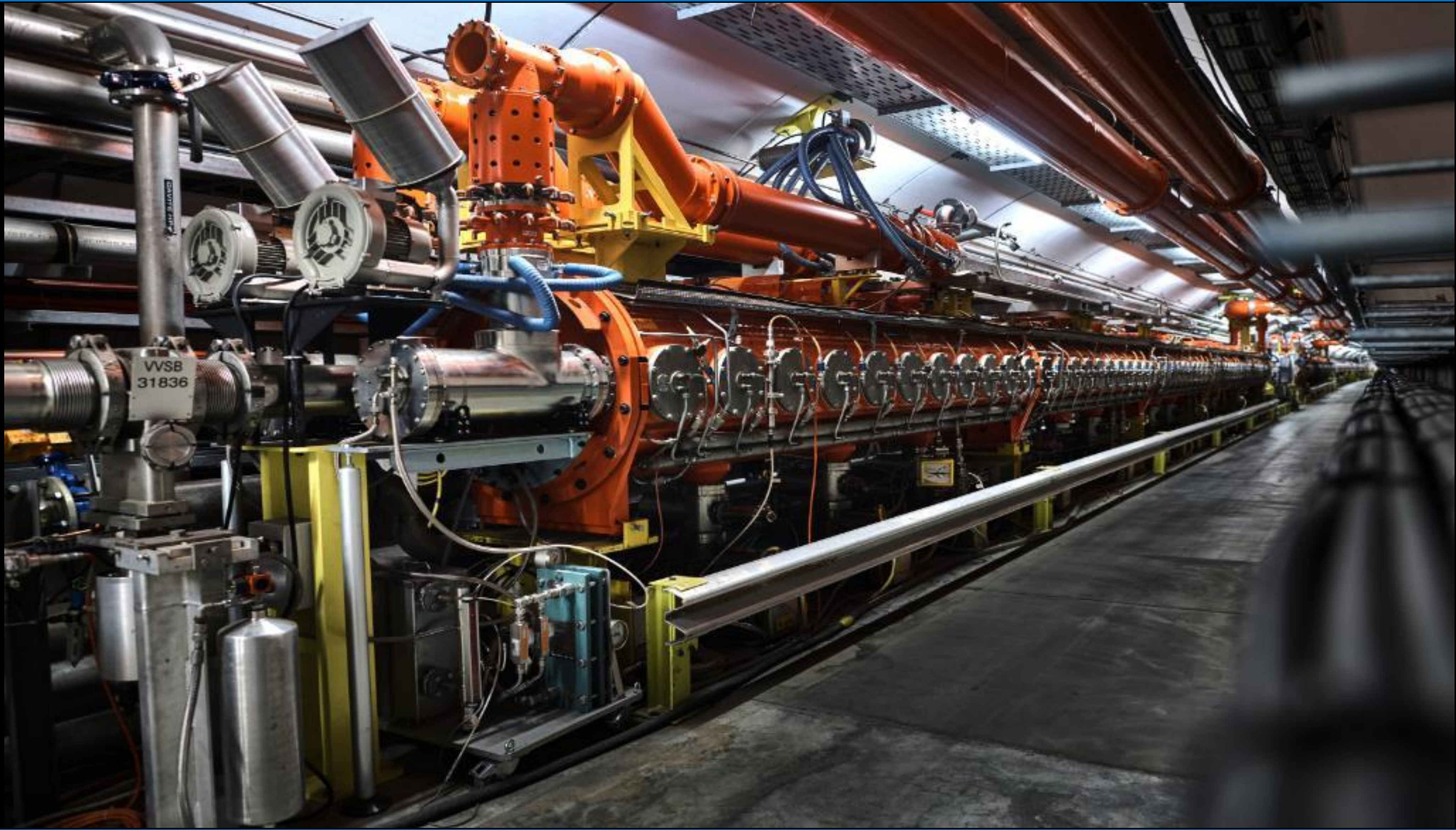
Maurizio Vretenar CERN

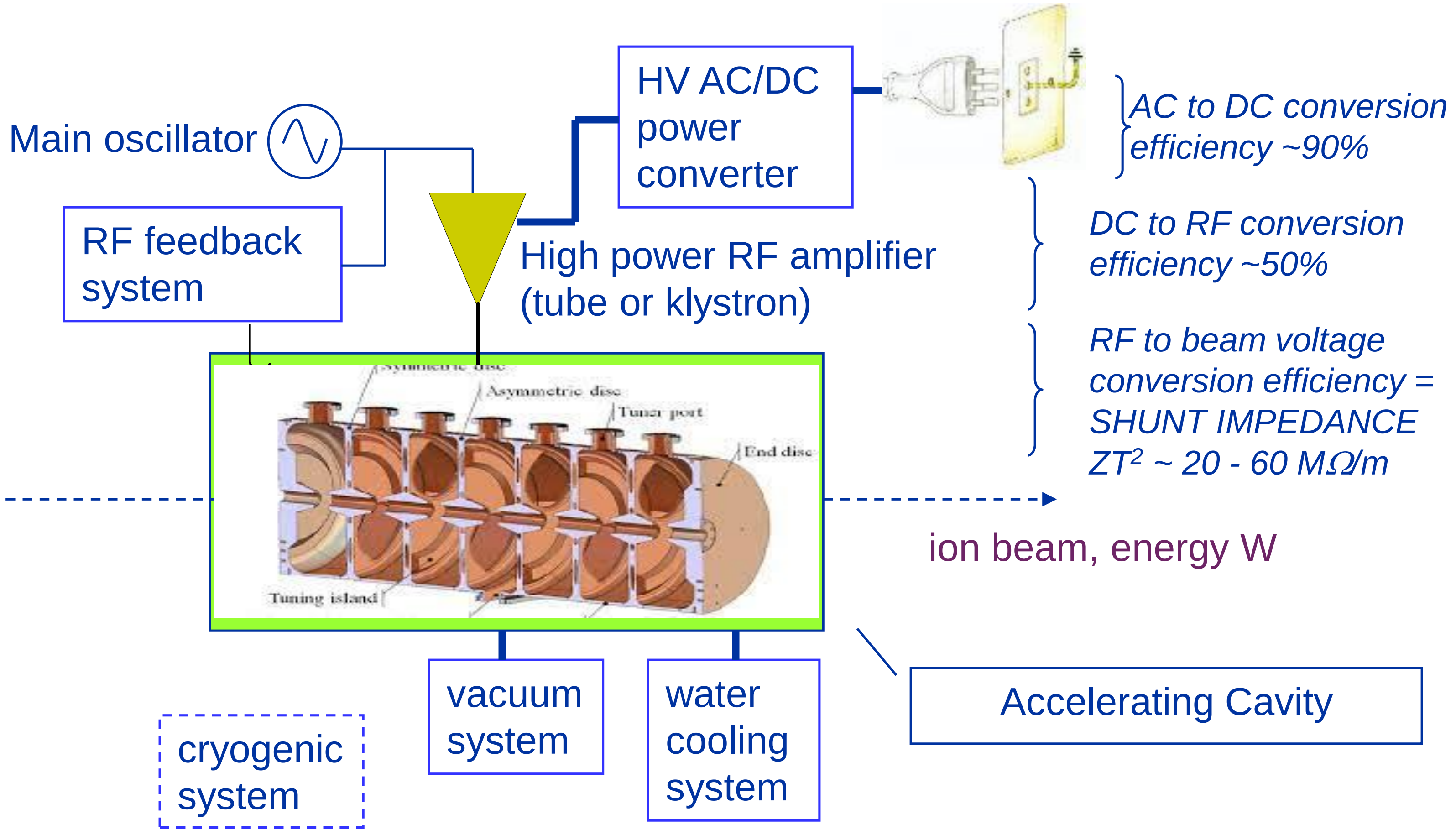
LECTURE 6

RTU April 2022



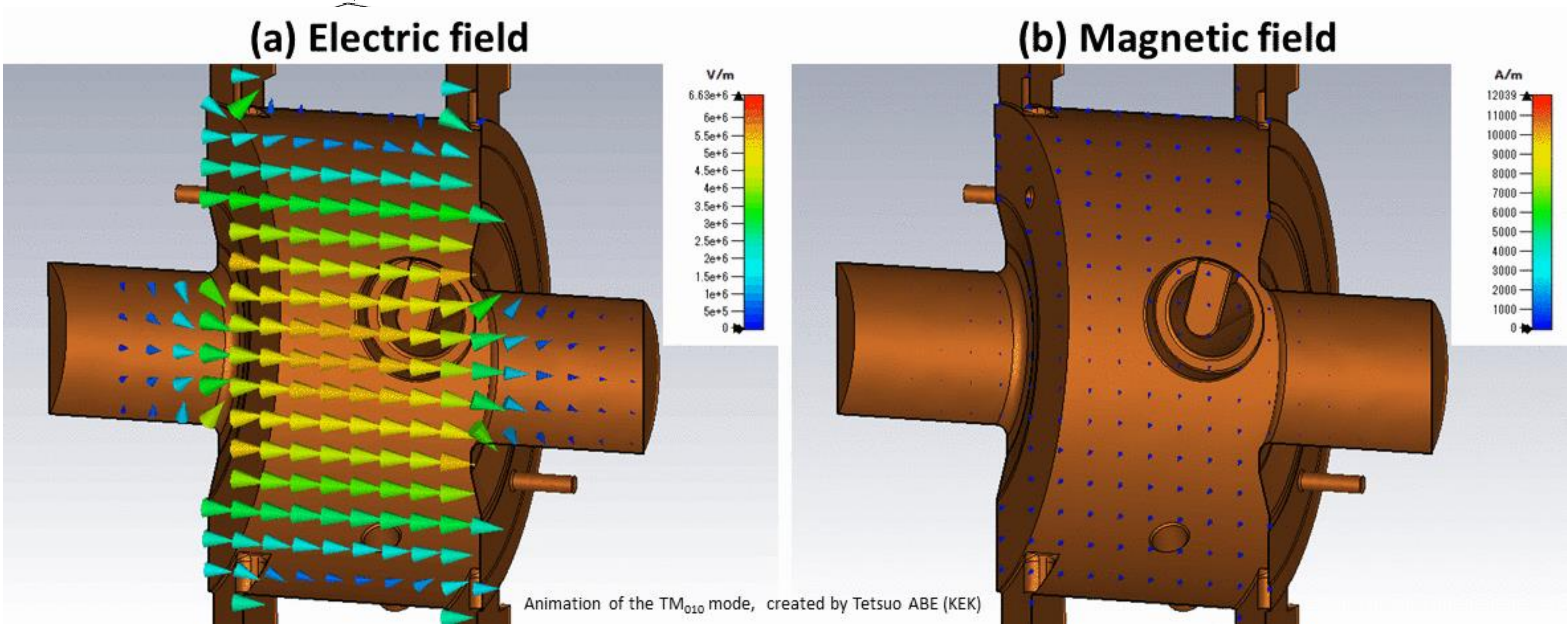
Radio Frequency Systems

Radio Frequency System building blocks



The basic («pillbox») Radio Frequency cavity

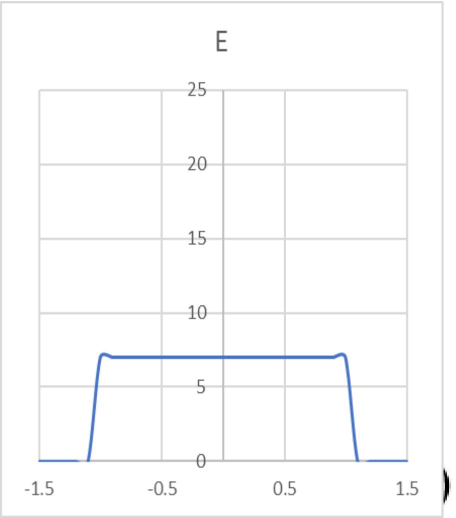
B_φ r

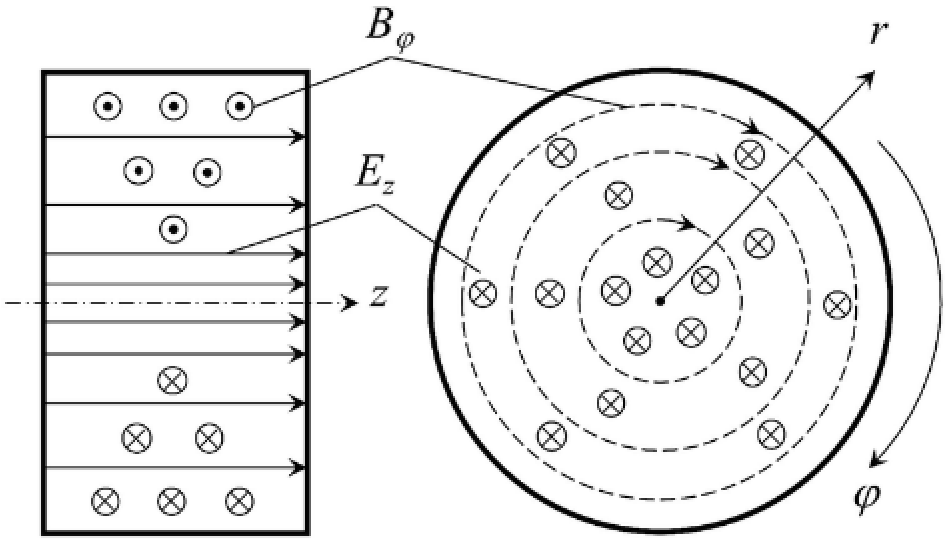


Animation of the TM_{010} mode, created by Tetsuo ABE (KEK)

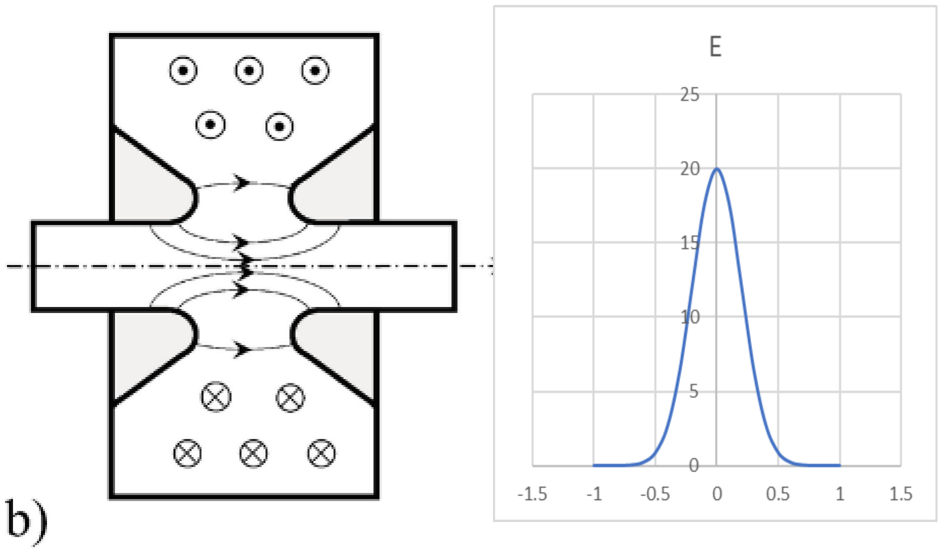
The basic («pillbox») Radio Frequency cavity

E-field on axis

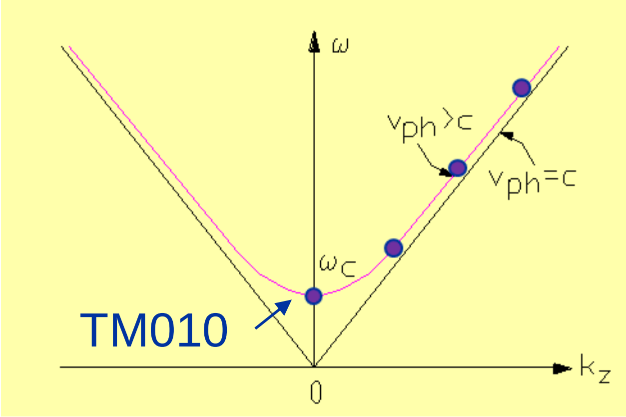




E-field on axis



The simplest (and at lowest frequency!) cavity mode with longitudinal electric field on axis is the TM010 (=0 space periods in *Φ*, 1 variation / half period in *r*, 0 space periods in *z*)



A particle traveling on the axis of a simple pillbox will have a low energy gain.
To improve it add some "noses" around axis, to increase the Transit Time factor, via:

a. An increase of electric field in the centre (Faraday law: integral of E = flux of B);
b. A decrease of the time spent by the beam in the maximum field region.

