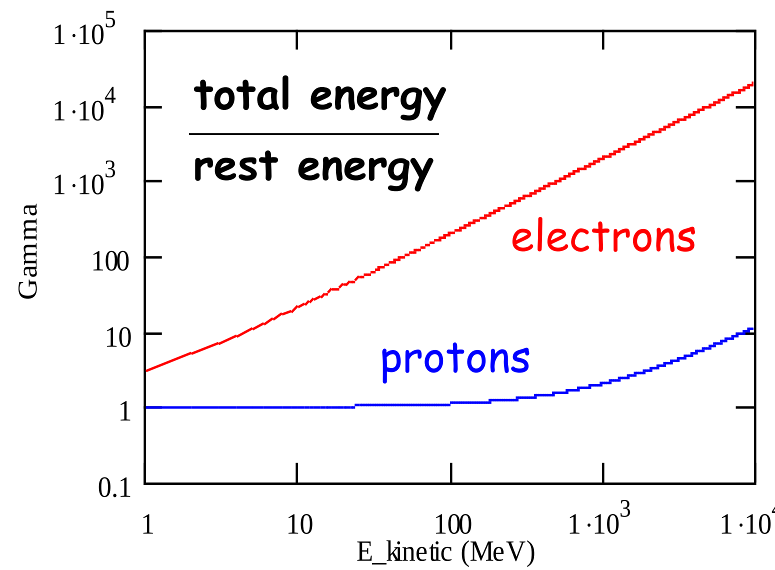
normalized velocity $\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$

=> electrons almost reach the speed of light very quickly (few MeV range)



$$\frac{\text{total energy}}{\text{rest energy}} \qquad E = \gamma m_0 c^2$$

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$$



Momentum $p = mv = \frac{E}{c^2}\beta c = \beta \frac{E}{c} = \beta\gamma m_0 c$

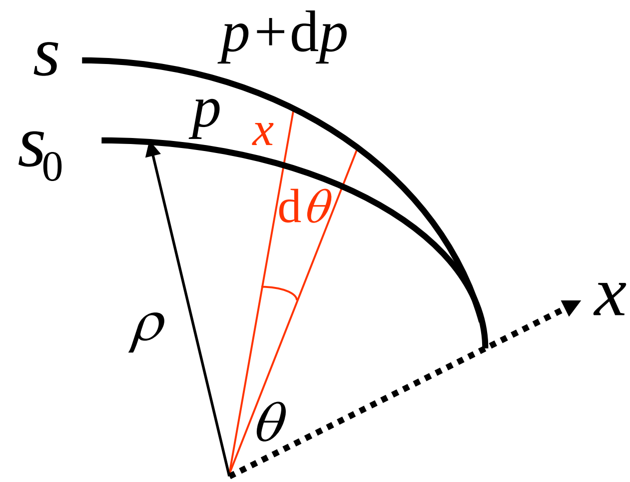
=> Magnetic field needs to follow the momentum increase

Appendix: Momentum Compaction Factor

$$\alpha_c = \frac{p}{L}\frac{dL}{dp}$$

$$ds_0 = r\,dq$$

$$ds = (r + x)\,dq$$



The elementary path difference from the two orbits is:

definition of dispersion D_x

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{r} = \frac{D_x}{r}\frac{dp}{p}$$

$$x = x_0 + D_x\frac{\Delta p}{p}$$

leading to the total change in the circumference:

$$dL = \oint_C dl = \oint \frac{x}{r}\,ds_0 = \oint \frac{D_x}{r}\frac{dp}{p}\,ds_0$$

$$\alpha_c = \frac{1}{L}\int_C \frac{D_x(s)}{\rho(s)}\,ds_0$$

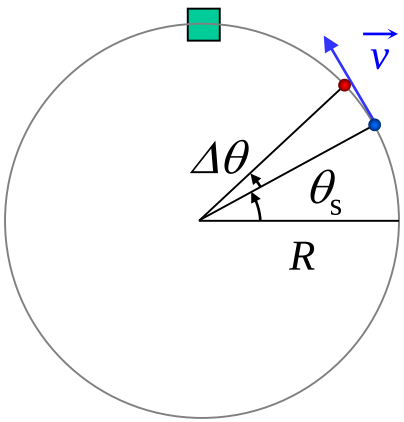
With ρ=∞ in straight sections we get:

$$\alpha_c = \frac{\langle D_x\rangle_m}{R}$$

$\langle\ \rangle_m$ means that the average is considered over the bending magnet only

Property of the **transverse** beam optics!

