Cosmological Perturbations in Hořava-Lifshitz Gravity

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Progress on Old and New Themes in cosmology
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based on:
AC, R. H. Brandenberger
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Goals

1. Determination of the **actual number of propagating scalar degrees of freedom** in both the projectable and non-projectable versions of Hořava-Lifshitz Gravity with scalar field matter.
2. Discussion on potential **ghost-like and/or tachyonic instabilities** of the **extra scalar degree of freedom** that the theory turns out to admit.

How?!

- By deriving the **quadratic action** for the linear fluctuation variables that are left once that gauge freedom and constraint equations are taken into full account and by then analyzing the kinetic terms.

Outline

1. Brief introduction to Hořava-Lifshitz Gravity
2. Presentation of original results

Hořava-Lifshitz Gravity (HLG): introduction (Horava, 2009)

HLG is a proposal for a **quantizable theory of gravitation** based on:

1. **Anisotropic scaling** between space and time:

$$\left\{ \begin{array}{ccc} \vec{x} & \rightarrow & b\,\vec{x} \\ t & \rightarrow & b^{z}\,t \end{array} \right. \rightsquigarrow \text{Lorentz-invariance is explicitly broken for any } z \neq 1$$

2. Power-counting **renormalizability**

$$\Rightarrow z = D = \#\text{ of spatial dimensions} = 3$$

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Remarks

1. The **full diffeomorphism invariance** of General Relativity is replaced by the invariance under **foliation preserving diffeomorphisms**:

$$x^k \to \tilde{x}^k = \tilde{x}^k(t, x^i)\,, \qquad \underline{t \to \tilde{t} = \tilde{t}(t)}$$

2. Scaling dimensions:

$$\boxed{[x^i]_s = -1; \quad [t]_s = -z} \quad \Longrightarrow \quad \boxed{[\partial_i]_s = 1; \quad [\partial_t]_s = z = 3}$$

$\rightsquigarrow$ improved graviton dispersion relation: $\omega^2 = k^2 + \alpha\,\frac{k^4}{\Lambda^2} + \beta\,\frac{k^6}{\Lambda^4}$

Hořava-Lifshitz Gravity: motivations

1. **Interesting cosmological implications**[1]**:**
   - alternative solutions to the
     - flatness problem,
     - horizon problem;
   - scale-invariant cosmological perturbations w/o inflation;
   - dark matter as an integration constant;
   - bouncing scenarios in the case of spatially curved background.

2. **Potential issues**[2]**:**
   - Is Lorentz symmetry recovered at low energies?
   - Does the reduced symmetry (with respect to GR) allow for extra gravitational degrees of freedom? If so, do they decouple in the infrared?

What about the most general version of HLG on the cosmological background?

[1] see *e.g.* Mukohyama (2010), *"Hořava-Lifshitz Cosmology: A Review"*
[2] see *e.g.* Sotiriou (2010): *"Hořava-Lifshitz gravity: a status report"*

ADM formalism

- It is convenient to adopt the Arnowitt-Deser-Misner (ADM) formalism for the metric (Arnowitt, Deser, and Misner, 1962):

$$ds^2 = -N^2 c^2 dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right).$$

- The metric variables are:

$N = N(t, \vec{x})$ lapse function	$N^i = N^i(t, \vec{x})$ shift vector	$g_{ij} = g_{ij}(t, \vec{x})$ 3D metric

- Their scaling dimensions are:

$$[N]_s = 0 \qquad [N^i]_s = z - 1 \qquad [g_{ij}]_s = 0$$

Projectability condition

$N = N(t) \rightsquigarrow$ **different constraint equations & gauge freedoms**

HLG + scalar field matter: action

(Sotiriou et al., 2009; Wang et al., 2010; Blas et al., 2010)

$$\boxed{S = \frac{M_{\rm Pl}^2}{2} \int dt\, d^3x\, N\sqrt{g} \left(\mathcal{L}_K - \mathcal{L}_V + \frac{2}{M_{\rm Pl}^2} \mathcal{L}_M \right)} \qquad [dt\, d^3x]_s = -6$$