Of course, we need to add as well an «aperture» for the beam to cross the cavity!

Transit time factor (reminder)

$$\Delta W = \int_{-L/2}^{L/2} eE(z)dz = eE_0 TL\cos\phi$$

Panofsky equation: energy gain of a particle crossing a cavity (gap)

$$T = \frac{\int_{-L/2}^{L/2} E(z)\cos\omega t(z)dz}{\int_{-L/2}^{L/2} E(z)dz}$$

Numerator: energy gain by a particle in the middle of the gap at t=0

Denominator: energy gain by a particle crossing the gap with infinite velocity at t=0

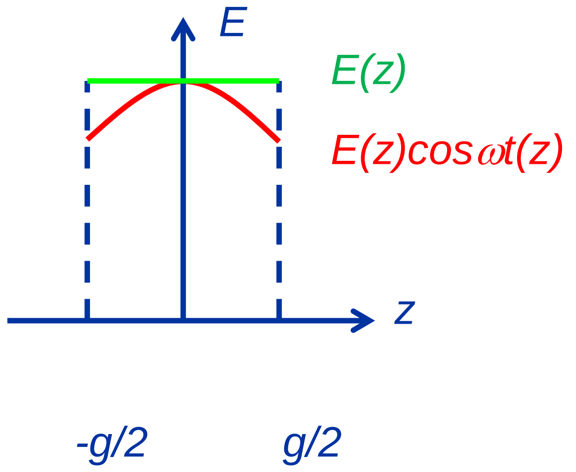
The Transit Time Factor T tells us how much of the E-field that we have provided on the gap has been really seen by a particle moving at velocity $v=\beta c$. Is the ratio between 2 integrals ($0 \leq T \leq 1$)

Neglecting the increase in velocity in the gap:

$$\omega z = \frac{2\pi c}{\lambda}\frac{z}{\beta c} = \frac{2\pi z}{\beta\lambda} \qquad T = \frac{\int_{-L/2}^{L/2} E(z)\cos(2\pi z/\beta\lambda)dz}{\int_{-L/2}^{L/2} E(z)dz}$$

For the simplest case $E(z)=E_0$ ($-g/2$, $g/2$):

$$T = \frac{\sin(\pi g/\beta\lambda)}{\pi g/\beta\lambda}$$



T is a characteristic of the cavity design; can be easily calculated by RF design codes

Shunt Impedance (reminder)

Design of an accelerating system → goal is to maximize acceleration ΔW for a given power delivered to the cavity P_c .
Can we define a figure of merit to compare different designs?

We define a « shunt impedance » Z as $Z = \frac{V_0^2}{P_c}$; exactly twice the shunt resistance in the equivalent parallel circuit

The figure of merit is (considering maximum acceleration at ϕ=0 and taking the square of the energy gain in volts, to give consistancy to the units):

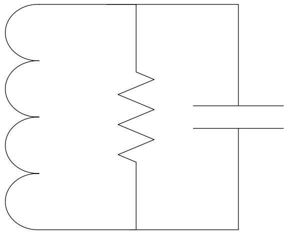
$$\frac{\Delta W^2}{e^2 P_c} = \frac{(eE_0TL)^2}{e^2 V_0^2 / Z} = Z\frac{(E_0TL)^2}{(E_0L)^2} = ZT^2$$

$ZT^2 \quad [\Omega]$

$ZT^2 / L \quad [\Omega / m]$

ZT^2 is the effective shunt impedance, depends only on the geometry (and on RF frequency and particle velocity). Unit is ohms (usually MΩ, or MΩ/m if referred to the unit length). Can be easily calculated by computer codes, and then used to calculate the cavity power and the final power efficiency:

$$P_c = \frac{V_0^2}{Z} = \frac{(E_0L)^2}{Z} = \frac{(\Delta W)^2 / \cos^2 \varphi}{ZT^2}$$

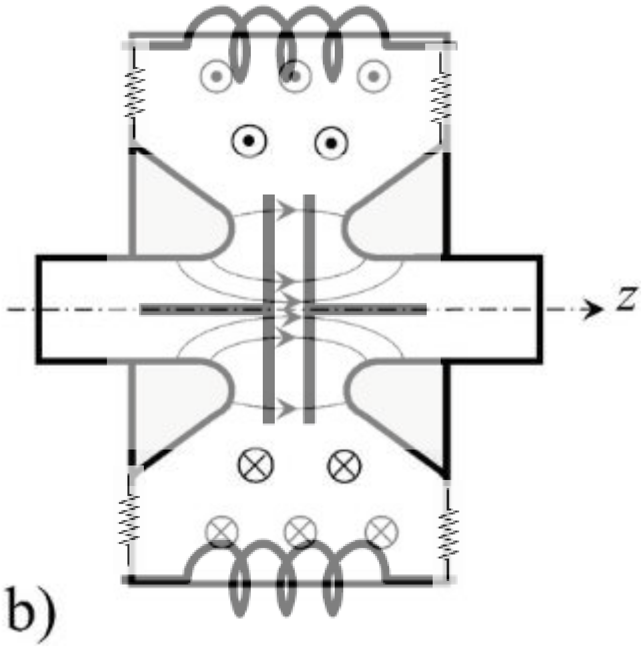


R_p

$$R_p = \frac{1}{2} Z$$

ATTENTION!!!
Usually in linacs we use Z, while for other RF systems and in other literatures is used R_p: difference by a factor 2 !

Capacity and Inductance

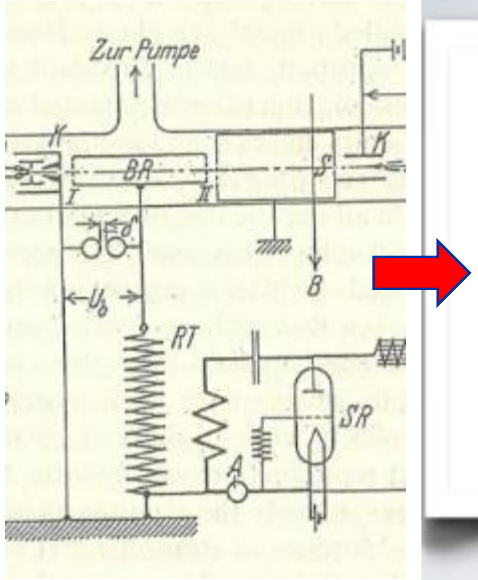


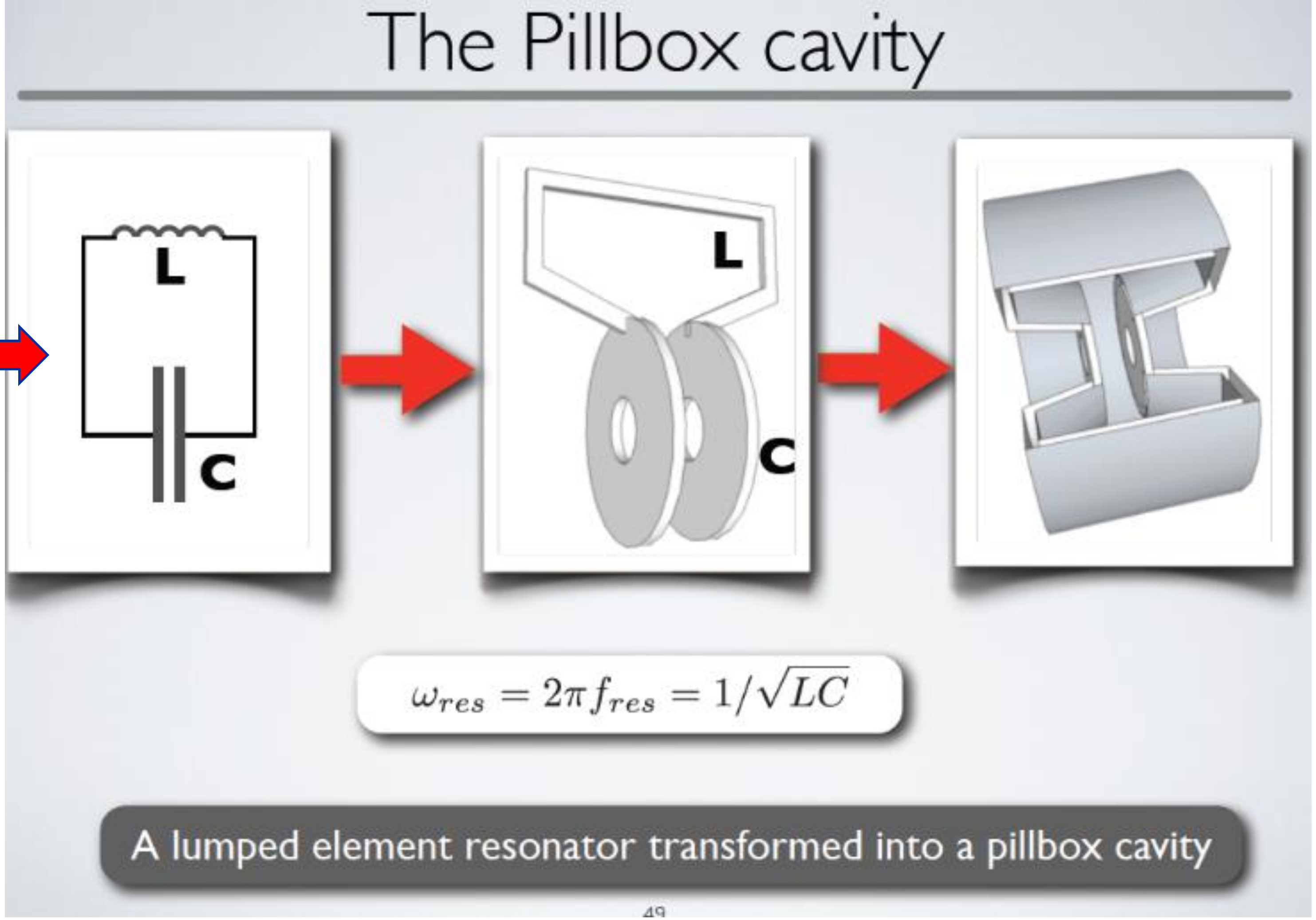
The «loaded pillbox» (with noses) is a very simple RLC resonator:

- the capacitance C is concentrated on the gap
- the inductance L is concentrated on the external part
- the resistance R is that of the internal walls of the cavity

The resonance frequency will be $f = \frac{1}{2\pi\sqrt{LC}}$

From the Wideröe gap to the pillbox cavity





The 3 cavity parameters

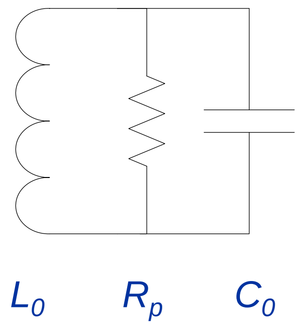
A resonator (= a mode in the accelerating cavity) is always defined by a set of 3 parameters.

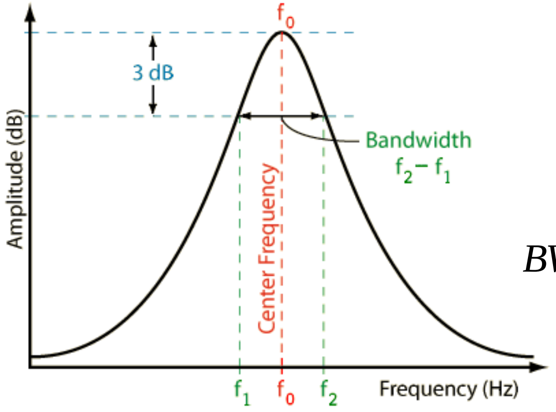
They can be:

(Capacitance, Inductance, Resistance)

(Frequency, Q-value, R/Q)

$$f_0 = \frac{1}{2\pi\sqrt{L_0 C_0}} \qquad Q_0 = 2\pi\, f_0\, R_p C_0$$





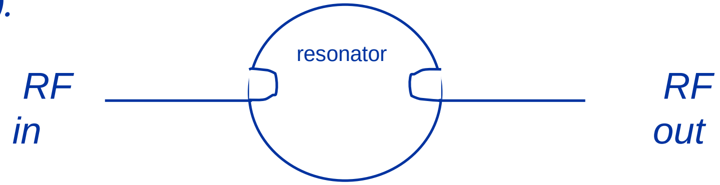
$$R/Q = \frac{1}{2\pi\, f_0 C_0} = \sqrt{\frac{L_0}{C_0}}$$

$$BW[3dB] = \frac{f_0}{Q_0}$$

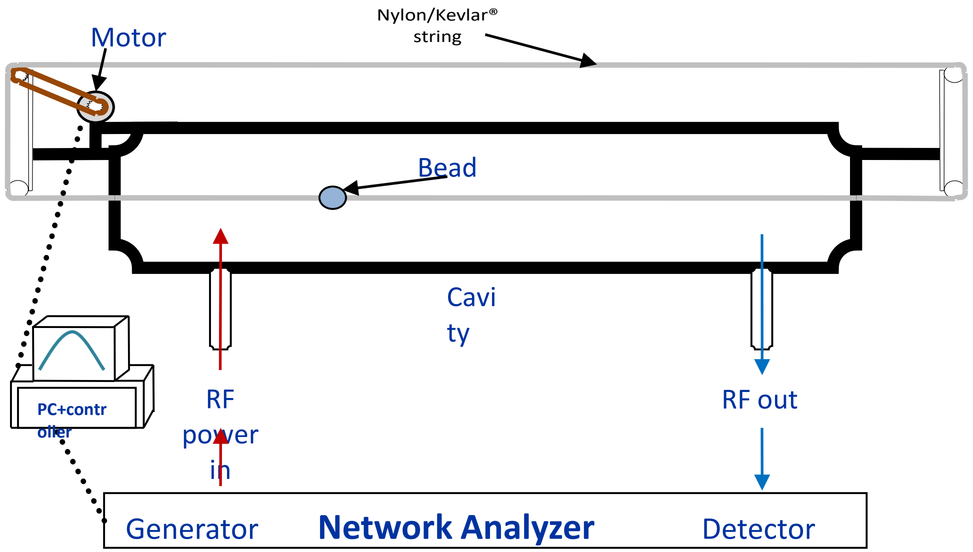
R/Q is a geometrical parameter, basically tells you the ratio between current and voltage in your system

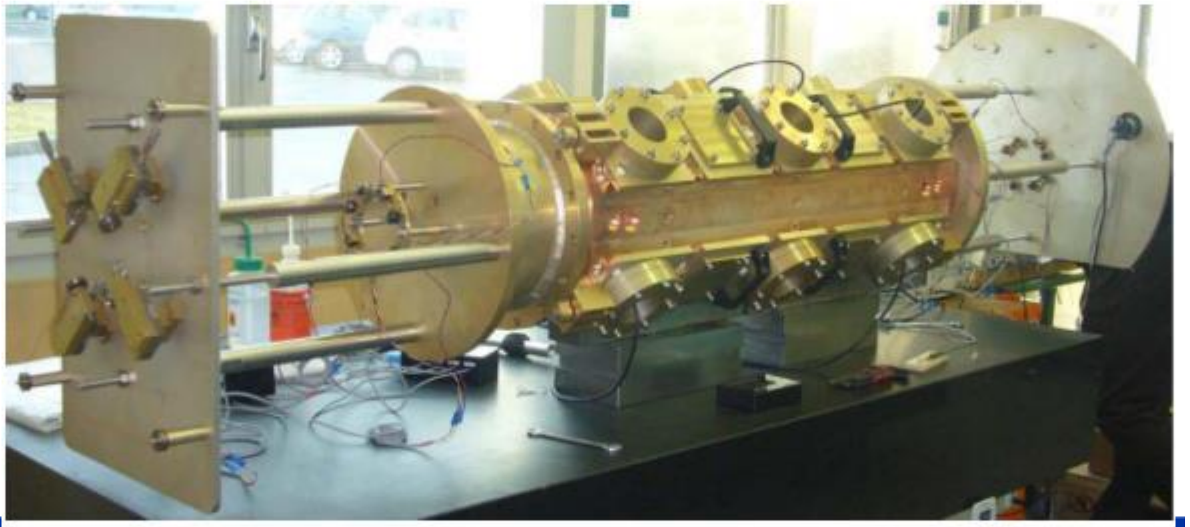
An RF transmission measurement can easily allow to get (f, Q) from the 3db bandwidth.