Appendix: First Energy-Phase Equation



$$f_{RF} = h f_r \quad \Rightarrow \quad \mathrm{D}f = -h\,\mathrm{D}q \quad \textit{with} \quad q = \int w\; dt$$

particle ahead arrives earlier
=> smaller RF phase

For a given particle with respect to the reference one:

$$\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $$\eta = -\frac{p_0}{\omega_0}\left(\frac{d\omega}{dp}\right)_s$$ and $$E^2 = E_0^2 + p^2c^2$$

$$\Delta E = v_s \Delta p = \omega_0 R \Delta p$$

one gets: $$\frac{\Delta E}{\omega_0} = \frac{p_0 R}{h\eta\omega_0}\frac{d(\Delta\phi)}{dt} = \frac{p_0 R}{h\eta\omega_0}\dot{\phi}$$

Appendix: Second Energy-Phase Equation

The rate of energy gained by a particle is: $$\frac{dE}{dt} = e\hat{V}\sin\phi \, \frac{\omega_r}{2\pi}$$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_0}\right) = e\,\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left-hand side to first order:

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}\,\Delta T_r + T_{rs}\,\Delta\dot{E} = \Delta E\,\dot{T}_r + T_{rs}\,\Delta\dot{E} = \frac{d}{dt}\left(T_{rs}\,\Delta E\right)$$

leads to the second energy-phase equation:

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right) = e\,\hat{V}(\sin\phi - \sin\phi_s)$$

Appendix: Stability condition for ϕ_s

$$\Omega_s^2 = \frac{-q\hat{V}_{RF}\eta h\omega_0}{2\pi R p_0}\cos\phi_s \Leftrightarrow \Omega_s^2 = \omega_0^2 \frac{-q\hat{V}_{RF}\eta h}{2\pi\beta^2 E}\cos\phi_s$$