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Kinetic term

NB: GR $\rightarrow \lambda \equiv 1$

$$\mathcal{L}_K = K_{ij}K^{ij} - \lambda K^2, \qquad K_{ij} \equiv \frac{1}{2N}\left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right) \text{ “extrinsic curvature”}$$

note that $[\partial_t]_s = 3$ & $[g_{ij}]_s = 0 \Rightarrow [K_{ij}]_s = 3 \Rightarrow [\mathcal{L}_K]_s = 6 \Rightarrow [M_{\rm Pl}^2]_s = 0$

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Scalar field matter

$$\mathcal{L}_M = \frac{1}{2N^2} \left(\dot{\varphi} - N^i \nabla_i \varphi \right)^2 - V(g_{ij}, \mathcal{P}_n, \varphi) \qquad \left(\text{note that } [\varphi]_s = \frac{D-z}{2} \right)$$

$$V = V_0(\varphi) + V_1(\varphi)\mathcal{P}_0 + V_2(\varphi)\mathcal{P}_1^2 + V_3(\varphi)\mathcal{P}_1^3 + V_4(\varphi)\mathcal{P}_2 + V_5(\varphi)\mathcal{P}_0\mathcal{P}_2 + V_6(\varphi)\mathcal{P}_1\mathcal{P}_2$$

$$\mathcal{P}_0 \equiv (\nabla\varphi)^2, \quad \mathcal{P}_i \equiv \Delta^i \varphi, \quad \Delta \equiv g^{ij}\nabla_i\nabla_j\,.$$

HLG + scalar field matter: action

...aka "healthy"/consistent extension (Blas, Pujolas, and Sibiryakov, 2010)

$$\boxed{S = \frac{M_{\rm Pl}^2}{2}\int dt\, d^3x\, N\sqrt{g}\left(\mathcal{L}_K - \mathcal{L}_V + \frac{2}{M_{\rm Pl}^2}\mathcal{L}_M\right)} \qquad [dt\, d^3x]_s = -6$$

power-counting renormalizability $\Longrightarrow [g\text{'s}]_s, [\eta\text{'s}]_s \geq 0$

$$\mathcal{L}_V = \underbrace{\color{red}{g_0\chi^2 + g_1 R} + \frac{1}{\chi^2}(g_2R^2 + g_3R_{ij}R^{ij})}_{\text{super-renormalizable / relevant}} + \underbrace{\frac{1}{\chi^4}\left(g_4R^3 + g_5RR_{ij}R^{ij} + g_6R^i_jR^j_kR^k_i\right)}_{\text{renormalizable / marginal}} +$$

$$+ \underbrace{\frac{1}{\chi^4}\left[g_7R\Delta R + g_8(\nabla_iR_{jk})(\nabla^iR^{jk})\right]}_{\text{renormalizable / marginal}} \underbrace{- \eta a_ia^i + \frac{1}{\chi^2}\left(\eta_2a_i\Delta a^i + \eta_3R\nabla_ia^i\right)}_{\text{super-renormalizable / relevant}} +$$

$$+ \underbrace{\frac{1}{\chi^4}\left(\eta_4a_i\Delta^2a^i + \eta_5\Delta R\nabla_ia^i + \eta_6R^2\nabla_ia^i\right)}_{\text{renormalizable / marginal}}$$

$$\boxed{a_i \equiv \frac{\partial_iN}{N};\ \chi^2 \equiv \frac{M_{\rm Pl}^2}{2}}$$

Once that

1) **constraints** and
2) **gauge freedom**

are taken into full account one is left with[3]

$$\delta_2 S^{(S)} = \frac{M_{\rm Pl}^2}{2}\int dt\,\frac{d^3k}{(2\pi)^3}a^3\Big\{\boxed{c_\varphi \dot{\delta\varphi}_k^2 + c_\psi \dot\psi_k^2 + c_{\varphi\psi}\dot\psi_k\dot{\delta\varphi}_k} +$$

$$+ f_\varphi \delta\varphi_k \dot{\delta\varphi}_k + f_\psi \psi_k\dot\psi_k + f_{\varphi\psi}\psi_k\dot{\delta\varphi}_k + \tilde f_{\varphi\psi}\dot\psi_k\delta\varphi_k \underline{-m_\varphi^2\delta\varphi_k^2 - m_\psi^2\psi_k^2 - m_{\varphi\psi}^2\psi_k\delta\varphi_k}\Big\}$$

fluctuations variables $\to$ ψ (metric) and $\delta\varphi$ (matter)