(precaution: to measure Q_0; we need a sufficiently low ioupling. If input coupling is too large, is measured the Q loaded (Q_l), given by the parallel of the cavity R and the transformed output impedance of the measurement device).



Measurement of electric field distribution

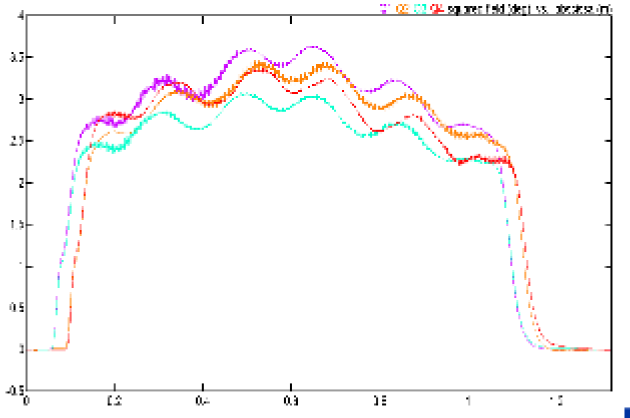




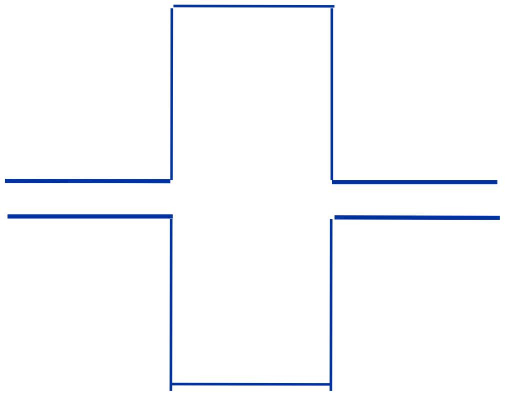
"bead-pull" perturbative measurement.

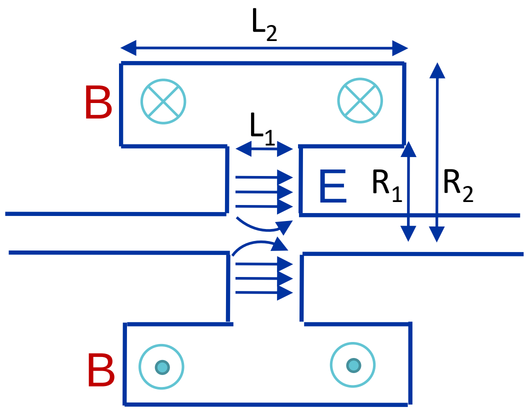
A small metallic bead is slowly moved inside the resonator.
Slater's perturbation theorem: the variation in resonant frequency is proportional to the difference between square of electric field and square of magnetic field at the position of the bead.

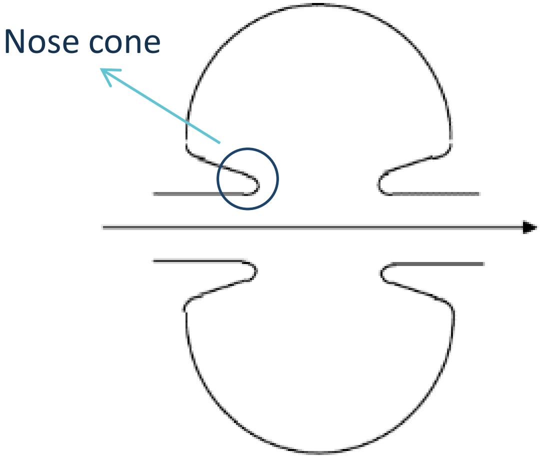
Bead in regions of pure E or pure B field: easy way to plot the longitudinal field distribution.
It is a relative measurement (or needs calibration of the bead)



Cavity optimization







Step 1: pillbox cavity

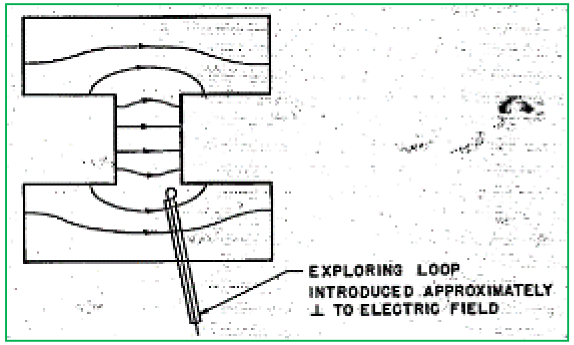
Step 2: **reentrant cavity**
E field is concentrated near beam axis and gap become shorter: T and gap capacitance increase. C_{gap} a function of L1 and R1.
Magnetic field is confined on the outer walls and inductance is function of L2 and R2-R1.
To get the same frequency it is possible to reduce cavity radius, increasing Q and Z.

Step 3: **optimization**
Rounding off edges close to beam axis allows to increase the ratio E0/Epeak. The protrusion inside the cavity (nose cone) increase T and ZT^2. A spherical shape of the inductive zone improve Q.

(courtesy of F. Grespan)

Mapping cavity fields and calculating cavity parameters

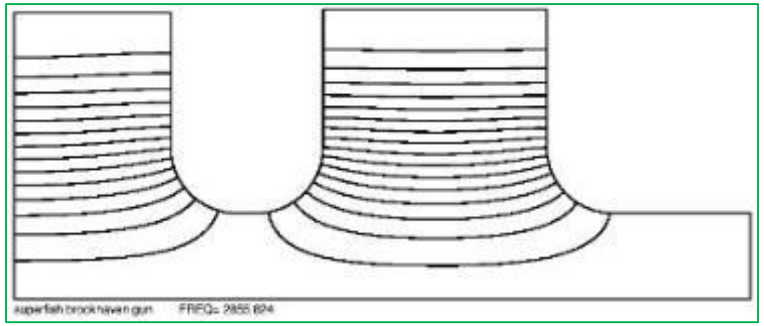
Knowledge of the field distribution and definition of the correct dimensions for the required frequency is essential to finalise the mechanical design, to compute the accelerating voltage, to dimension the ancillary elements coupler, etc.

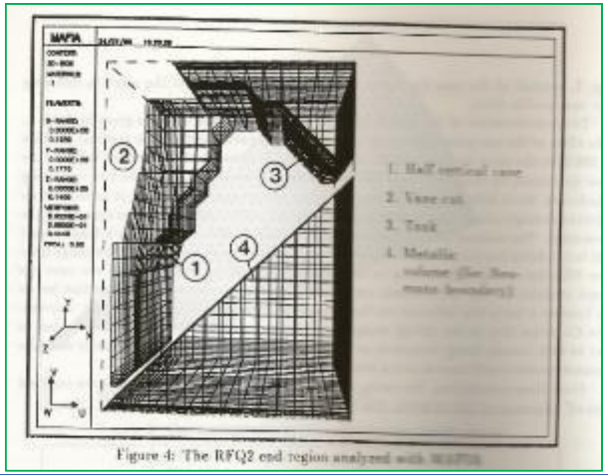


1. The good old way: build a model and then measure frequency and explore fields with a probe.

(illustration from the original Alvarez paper on DTL)

2. From the beginning of the 80's, modern computers allow to calculate frequency and fields in 2D (axis-symmetric cavities): SUPERFISH, URMEL.





3. At the end of the 80's comes the first 3D software: the MAFIA package (DESY and LANL). Constantly improved, 3D packages allow nowadays to calculate complex shapes with amazing precision.

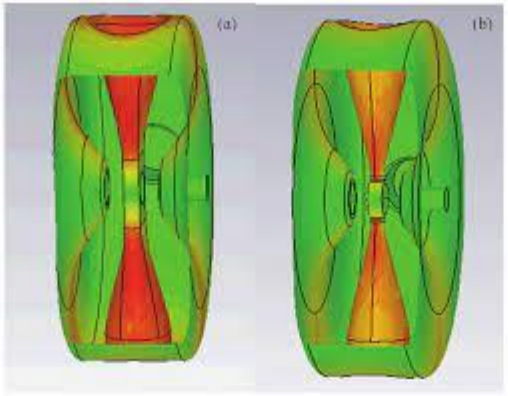
(1st 3D simulation of the CERN RFQ2 – 1987, 6000 mesh points)

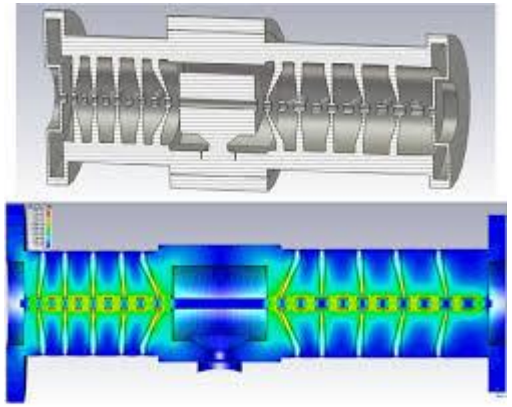
Modern cavity simulation software

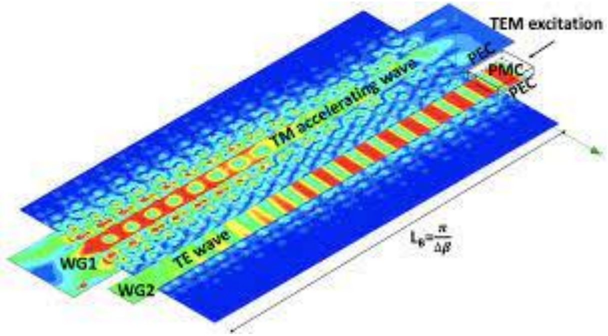
Modern 3d simulation software (e.g. CST Studio Suite) can analyse every geometry made with a large variety of materials (metallic and dielectric), providing you with information like:
Frequency, field distribution, acceleration parameters, shunt impedance, sensitivities, etc.

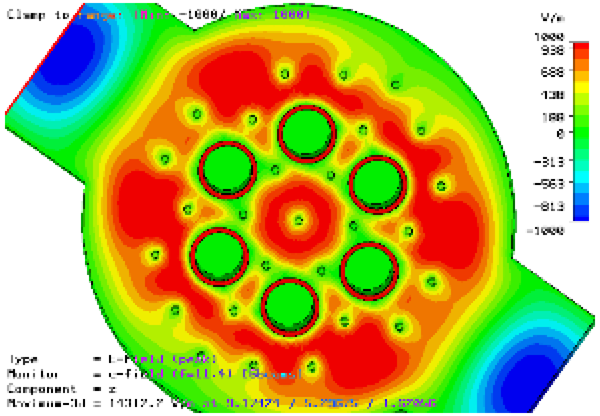
The codes are integrated into modern CAD systems and interfaced with thermo-mechanical simulation software.

CST Studio Suite contains finite integration technique (FIT), finite element method (FEM), transmission line matrix (TLM), multilevel fast multipole method (MLFMM) and particle-in-cell (PIC).

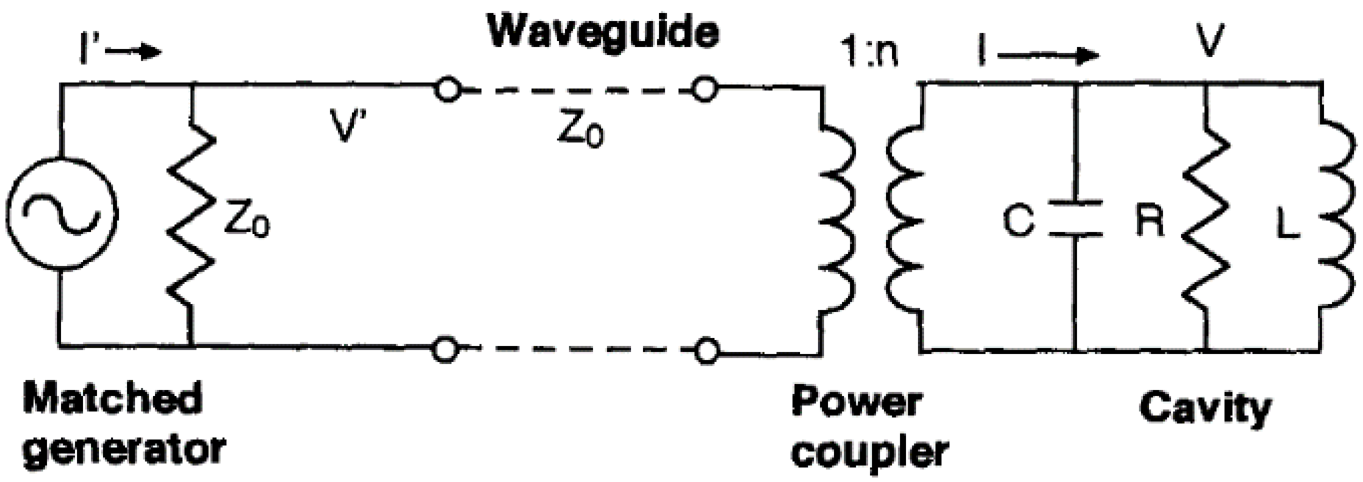








Coupling RF power to cavity



At resonance, the cavity behaves like a pure resistance ***R***.
A **coupler** is required to:

1. allow RF power to flow into the cavity, and
2. transform the cavity impedance *R* into the characteristic impedance of the transmission line, Z_0, to avoid power reflections.

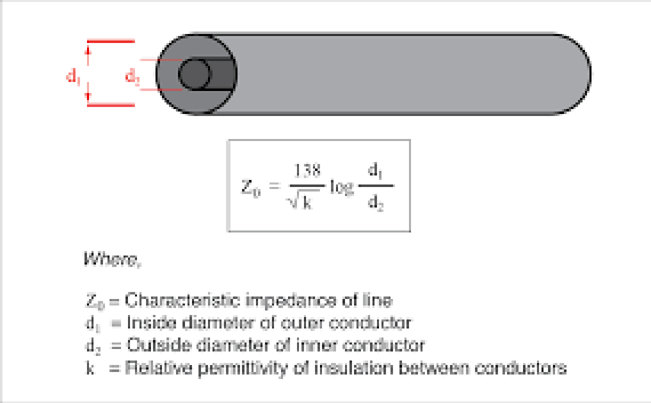
Transmission lines (coaxial, waveguides):

- Each line has a characteristic impedance Z_0 (ratio energy in electric and magnetic fields) that depends on its dimensions.