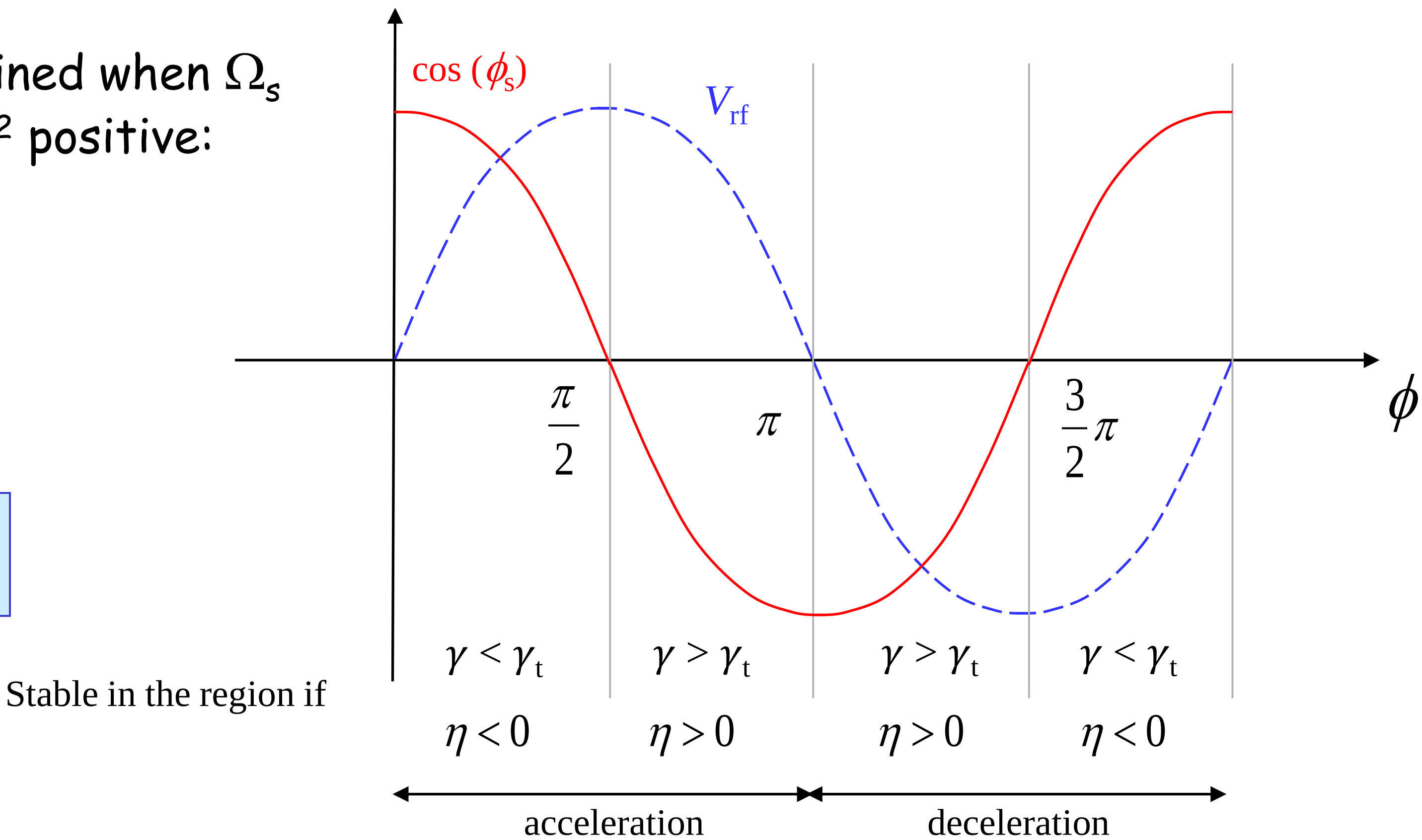
with $Rp = \frac{\beta^2 E}{\omega}$

Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\Omega_s^2 > 0$$

$$\Updownarrow$$

$$\eta \cos\phi_s < 0$$



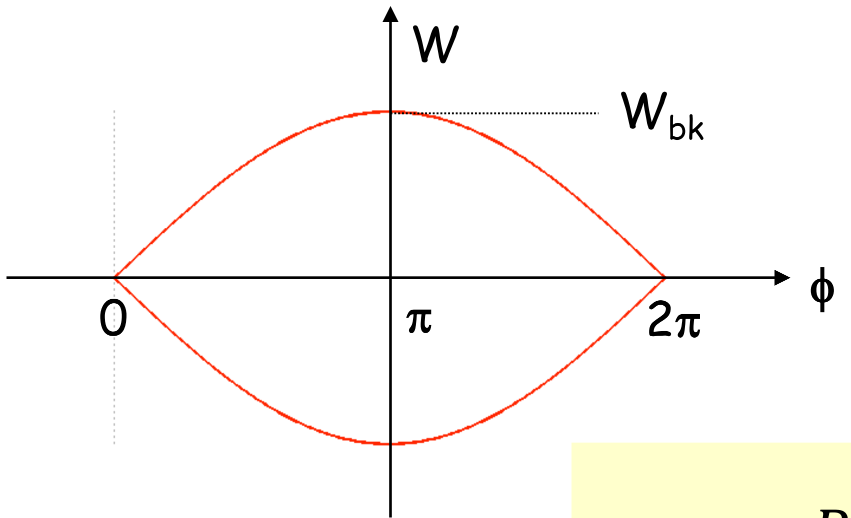
Appendix: Stationnary Bucket - Separatrix

This is the case $\sin\phi_s=0$ (no acceleration) which means $\phi_s=0$ or π . The equation of the separatrix for $\phi_s=\pi$ (above transition) becomes:

$$\frac{\dot{\phi}^2}{2}+\Omega_s^2\cos\phi=\Omega_s^2$$

$$\frac{\dot{\phi}^2}{2}=2\Omega_s^2\sin^2\frac{\phi}{2}$$

Replacing the phase derivative by the (canonical) variable W:



$$W=\frac{\Delta E}{\omega_0}=\frac{p_0 R}{h\eta\omega_0}\dot{\phi}$$

and introducing the expression for Ω_s leads to the following equation for the separatrix:

with $C=2\pi R$

$$W=\pm\frac{R}{c}\sqrt{\frac{2q\hat{V}E_0}{\pi h\eta}}\sin\frac{\phi}{2}=\pm W_{bk}\sin\frac{\phi}{2}$$

Bucket height - bucket area

Setting $\phi=\pi$ in the previous equation gives the height of the stationary bucket:

$$W_{bk} = \frac{R}{c}\sqrt{\frac{2q\hat{V}E_0}{\pi h|\eta|}}$$

The bucket area is: $A_{bk} = 2\int_0^{2\pi} W\,d\phi$

Since: $\int_0^{2\pi} \sin\frac{\phi}{2}\,d\phi = 4$

one gets:

$$A_{bk} = 8\,W_{bk} = \frac{8R}{c}\sqrt{\frac{2q\hat{V}E_0}{\pi h|\eta|}}$$

→ $W_{bk} = \frac{A_{bk}}{8}$

For an accelerating bucket, this area gets reduced by a factor depending on Φ_s:

$$\alpha(\phi_s) \approx \frac{1-\sin\phi_s}{1+\sin\phi_s}$$

Appendix: Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

which for small amplitudes reduces to:

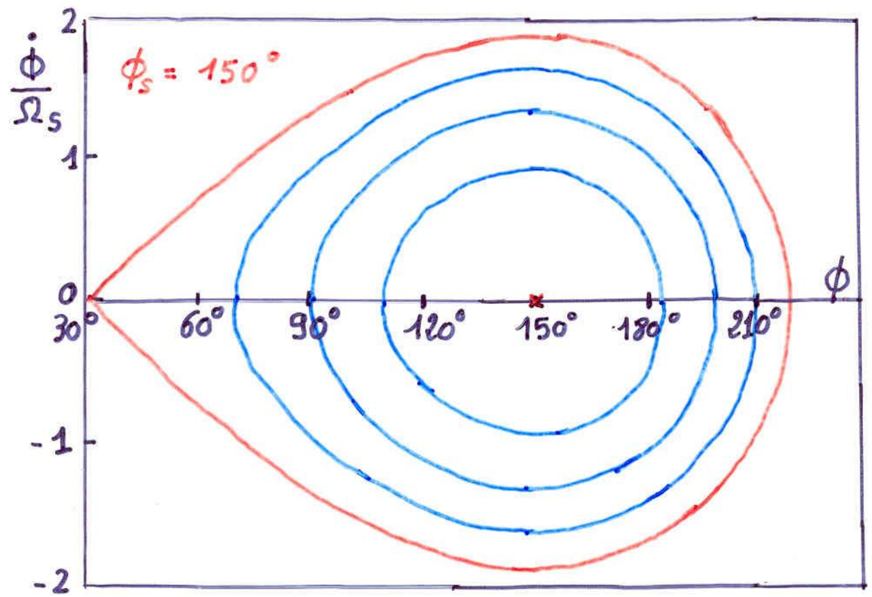
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{\left(\Delta\phi\right)^2}{2} = I'$$

(the variable is $\Delta\phi$, and ϕ_s is constant)

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches $\pi-\phi_s$ the force goes to zero and beyond it becomes non restoring.
Hence $\pi-\phi_s$ is an extreme amplitude for a stable motion which in the phase space($\frac{\dot{\phi}}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s\right)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$

Energy Acceptance

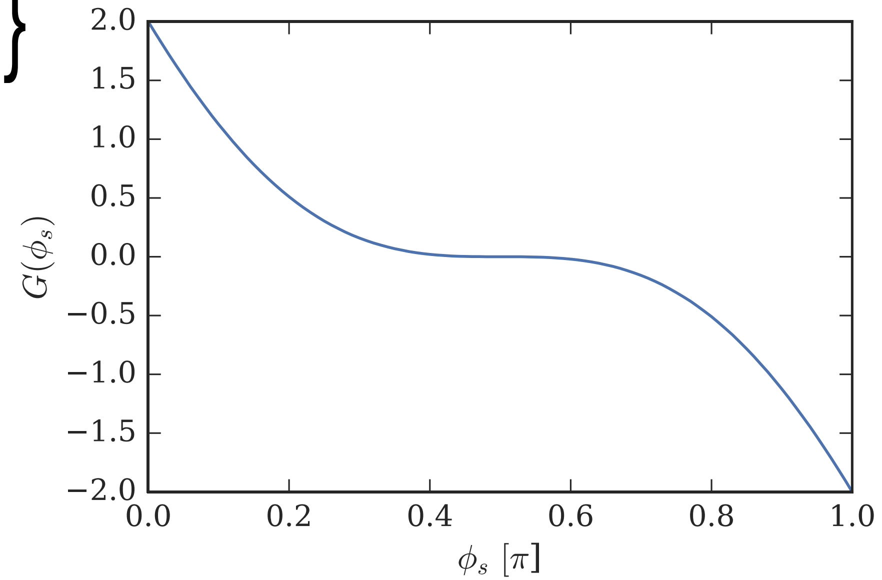
From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme at $\phi = \phi_s$.
Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{max} = 2\Omega_s^2 \left\{ 2 + (2\phi_s - \pi)\tan\phi_s \right\}$$

That translates into an energy acceptance:

$$\left(\frac{\Delta E}{E_0}\right)_{max} = \pm\beta\sqrt{\frac{-q\hat{V}}{\pi h \eta E_0} G(\phi_s)}$$

$$G(\phi_s) = \left[2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s \right]$$



This "RF acceptance" depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

It's largest for ϕ_s=0 and ϕ_s=π (no acceleration, depending on η).