[3] in Fourier-space: $\delta\varphi_k(t) \equiv \mathcal{F}\left[\delta\varphi(t,\vec{x})\right]$

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$$\left. + f_\varphi \delta\varphi_k \dot{\delta\varphi}_k + f_\psi \psi_k \dot\psi_k + f_{\varphi\psi}\psi_k \dot{\delta\varphi}_k + \tilde f_{\varphi\psi}\dot\psi_k \delta\varphi_k \underline{- m_\varphi^2 \delta\varphi_k^2 - m_\psi^2 \psi_k^2 - m_{\varphi\psi}^2 \psi_k \delta\varphi_k} \right\}$$

fluctuations variables $\rightarrow \psi$ (metric) and $\delta\varphi$ (matter)

How many propagating degrees of freedom?

Note that $\left. c_\varphi \dot{\delta\varphi}_k^2 + c_\psi \dot\psi_k^2 + c_{\varphi\psi}\dot\psi_k \dot{\delta\varphi}_k \right|_{k=0} \propto \left(\frac{H}{\dot\varphi_0}\dot{\delta\varphi}_k + \dot\psi_k \right)^2 = \dot{\mathcal{R}}^2 + \dots$

$\mathcal{R} \equiv \psi + \frac{H}{\dot\varphi}\delta\varphi \rightarrow$ comoving curvature perturbation (gauge invariant)

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Characterization of HLG's degrees of freedom

the kinetic matrix has **no zero eigenvalue** $\Rightarrow$ 2 scalar degrees of freedom

$c_{1,2}$: eigenvalues of the kinetic matrix

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- **infrared limit** ($k \to 0$)

$$c_1 \simeq \frac{1}{2}, \quad c_2 \simeq \eta \left(\frac{k}{aH}\right)^2 \qquad \Longrightarrow \text{ghost-freeness for } \eta \geq 0$$

- **ultraviolet limit** ($k \to \infty$)

$$c_1 \simeq \frac{1}{2}, \quad c_2 \simeq 2\frac{3\lambda - 1}{\lambda - 1} \qquad \Longrightarrow \text{ghost-freeness for } \lambda \notin (1/3, 1) \rightsquigarrow \lambda > 1$$

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Note

The mass matrix coefficients $m_\varphi, m_\psi, m_{\varphi\psi}$ do not tend to zero as $k \to 0$

$\Longrightarrow$ **the extra scalar mode decouples in the IR limit**

Projectable version: a summary of results

NB: lapse function $\rightarrow N \equiv N(t)$

$$\delta_2 S^{(S)} = \frac{M_{\rm Pl}^2}{2}\int dt\, d^3x\, a^3 \left\{ \frac{4(3\lambda-1)}{\lambda-1}\frac{\dot{\psi}^2}{2} + \frac{\dot{\delta\varphi}^2}{M_{\rm Pl}^2} + \dots \right\}$$

the kinetic matrix has **no zero eigenvalue** $\Rightarrow$ 2 scalar degrees of freedom

Quantum instability

$\frac{1}{3} < \lambda < 1 \;\Rightarrow\;$ the extra d.o.f. is **ghost-like** $\quad \rightsquigarrow \lambda > 1$

Classical instability

$\lambda > 1 \;\Rightarrow\;$ the extra d.o.f. is **tachyonic** $(m^2 < 0)$

Hořava-Lifshitz Gravity: conclusions

Projectable version, $N = N(t)$

- ☹ There exist an **unwanted d.o.f.** of gravitational origin;
- ☹ the extra d.o.f. is either **ghost-like** for $1/3 < \lambda < 1$ or **tachyonic**.

Non-projectable version, $N = N(t, \vec{x})$

- ☹ There exist an **unwanted d.o.f.** of gravitational origin;
- ☺ it **decouples** from the low-energy Physics;
- ☺ it is **stable in the infrared** as long as $\eta \geq 0$;
- ☺ it is **stable in the ultraviolet** for $\lambda > 1$.