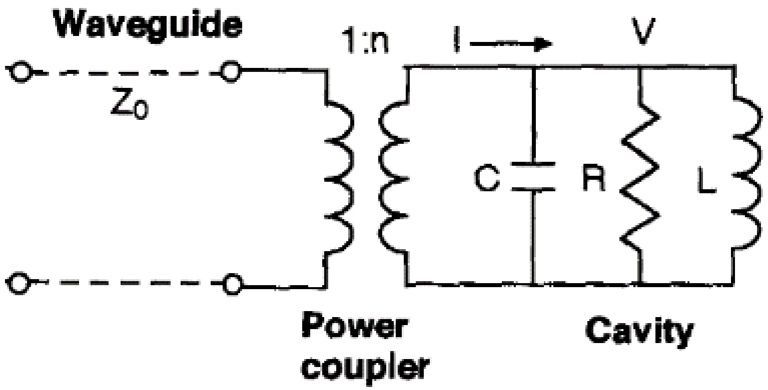
Examples:

$Z_0 = 75\ \Omega$ (TV cables, low loss)
$Z_0 = 50\ \Omega$ (accelerators, high voltage)

- To avoid reflections of the transmitted wave, a transmission line must always be terminated on its characteristic impedance!



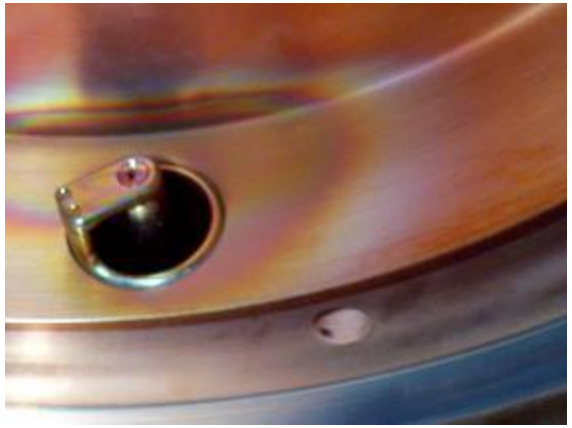
Power couplers: loops and waveguide slots



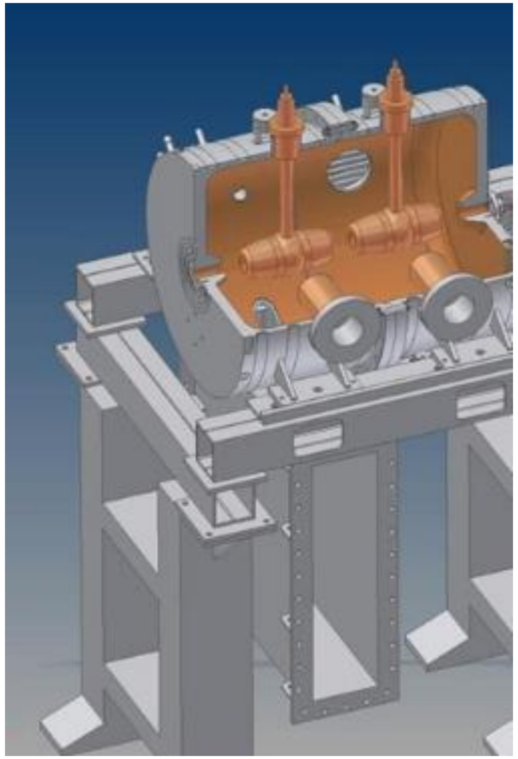
We need to inductively couple 2 magnetic field distributions.

Two options:

1. A coaxial line is terminated in a small loop immersed in the cavity magnetic field;
2. A waveguide is terminated in a small slot that allows some magnetic field to enter the cavity.



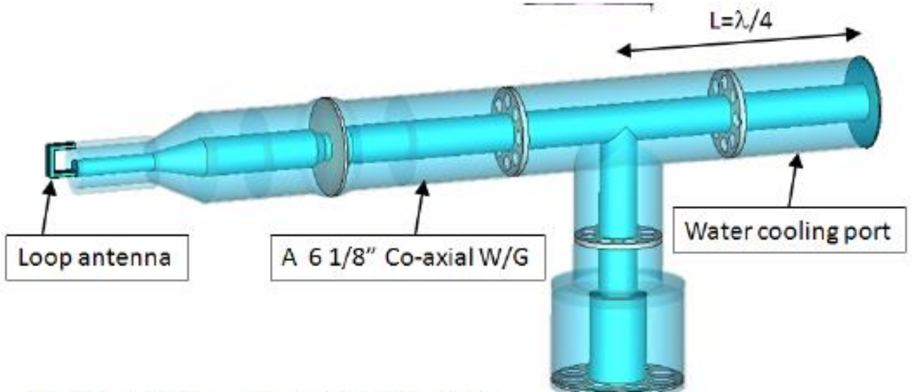
Small loop coupler to a 400 MHz buncher after 15 years of operation (traces of multipactoring)



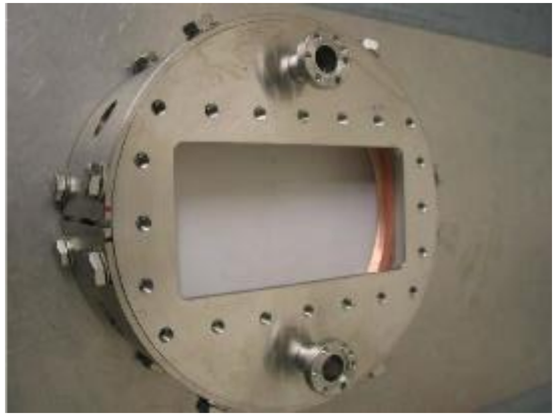
Waveguide coupling slot to the Linac4 CCDTL accelerating structure

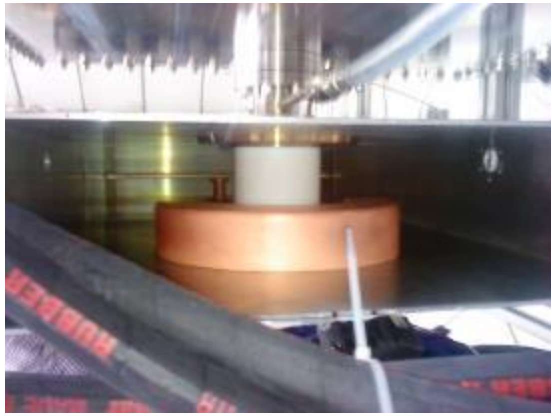
RF windows

RF windows provide the barrier for vacuum between transmission line (coaxial or waveguide) and cavity. Made of ceramic material.



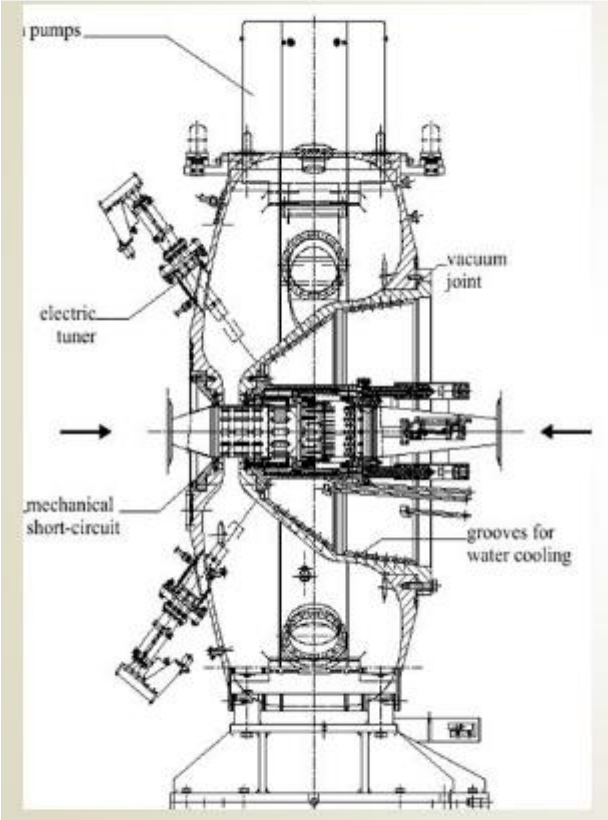
(b) An RF coupler with water cooling port





Examples of cavities

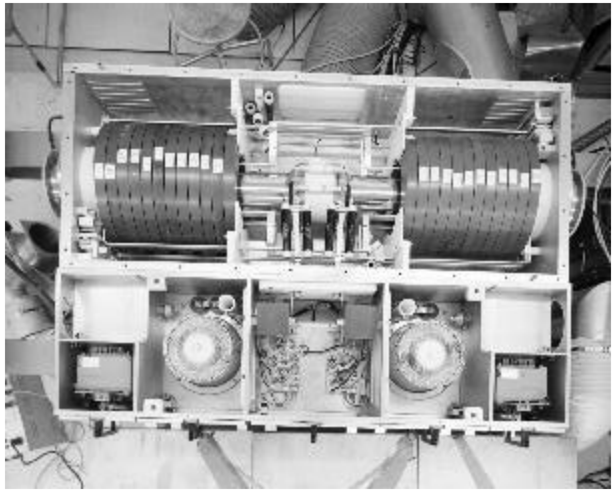
Synchrotron cavities (high energy)



The pillbox becomes quite complex…
CERN PS 80 MHz cavity (for LHC beams)

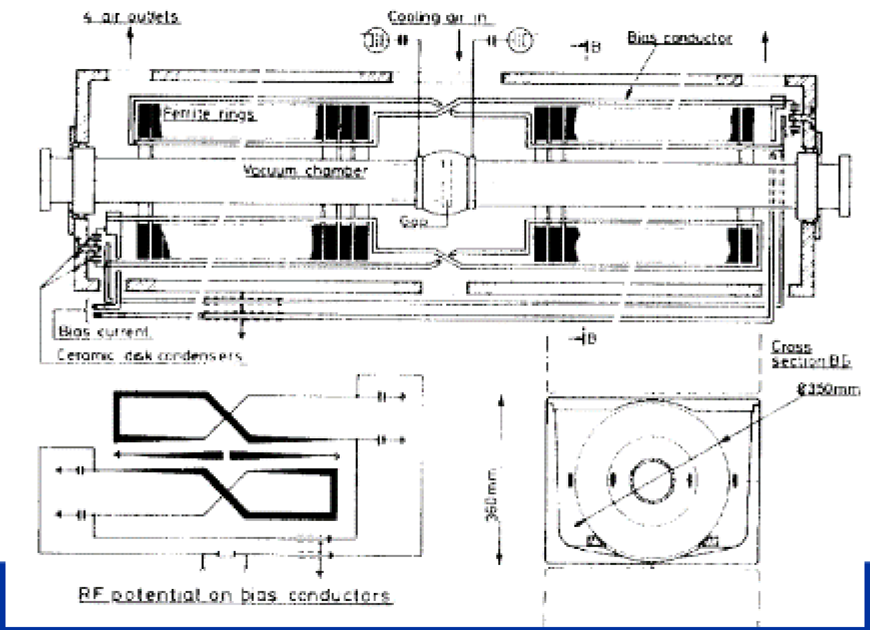
One of the 16 cavities installed in the old LEP collider: 352.2 MHz, 5 coupled cells, equipped with a spherical storage cavity on top to reduce losses

Synchrotron cavities (low energy, tunable)



CERN PS Booster:
4 rings with a single accelerating cavity.
2 quarter-wave ferrite-loaded resonators: 2 figure-of-eight loops on the ferrite loads for tuning the frequency throughout the acceleration cycle, from 3 to 8 MHz (from 50 MeV at injection to the original Booster energy of 800 MeV).

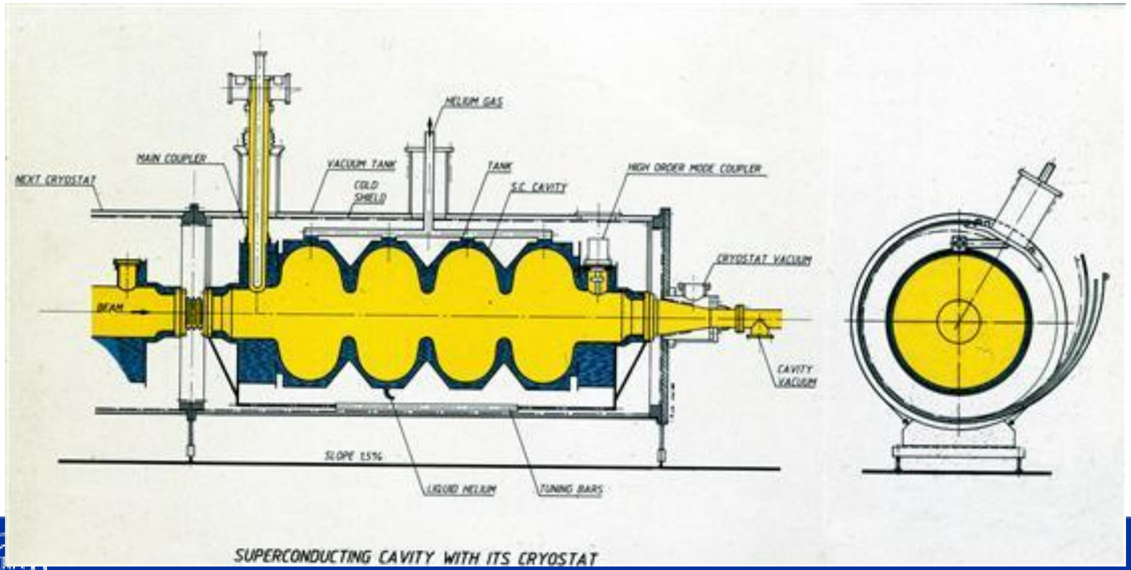
Finemet cavities
new type of magnetic alloy with wideband frequency response.
State-of-art technology for modern small synchrotrons (e.g. medical)



5 gap prototype ... installed in PSB ring 4

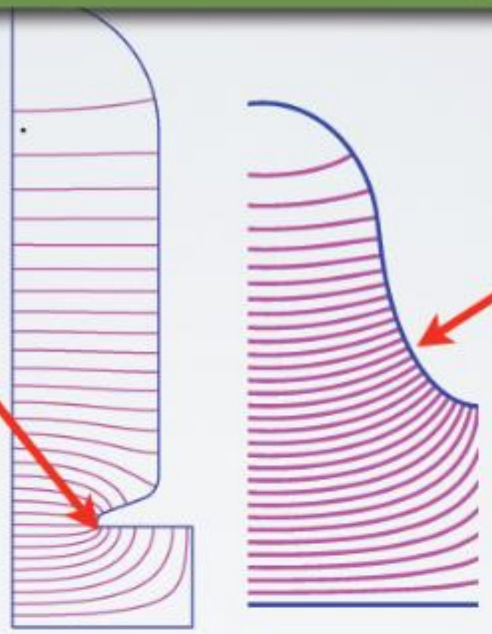
Superconducting cavities

Used in high-energy synchrotrons (protons and electrons) and in linacs (fixed frequency, not tunable).
Made in niobium or niobium-sputtered copper immersed in a liquid helium bath at 4 degK.



Comparing normal conducting and superconducting

NC and SC half cells (typical shapes)



normal conducting:

- nose cones reduce the gap length & increase the transit time factor and eff. shunt impedance ZT^2,
- high peak fields,
- $P_{beam} \lesssim P_{diss}$
- design goal: maximise ZT^2 and keep Kilpatrick below a certain value (1.2 - 2.4)

$$P_d = \frac{V_{acc}^2}{ZT^2 L}$$

superconducting:

- ZT^2 has no big importance ($P_{beam} \gg P_{diss}$),
- cryogenic losses (P_{diss}) can be optimised with the temperature (2 K/ 4.5 K),
- keep the ratio $E_{peak,surface}/E_{peak,axis}$ as small as possible (for $\beta=1 \Rightarrow P_s/P_a \approx 2$),

$$P_d = \frac{V_{acc}^2}{(R/Q)Q_0}$$

This and next 2 slides courtesy of F. Gerigk, CERN

When are SC cavities attractive?