It becomes smaller during acceleration, when ϕ_s is changing

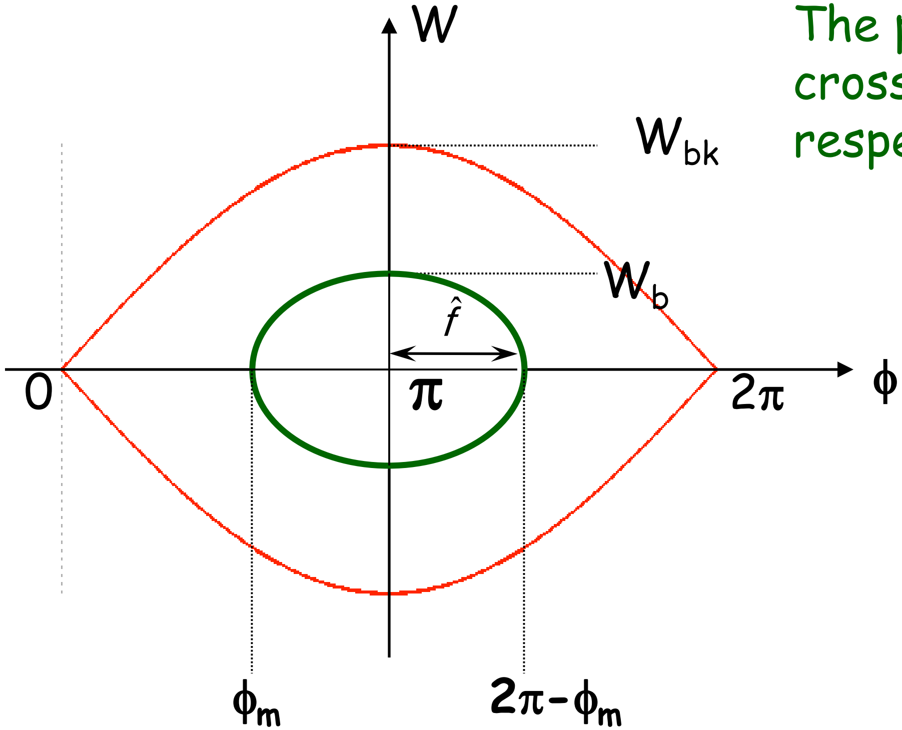
Need a higher RF voltage for higher acceptance.

For the same RF voltage it is smaller for higher harmonics h.

Bunch Matching into a Stationary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I \quad \xrightarrow{\phi_s = \pi} \quad \frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = I$$



The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = \Omega_s^2\cos\phi_m$$

$$\dot{\phi} = \pm\Omega_s\sqrt{2\left(\cos\phi_m - \cos\phi\right)}$$

$$W = \pm W_{bk}\sqrt{\cos^2\frac{\phi_m}{2} - \cos^2\frac{\phi}{2}}$$

$$\cos(\phi) = 2\cos^2\frac{\phi}{2} - 1$$

Bunch Matching into a Stationary Bucket (2)

Setting $\phi = \pi$ in the previous formula allows to calculate the bunch height:

$$W_b = W_{bk} \cos\frac{\phi_m}{2} = W_{bk} \sin\frac{\hat{\phi}}{2}$$

or:

$$W_b = \frac{A_{bk}}{8} \cos\frac{\phi_m}{2}$$

$$\longrightarrow \left(\frac{\Delta E}{E_s}\right)_b = \left(\frac{\Delta E}{E_s}\right)_{RF} \cos\frac{\phi_m}{2} = \left(\frac{\Delta E}{E_s}\right)_{RF} \sin\frac{\hat{\phi}}{2}$$

This formula shows that for a given bunch energy spread the proper matching of a shorter bunch (ϕ_m close to π, $\hat{\phi}$ small) will require a bigger RF acceptance, hence a higher voltage

For small oscillation amplitudes the equation of the ellipse reduces to:

$$W = \frac{A_{bk}}{16}\sqrt{\hat{\phi}^2 - (\Delta\phi)^2} \longrightarrow \left(\frac{16\,W}{A_{bk}\hat{\phi}}\right)^2 + \left(\frac{\Delta\phi}{\hat{\phi}}\right)^2 = 1$$

Ellipse area is called longitudinal emittance

$$A_b = \frac{\pi}{16} A_{bk} \hat{\phi}^2$$