Instead of Q values in the range of ~10^4, we can now reach 10^9 - 10^{10}, which drastically reduces the surface losses (basically down to ~0) ➜ high gradients with low surface losses

$$P_d = \frac{V_{acc}^2}{(R/Q)Q_0}$$

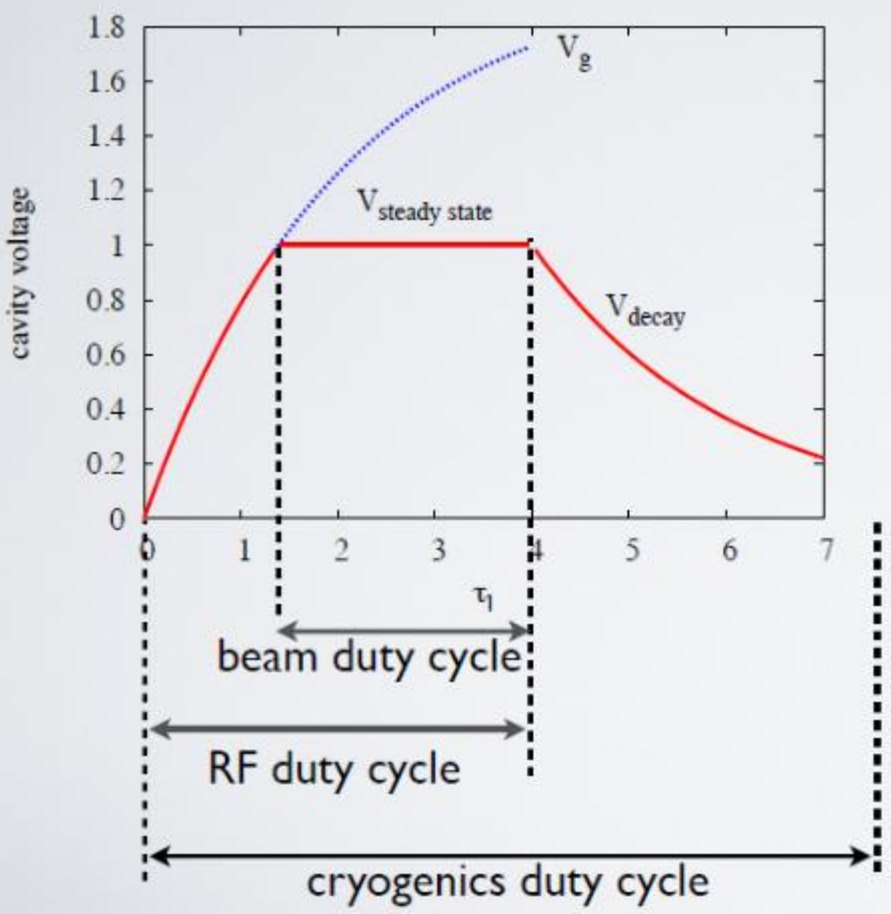
However, due to the large stored energy, also the filling time for the cavity increases (often into the range of the beam pulse length):

$$\tau_l = \frac{Q_l}{\omega_0} = \frac{Q_0}{\omega_0(1+\beta)} \approx \frac{Q_0}{\omega_0 \cdot P_b/P_d}$$

using: $\beta = 1 + \frac{P_b}{P_d} \approx \frac{P_b}{P_d}$

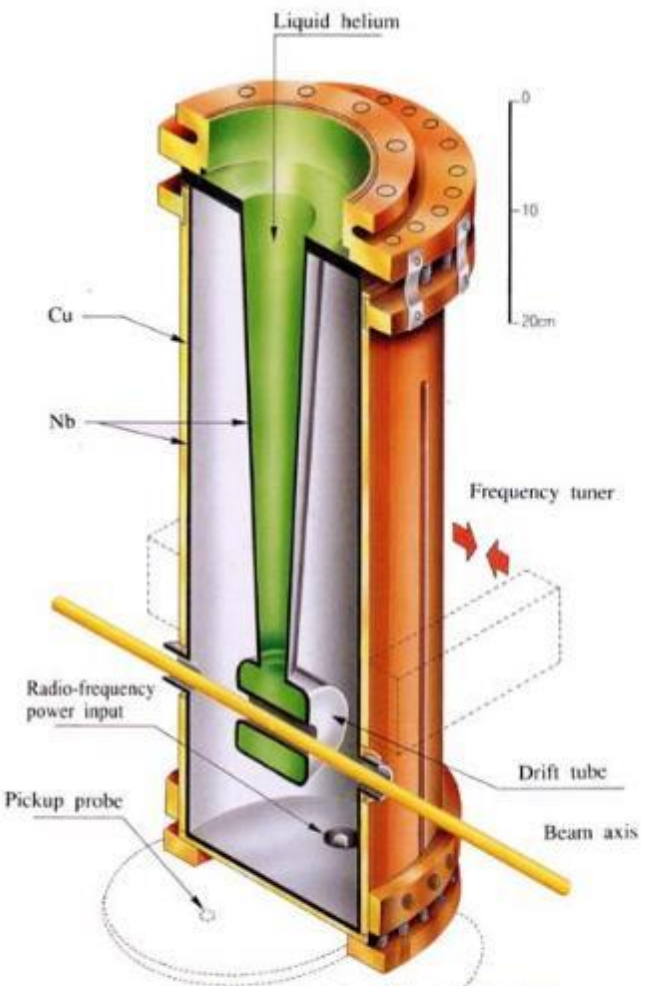
only for SC cavities!

Pulsed operation for SC cavities, overall efficiency



- **beam duty cycle:** covers only the beam-on time,
- **RF duty cycle:** RF system is on and needs power (modulators, klystrons)
- **cryo-duty cycle:** cryo-system needs to provide cooling (cryo-plant, cryo-modules, RF coupler, RF loads)
- RF and cryo-duty cycle have to be calculated as **integrals** of voltage over time.

Superconducting Quarter Wave Resonators



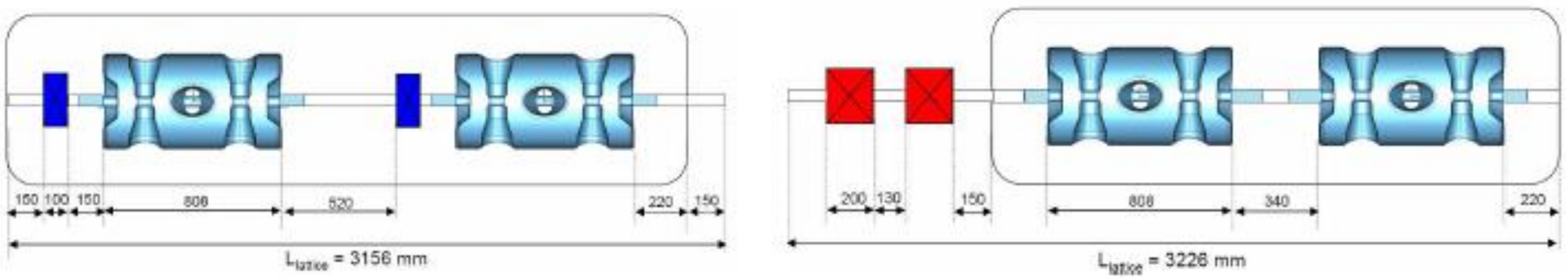
▲ Cross section of the quarter wave superconducting resonator

Simple 2-gap cavities commonly used in SC version (lead, niobium, sputtered niobium) for low beta protons or ion linacs, where ~CW operation is required.

The Spoke cavity

HIPPI Triple-spoke cavity prototype built at FZ Jülich, now under test at IPNO

Another option:
Double or triple-spoke cavity, can be used at higher energy (100-200 MeV for protons and triple-spoke).



The superconducting zoo

Spoke (low beta)
[FZJ, Orsay]

CH (low/medium beta)
[IAP-FU]

QWR (low beta)
[LNL, etc.]

HWR (low beta)
[FZJ, LNL, Orsay]

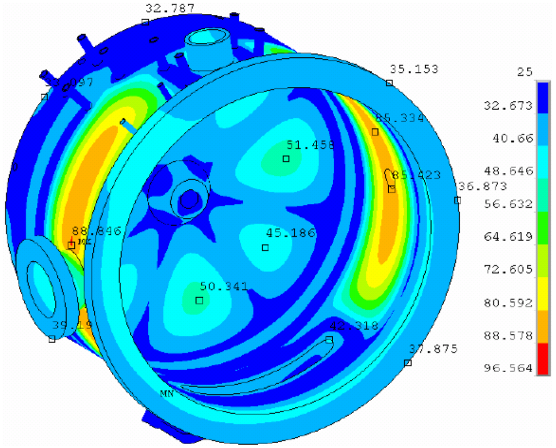
Re-entrant
[LNL]

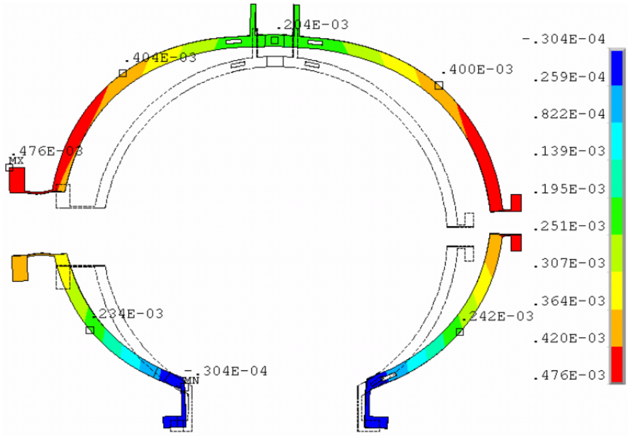
Superconducting linacs for low and medium beta ions are made of multi-gap (1 to 4) individual cavities, spaced by focusing elements.

Cavity construction technologies

Set of parts joined together; needs to have:

- A well defined vacuum envelope (no trapped volumes, special vacuum joints)
- A well defined RF envelope (no leaking RF, good RF contacts between parts)
- A stable mechanical structure under vibrations and thermal deformations
- An effective cooling of the heat produced by the RF.



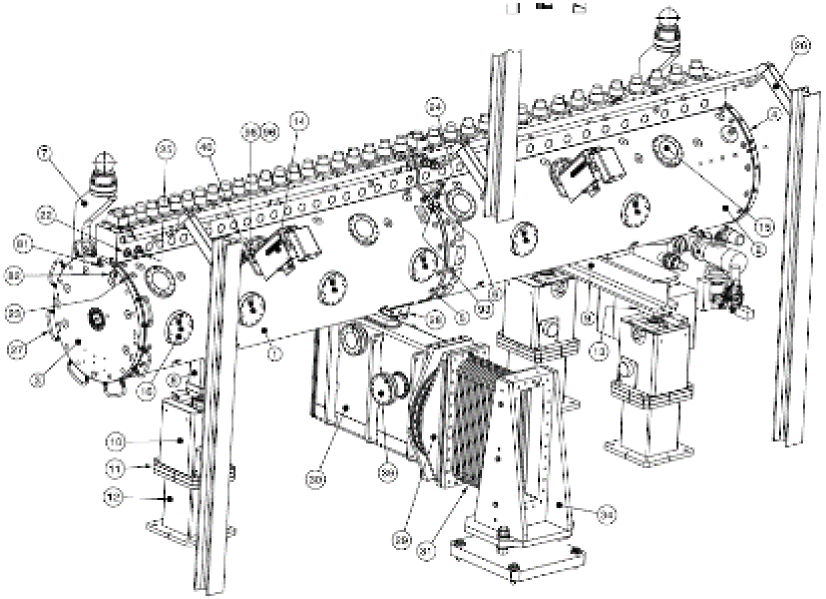


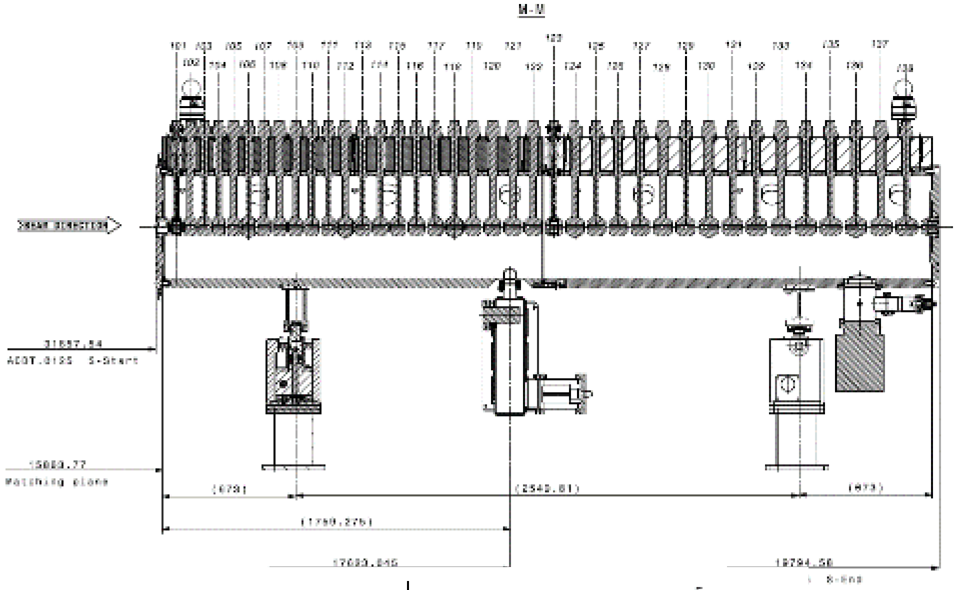
Construction technology depends on dimensions (→on frequency):

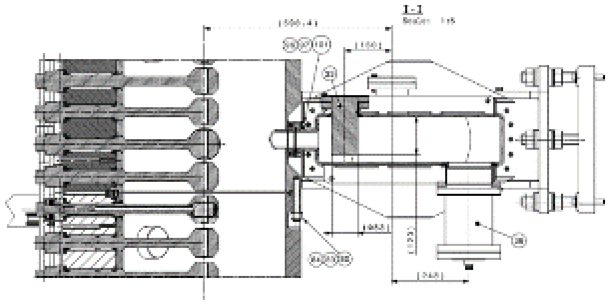
- ☞ brazed copper elements (>500 MHz) commonly used for electron linacs.
- ☞ copper or copper plated welded/bolted elements commonly used for ion linacs (<500 MHz).

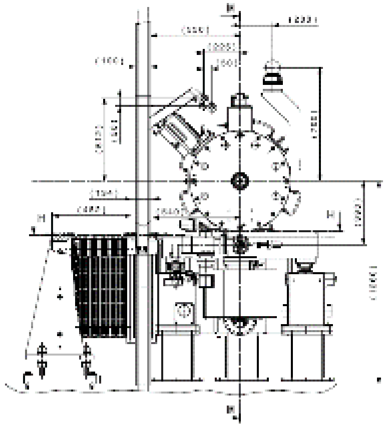
In most of the cases,
it is on the mechanical construction that you will win or lose your project…

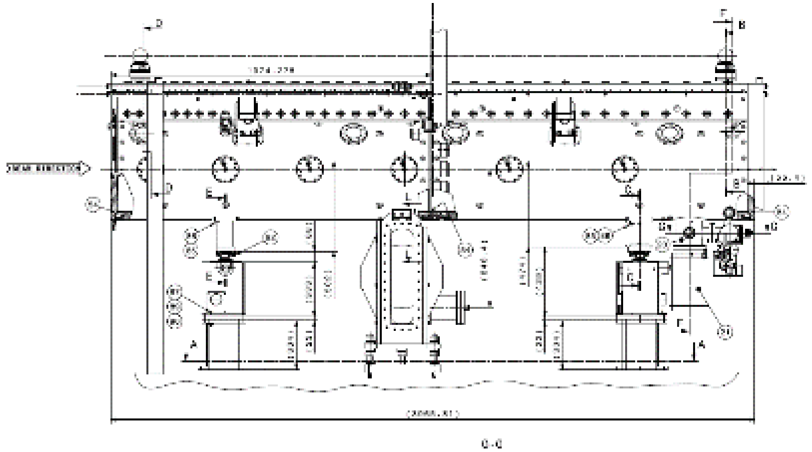
Example: Linac4 DTL Tank1











a puzzle of several thousand elements…

Cavity coatings

$$Skin\ Depth = \delta = \sqrt{\frac{\rho}{\pi f \mu}} = \sqrt{\frac{\rho}{\pi f \mu_r \mu_o}}$$

Where,

ρ = Resistivity of the Material

f = Frequency

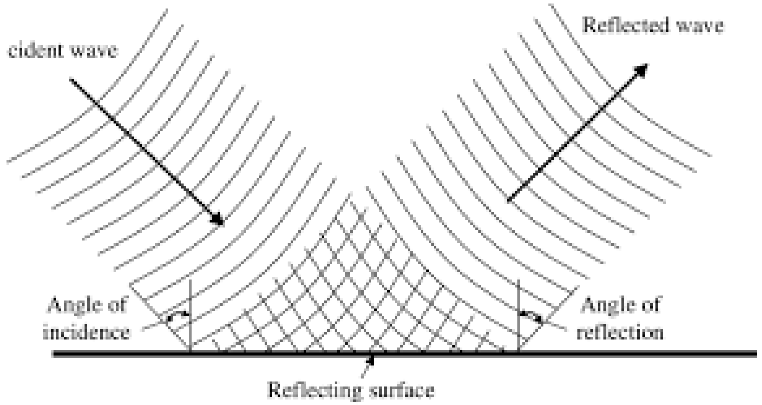
μ_r = Relative Permeability (usually 1)

μ_o = Permeability Constant = $4\pi \times 10^{-7}$

Skin depth:

In the reflection of a wave from a metallic surface, if the resistivity of the metal is not zero the wave will penetrate in the metal with an amplitude that decreases exponentially.

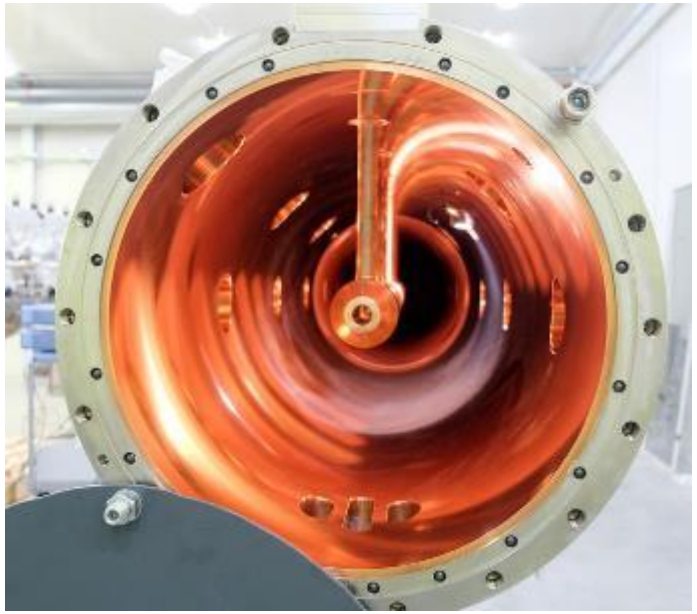
The depth at which the amplitude is 1/e (37%) of the original amplitude is called “skin depth”.



Because at high frequencies the power dissipation in a cavity occurs only in a thin layer below the surface, this property is used to realize cavities that are not entirely made in copper (expensive and difficult to machine) but made of a stainless-steel substrate electrochemically plated with a thin layer of copper.

Example: at 200 MHz, skin depth is 4.5 μm. A 30 μm layer of copper (about 5 skin depths) will intercept 99.9% of RF current.

In a similar way, often SC cavities are made in copper sputtered with a thin layer of niobium (resistivity in this case is very low and only few atomic layers are sufficient!)

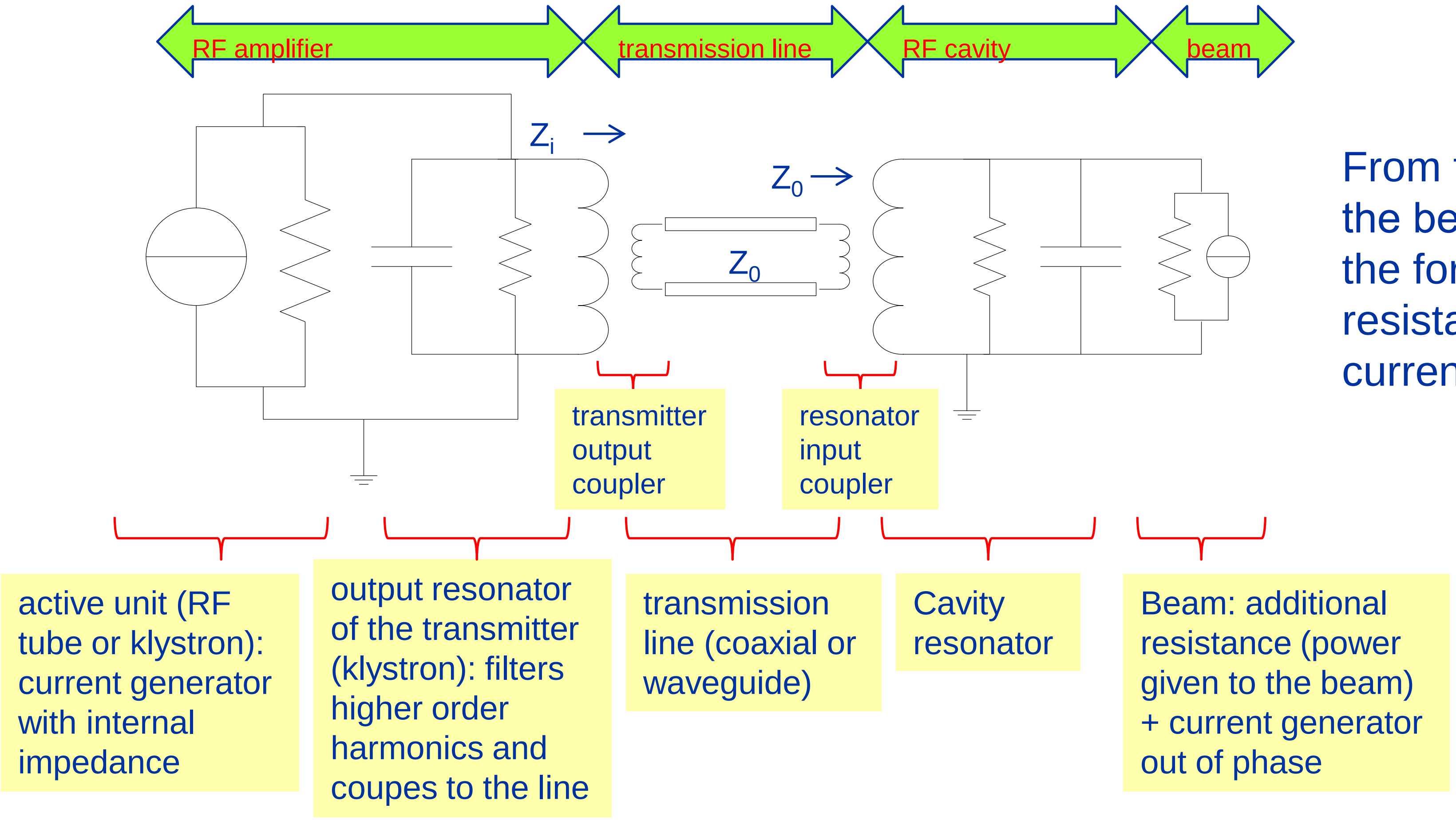


The RF power system

The RF system – the complete circuit diagram

A closer insight of what the RF does is given by the equivalent circuit of an amplifier/cavity system

RF is the art of transporting energy at given frequency by matching impedances



From the electrical point of view, the beam is a "perturbation", in the form of an additional parallel resistance and a (dephased) current generator!

RF and construction technologies

→ Type of RF power source depend on frequency:

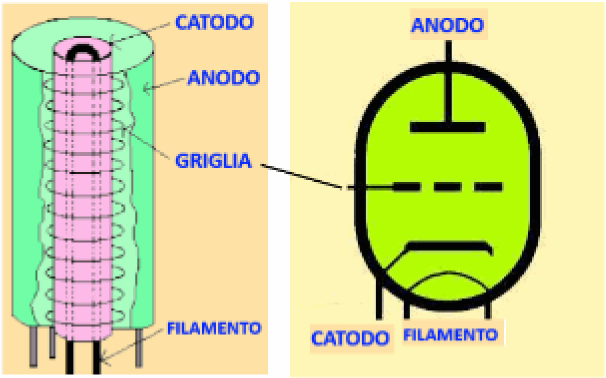
- ☞ Klystrons (>350 MHz) for electron linacs and modern proton linacs. RF distribution via waveguides.
- ☞ RF tube (<400 MHz) or solid state amplifiers for proton and heavy ion linacs. RF distribution via coaxial lines.

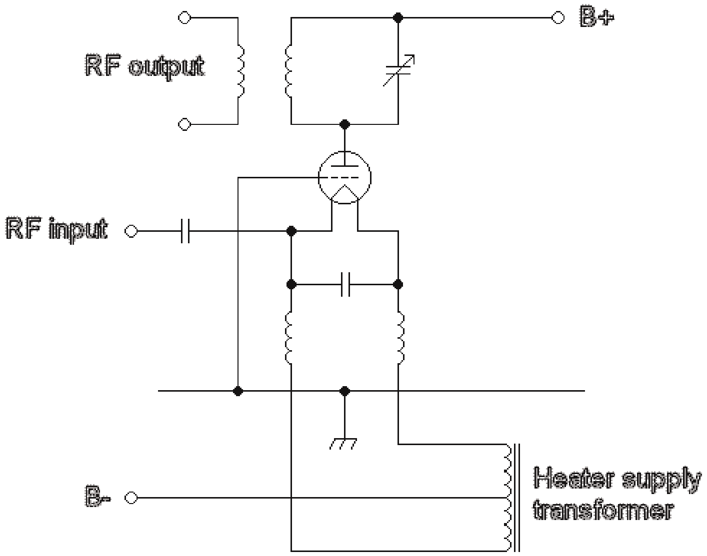
3 GHz klystron (CERN LPI)

200 MHz triode amplifier (CERN Linac3)

RF tubes

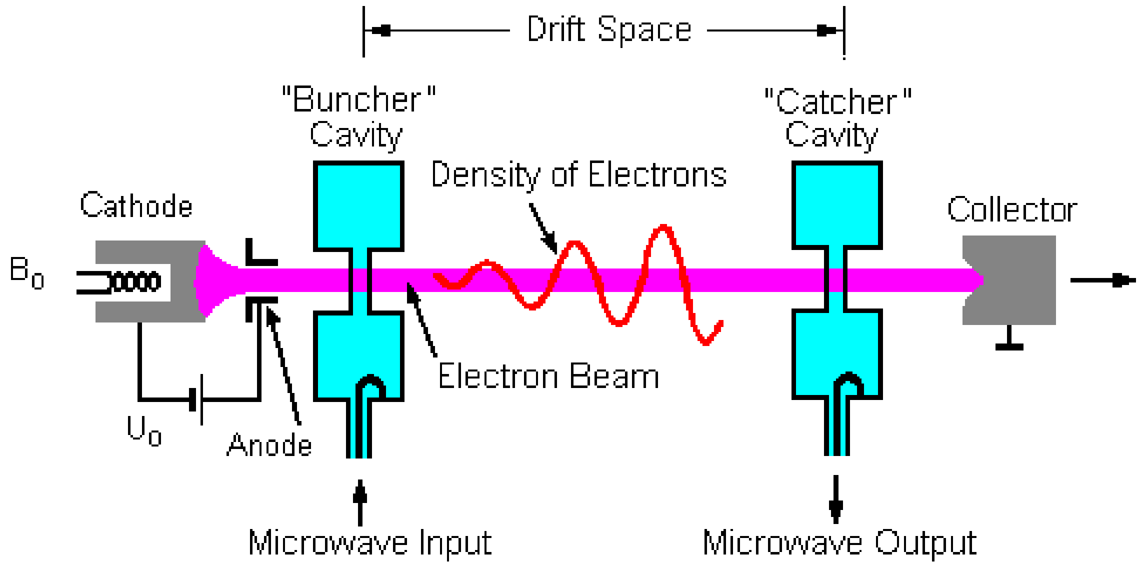
High power triodes and tetrodes :
Used as active elements to amplify an RF signal (typical gain 10-30 dB).
To reach peak powers in the MW range are used cascades of RF amplifiers using different types of tubes.





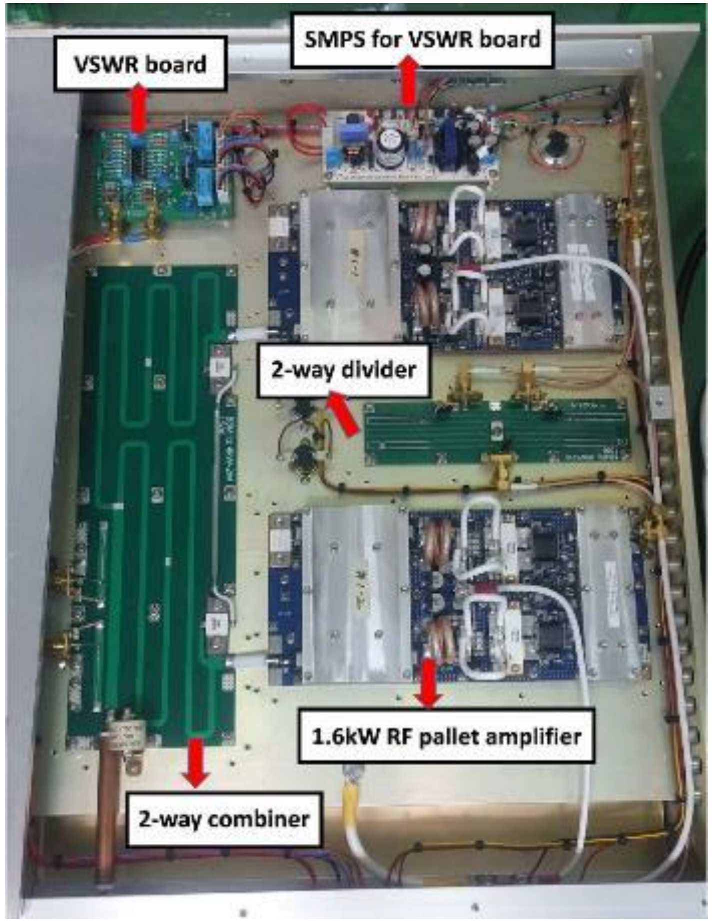
Courtesy
JAES
company

Klystrons



High frequency RF power amplifier :
High efficiency and high gain.
Need bunching length of a few wavelength,
used only at frequencies > 350 MHz.

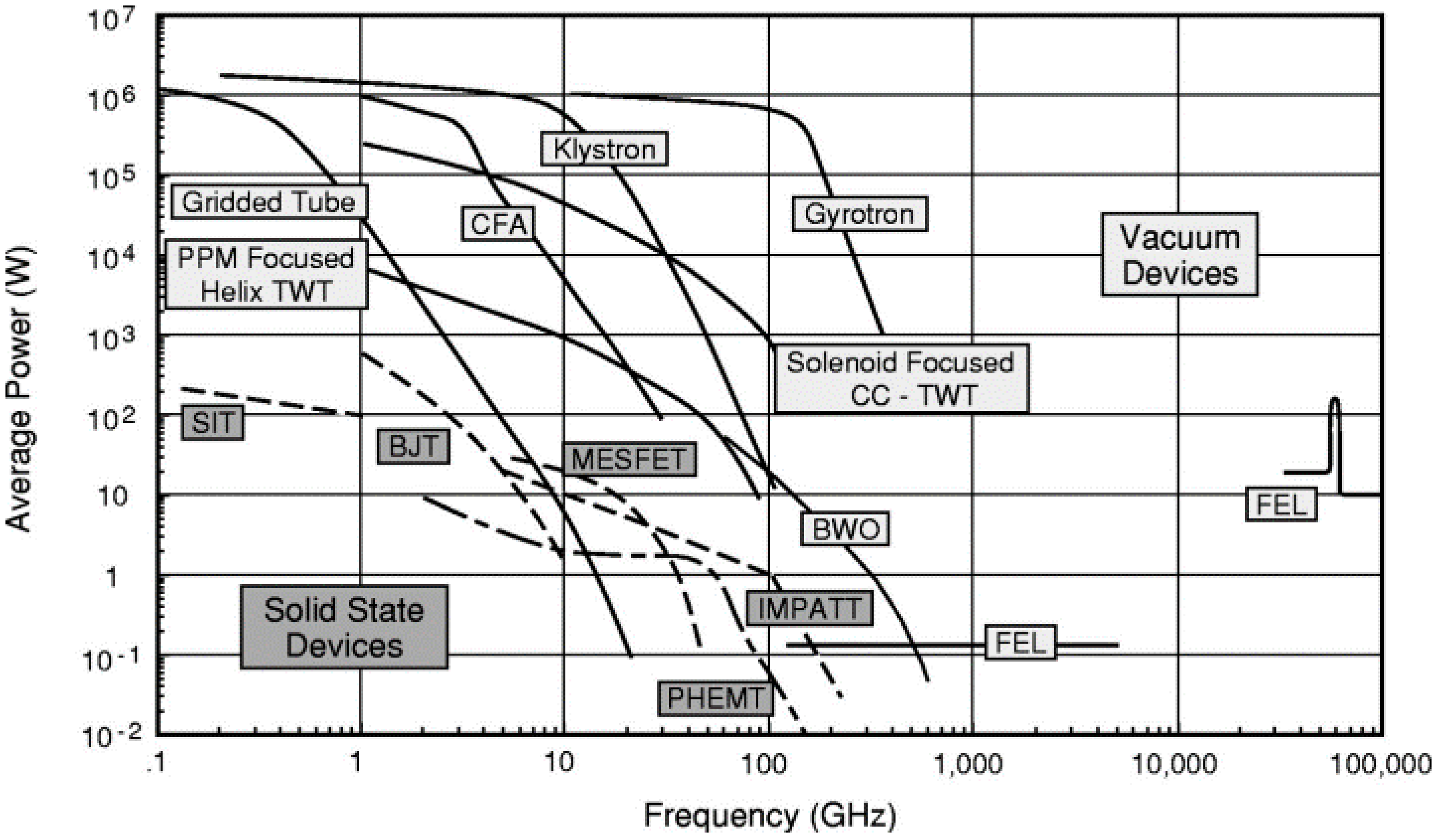
Solid State

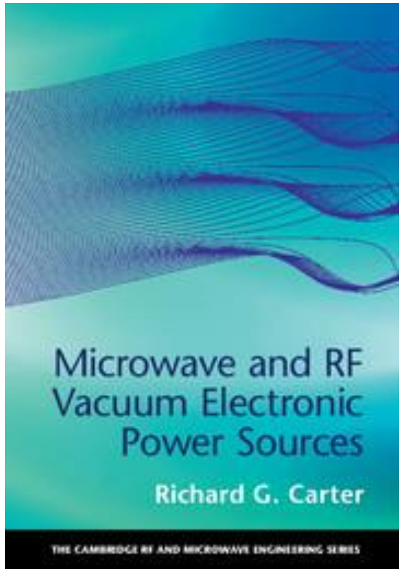


3.2-kW RF amplifier module with two-way divider and combiner
H. Song, S. Lee, J. Chai, Modular 20 kW, 83.2-MHz Solid-State RF Amplifier for a 10-MeV Cyclotron

New solid-state RF amplifier system built by Thales for the CERN SPS, to provide the power for the high-luminosity LHC upgrade, commissioned in 2020.
Transistors are assembled in sets of four on 2 kW modules.
A total of 2560 modules (10 240 transistors) are spread across 32 towers. Power from 16 towers is combined to reach two times 1.6 MW.

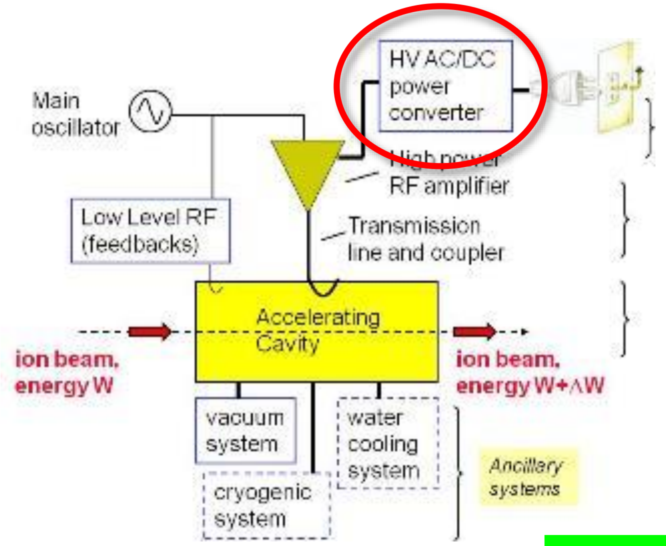
Comparing different RF power sources





(from R. Carter's book)

Power converter

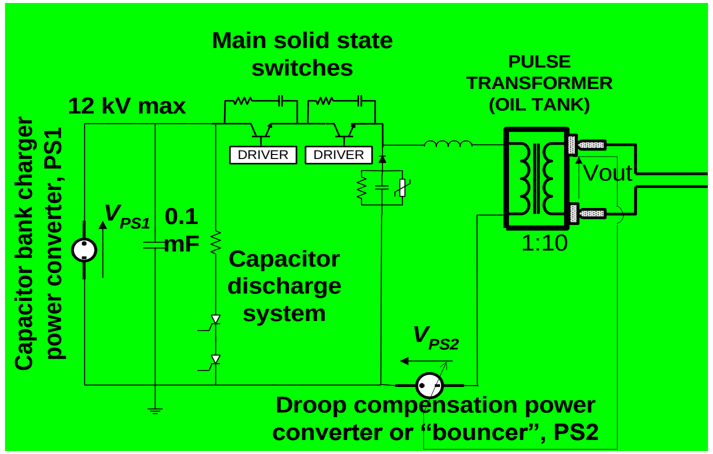


The Power Converter

An essential ingredient for the reliability (high voltages!), stability (noise) and cost of a linac!

Made of rectifiers + HV transformer + energy storage when pulsed (can be a PFN).

If pulsed and HV, called "modulator".



Topology and layout of the Linac4 modulator

Transmission lines

Power has to be transported to the cavity without reflections (matching), with minimum loss and reliably (no arcs!).
Coaxial (rigid or cable) or waveguide.

Rigid coaxial lines for the CERN TW 400 MHz RF system at the SPS

Semi-rigid coaxial cable

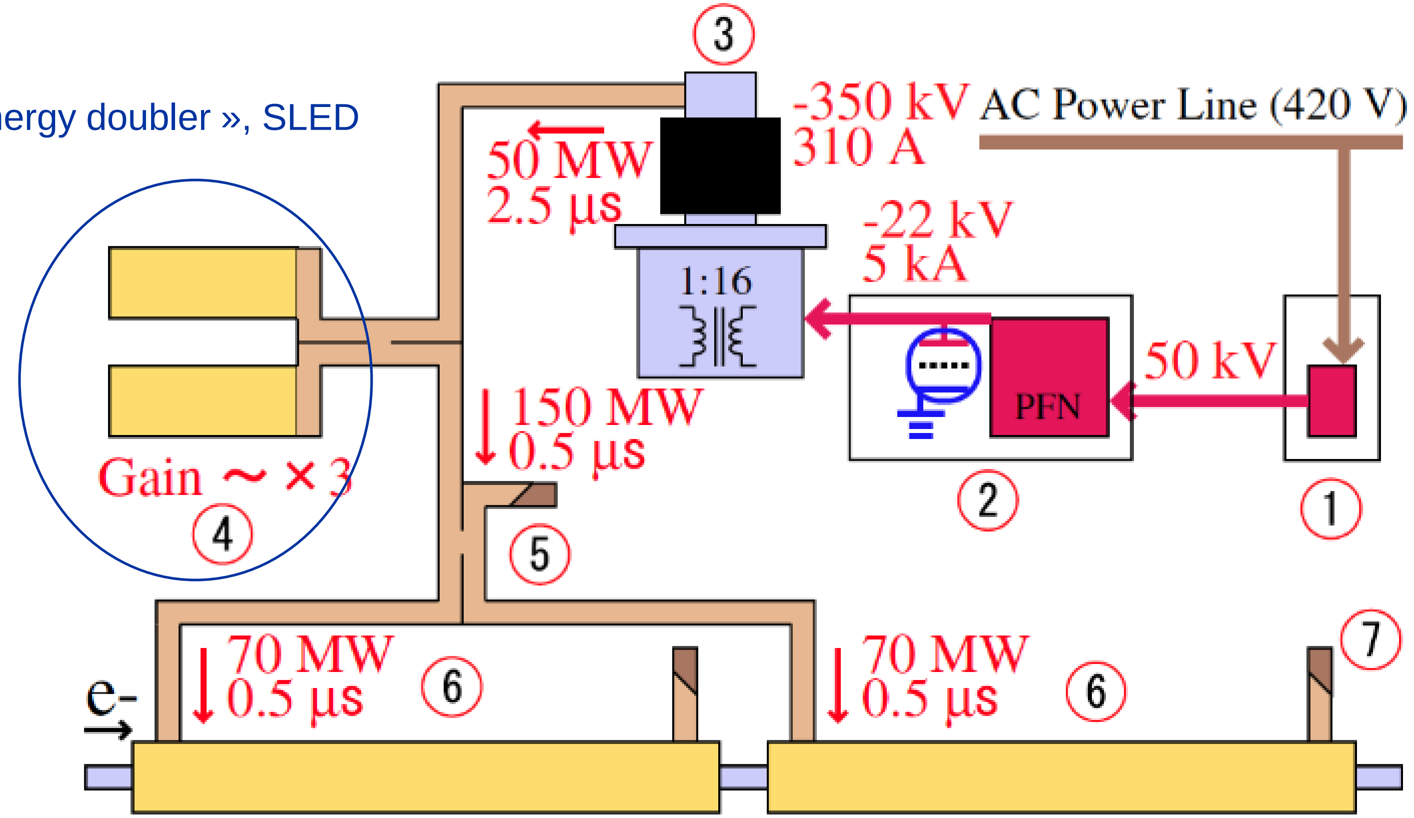
Rectangular waveguide

Rigid coaxial lines

Traveling wave building blocks

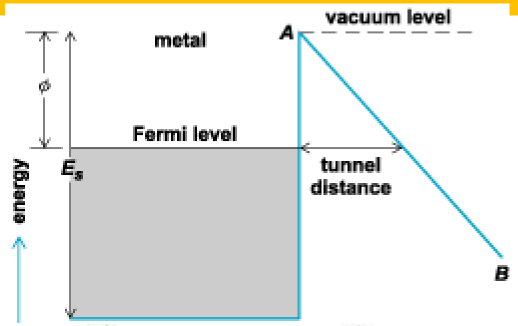
TSUMORU SHINTAKE *et al.*

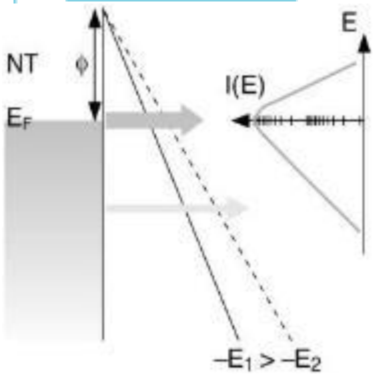
« Energy doubler », SLED

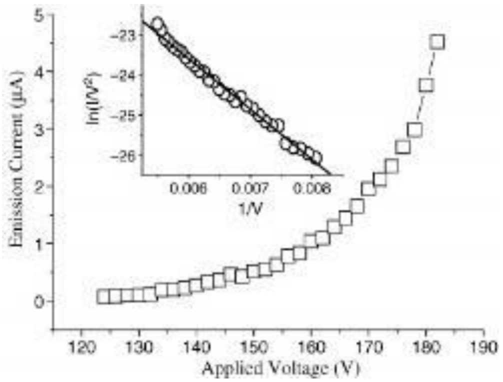


Operational limitations for RF systems

The breakdown limit (in vacuum)







The origin of breakdowns is FIELD EMISSION:
Tunneling of electrons through the potential barrier at the boundary metal/vacuum in presence of an applied voltage. Increasing the surface electric field reduces the barrier and increases the number of escaping electrons. The temperature enhances the emitted current (field assisted thermoionic emission).

Quantum mechanics phenomenon, can be calculated, Fowler-Nordheim relation:

$$J(E) = \frac{1.54 \cdot 10^{-10}}{\phi} \cdot E^2 \cdot \exp\left(-\frac{3.21 \cdot 10^{-9} \cdot \phi^{3/2}}{E}\right)$$

Current density (A/cm²) emitted by a metal of extraction potential Φ with an applied electric field E, in the limit case of $T \to 0$ (with some additional approximations).

The current goes up very quickly (exponential); a F.-N. behaviour is characterised by a linear ln(I/V)=f(1/V)

Breakdown and impurities

Looking at the numbers, for copper the F.E. current starts to become important only for fields in the region of few GV/m, well beyond normal operating fields in the order of 10-40 MV/m.

(E=1 GV/m → J=5e-17 A/m^2, E=5 GV/m → J=7e11 A/m^2 !!)

But: the theory is valid for smooth and perfect surfaces. A real electrode has marks coming from machining (finite surface roughness) and contains impurities incrusted on the surface (grains). Both these elements increase the surface field (edges for the roughness and dielectric constant for the impurities).

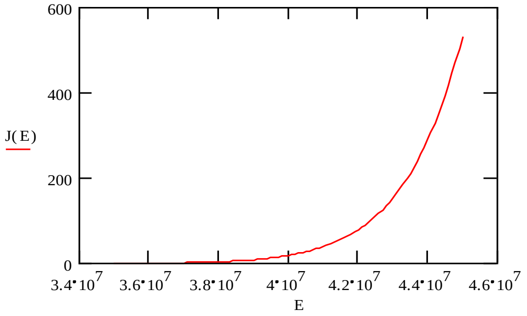
"Mountain and snow": The real field on the surface is βE, adding an "enhancement" factor β.

$$J(E) = 4.83 \cdot 10^{-11} \cdot (\beta \cdot E)^{2.5} \cdot \exp\left(-\frac{6.55 \cdot 10^{10}}{\beta \cdot E}\right)$$

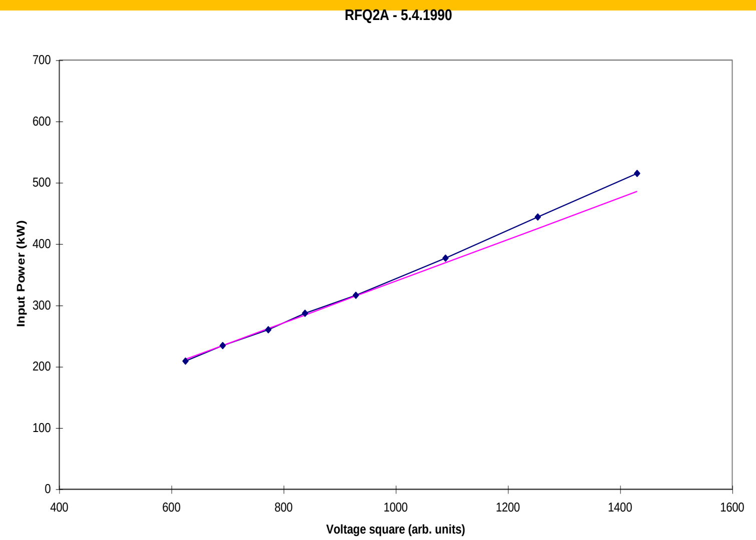
F.-N. formula for copper, with the enhancement factor

Field emission current is a pre-breakdown phenomenon. When the current at a certain spot goes beyond a certain limit a breakdown starts (the real physics of the phenomenon is still under discussion).

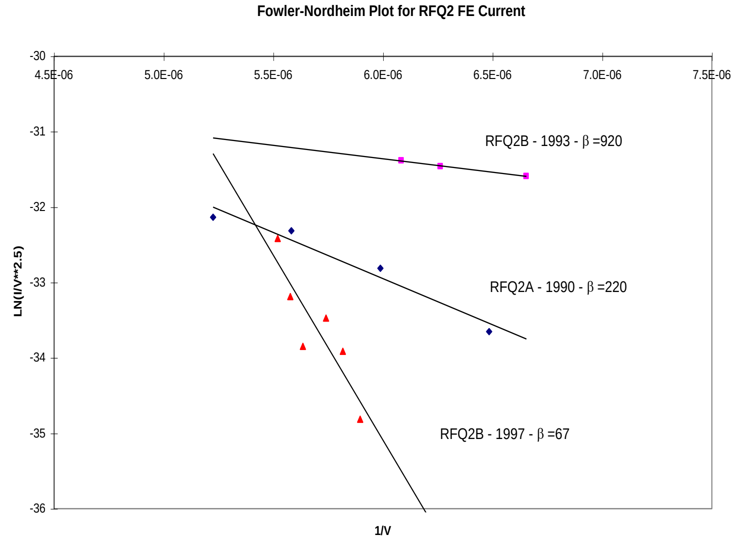
Field emission current is often called « dark current » and can be measured.



Measurement of dark current



Plotting RF power measured at the input of the cavity as function of cavity voltage square (arbitrary units, measures on a pick-up loop at the cavity) we see above a certain power the appearance of dark current accelerated on the gap and absorbing power.



Considering that the dark current power is proportional to gap voltage and to current intensity, we can plot ln(I/V)=f(1/V); the slope of the curve is the enhancement factor.

CERN RFQ2:

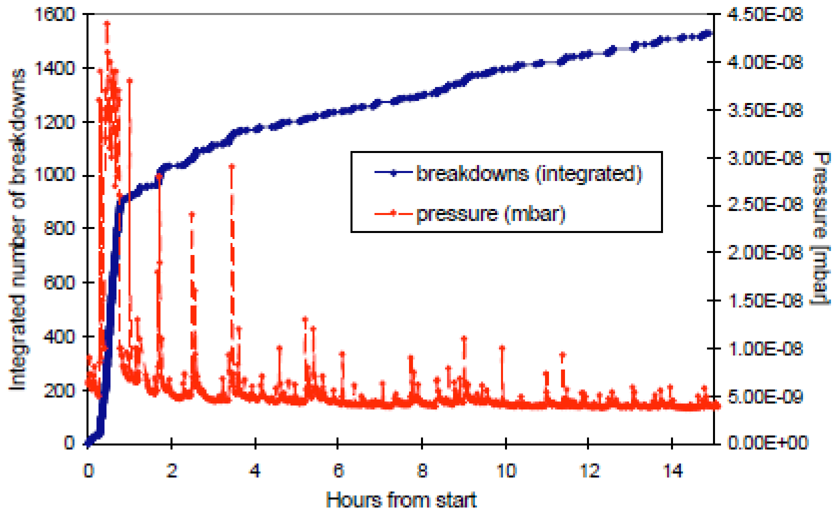
β=220 electrodes as out of the workshop

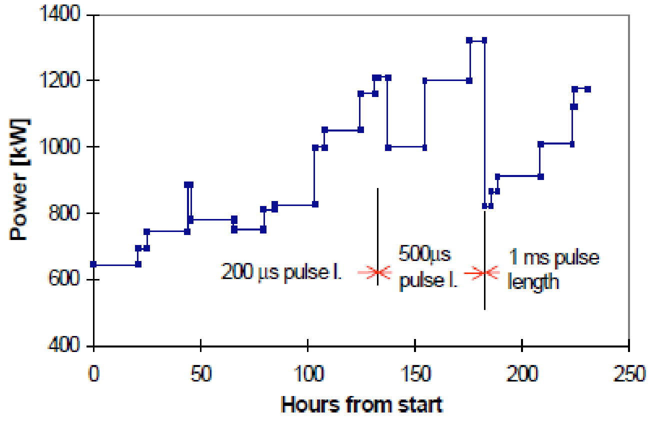
β=920 after a heavy pollution from hydrocarbons from the vacuum system

β=67 after long conditioning

Breakdowns and conditioning

Increasing the voltage, in one or few points the F.-E. current will go above the threshold and start a breakdown.
But the breakdown will reconfigure the surface and there is a certain probability that the new surface will be better (melted spikes, degazed impurities) → we can continue and condition the cavity.
At some point, the number of emitting spots will be so high that we will always find a sparking point → limit of conditioning.
Note that breakdown is a statistic phenomenon. We cannot define a breakdown threshold, instead we can speak of sparking rate (number of breakdown per unit time) as funtion of voltage. The rate is decreased by conditioning; it will decrease asymptotically towards a limiting value.





The Kilpatrick field

W. Kilpatrick in 1956 fitted some experimental data on breakdown "levels" in RF regime with a F.-N. type formula, assuming that the breakdown is ignited by the impact on the surface of an ion accelerated on the gap, with energy W:

$$WE^2 \exp(-17/E) = 1.8$$

In the late 60's at Los Alamos they introduced a calculation of the ion velocity based on gap and frequency (the ion has a transit time factor!) → frequency dependant version of the Kilpatrick criterion:

$$f = 1.64E^2 \exp(-8.5/E)$$

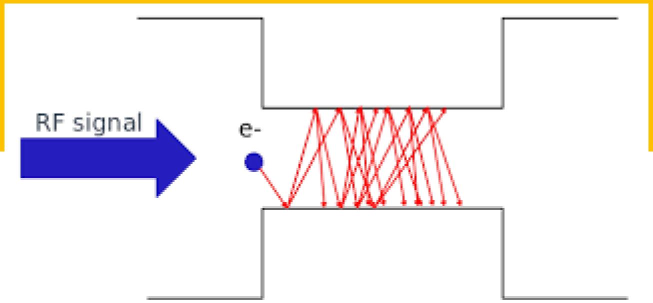
This formula is still used as a reference in the design of linear accelerators! For a given frequency, allows to calculate the "Kilpatrick field".

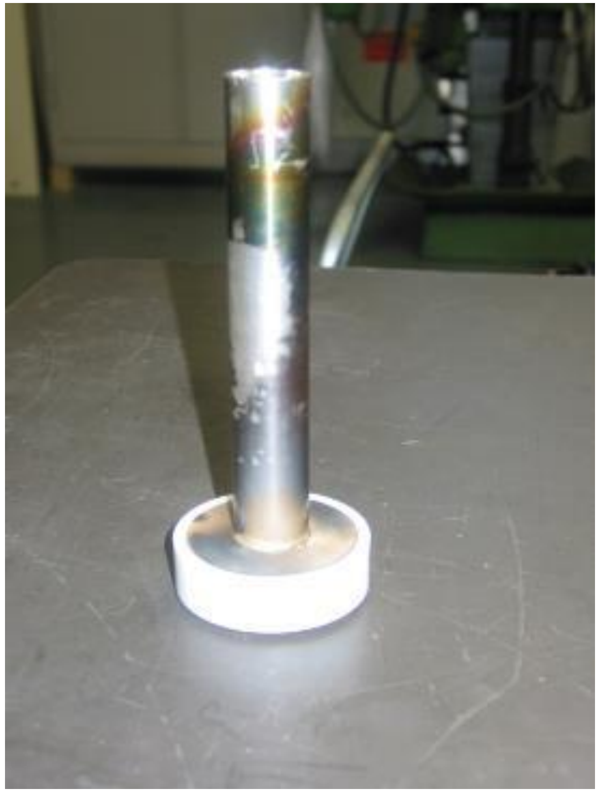
Very useful and a good reference, with some caveats:

1. It gives the famous result that maximum field goes as $\sqrt{f}$; although demonstrated in several cases, now it is not considered as a fixed law and in particular does not hold >10 GHz, where other phenomena take place.
2. It is not valid for small gaps and low frequencies, where we approach instead the limits for DC field;
3. It gives the wrong message that there is a breakdown threshold;
4. The experimental data were taken in the 50's with bad vacuums, nowadays a cavity can operate at a few times the Kilpatrick limit.

Usual maximum fields range from about 1 Kilpatrick (CW systems high reliability) to 2.5 Kilpatrick (RFQ2)

Multipactoring





The inner conductor of a coaxial coupler that was (probably by mistake) silver plated and then installed in a CERN linac cavity in 1978. Multipactoring went on for 25 years during normal operation sputtering silver on the window, until it eventually short circuited a Friday night in 2003 stopping all CERN accelerators…

Electron resonance due to secondary emitted electrons from the surfaces.
Occurs at low voltages and can completely stop normal operation of a cavity.

Some causes of multipactoring:

- Dirty surfaces (impurities emitting electrons)
- Air pockets (providing ions and electrons)
- Parallel plate and coaxial geometries (well-defined electron path)
- High pulsing rates (remaining electrons)
- Presence of silver (secondary emission >1)
- Vicinity of a ceramic insulator (high sec. emission)
- Bad luck…

Some cures:

- Conditioning: long times in the mp. region, possibly at higher repetition frequency, heating the surfaces to oxidize them and thus reduce the secondary emission coefficient below 1. Adding a frequency modulation increases the conditioned surface.
- Some paints (aquadag) were used in the past but are not very good for vacuum.

End of Lecture 6

Elwood Smith

Additional slides on coupling

Coupling between two cavities

In the PIMS, cells are coupled via a slot in the walls. But what is the meaning of coupling, and how can we achieve a given coupling?

Simplest case: 2 resonators coupled via a slot

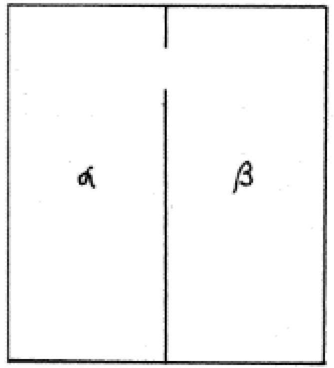


FIG. 1. Two cavities, α and β, coupled by a small hole.

E-field mode 1
E-field mode 2

Described by a system of 2 equations:

$$\begin{cases} X_1\left(1-\frac{\omega_1^2}{\omega^2}\right)+kX_2=0 \\ kX_1+X_2\left(1-\frac{\omega_2^2}{\omega^2}\right)=0 \end{cases} \quad \text{or} \quad \begin{vmatrix} 1-\frac{\omega_1^2}{\omega^2} & k \\ k & 1-\frac{\omega_2^2}{\omega^2} \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = 0$$

If $\omega_1=\omega_2=\omega_0$, usual 2 solutions (mode 0 and mode π):

$$\omega_{c,1}=\frac{\omega_0}{\sqrt{1+k}} \quad \text{with} \quad \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \end{vmatrix} \qquad \text{and} \qquad \omega_{c,2}=\frac{\omega_0}{\sqrt{1-k}} \quad \text{with} \quad \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \begin{vmatrix} 1 \\ -1 \end{vmatrix}$$

Mode + + (field in phase in the 2 resonators) and mode + - (field with opposite phase)

Taking the difference between the 2 solutions (squared), approximated for k<<1

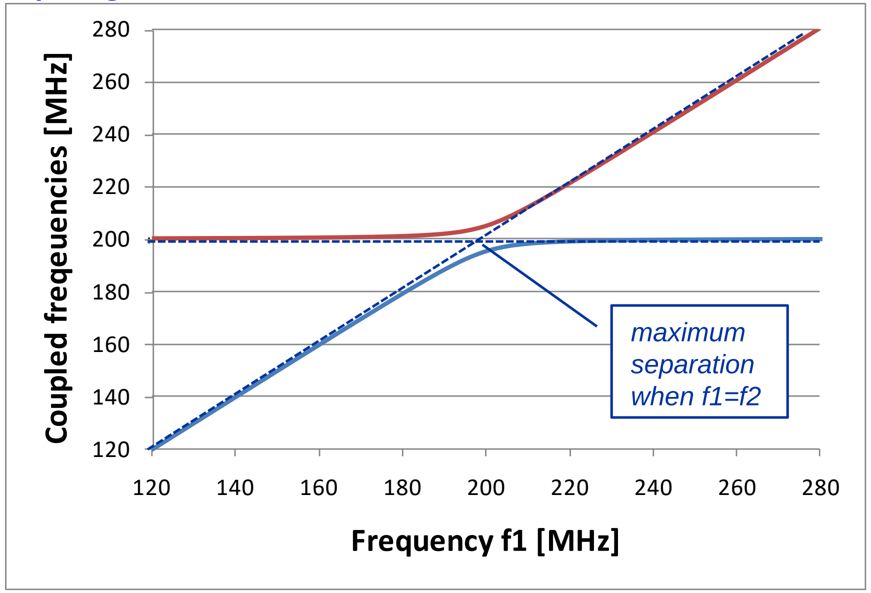
$$\frac{\omega_{c,2}^2-\omega_{c,1}^2}{\omega_0^2}=\frac{1}{1-k}-\frac{1}{1+k}\approx 2k \quad \text{or} \quad \boxed{\frac{\omega_{c,2}-\omega_{c,1}}{\omega_0}\approx k}$$

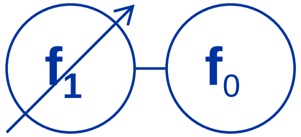
The coupling k is equal to the difference between highest and lowest frequencies.

→ k is the bandwidth of the coupled system.

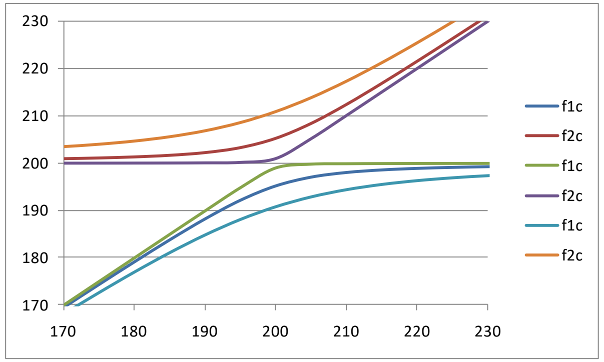
More on coupling

Solving the previous equations allowing a different frequency for each cell, we can plot the frequencies of the coupled system as a function of the frequency of the first resonator, keeping the frequency of the second constant, for different values of the coupling k.





- "Coupling" only when the 2 resonators are close in frequency.
- For $f_1=f_2$, maximum spacing between the 2 frequencies ($=kf_0$)



case of 3 different coupling factors (0.1%, 5%, 10%)

For an elliptical coupling slot:

$$k \approx F\, l^3 \left(\frac{H_1}{\sqrt{U_1}}\right)\left(\frac{H_2}{\sqrt{U_2}}\right)$$

F = slot form factor
l = slot length (in the direction of H)
H = magnetic field at slot position
U = stored energy

The coupling k is:
- Proportional to the 3rd power of slot length.
- Inv. proportional to the stored energies.

Measurement of R (R/Q)

The most difficult measurement (and for this reason one usually relies on simulation codes).

1. Direct voltage calibration: $Z = \frac{V_0^2}{P_c}$: we need to measure the power and then the voltage from an X-ray measurement

2. Perturbation measurement: $\frac{\Delta f}{f} = \frac{\int_{\Delta V} (\mu H^2 - \varepsilon E^2) dV}{2U} = \frac{1}{2} \frac{\Delta U}{U}$

a small perturbation (introduction of an object in a resonant cavity) shifts the frequency by one half of the ratio between change in magnetic minus electric energy and the total stored energy → knowing ΔU it is possible to calculate U !!!

R/Q:

$$Q = \frac{\omega U}{P} \qquad \frac{R}{Q} = \frac{P}{\omega U} \frac{E_0^2 L^2}{P}$$