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Thank you for your attention!

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From GR to HLG

ADM metric

$$ds^2 = -N^2 c^2 dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$

General Relativity action

$$S_{EH} = \frac{M_{\rm Pl}^2}{2}\int d^4x\,\sqrt{-{}^{(4)}g}\,\left[{}^{(4)}R - 2\Lambda\right] =$$

$$= \frac{M_{\rm Pl}^2}{2}\int dt\,d^3x\,N\sqrt{{}^{(3)}g}\left[\underbrace{K_{ij}K^{ij} - K^2}_{\text{kinetic term}} + \underbrace{{}^{(3)}R - 2\Lambda}_{\text{potential term}}\right] \qquad [dt\,d^3x]_s = -4$$

$$K_{ij} \equiv \frac{1}{2N}\left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right) \qquad \text{“extrinsic curvature”}$$

$$\boxed{\text{note that } [K_{ij}]_s = 1 \Rightarrow [M_{\rm Pl}^2]_s = 2}$$

HLG: constraints

N and N^i are not dynamical. They act as constraints.

1. **Hamiltonian constraint**

projectable version	non-projectable version
$\int d^3x\sqrt{g}\left(\mathcal{L}_K+\mathcal{L}_V-\frac{1}{2\chi^2}J^t\right)=0$	$\mathcal{L}_K+\mathcal{L}_V-\frac{1}{2\chi^2}J^t=0$

where $J^t \equiv 2\left(N\frac{\delta\mathcal{L}_M}{\delta N}+\mathcal{L}_M\right)$.

2. **Super-momentum constraint**

$$\pi^{ij}{}_{;i}=\frac{1}{2\chi^2}J^j$$

where $\pi^{ij}\equiv N\frac{\delta\mathcal{L}_K}{\delta\dot{g}_{ij}}=-K^{ij}+\lambda K g^{ij}$ and $J_i\equiv -N\frac{\delta\mathcal{L}_M}{\delta N^i}=\frac{1}{N}\left(\dot{\varphi}-N^k\nabla_k\varphi\right)\nabla_i\varphi$.

Cosmology in HLG

$$
\begin{aligned}
N(t) &= 1 + \delta N(t), \\
N^i(t, x^k) &= 0 + \delta N^i(t, x^k), \\
g_{ij}(t, x^k) &= \frac{a^2(t)}{\left(1 + \frac{\mathcal{K}}{4} r^2\right)^2} \delta_{ij} + \delta g_{ij}(t, x^k) \,.
\end{aligned}
$$

0th-order Hamiltonian constraint

$$
\frac{(3\lambda - 1)}{2} H^2 = \frac{1}{3}\left(\frac{\rho_M}{2\chi^2} + \Lambda\right) - \frac{\mathcal{K}}{a^2} + \frac{2\beta_1 \mathcal{K}^2}{a^4} + \frac{4\beta_2 \mathcal{K}^3}{a^6}
$$

($\sim$ Friedmann equation)

- ρ_M is the (background) energy density associated with matter
- $\beta_1 = (3g_2 + g_3)/\chi^2$, $\beta_2 = (9g_4 + 3g_5 + g_6)/\chi^4$

Cosmology in HLG

The dynamical equation for the scale factor $a(t)$ can be obtained by varying the action with respect to g_{ij} and evaluating the result in the homogeneous limit.

$$(3\lambda - 1)\frac{\ddot{a}}{a} = -\frac{1}{6\chi^2}(\rho_M + 3p_M) + \frac{2}{3}\Lambda - \frac{4\beta_1\mathcal{K}^2}{a^4} - \frac{16\beta_2\mathcal{K}^3}{a^6}$$

($\sim$ 2nd Friedmann equation)

- ρ_M is the (background) energy density associated with matter
- p_M is the (background) pressure associated with matter
- $\beta_1 = (3g_2 + g_3)/\chi^2$, $\beta_2 = (9g_4 + 3g_5 + g_6)/\chi^4$

Cosmological (scalar) perturbations

$\delta N(t) = \nu(t)$ (projectable) $\qquad \delta N(t,\vec{x}) = \nu(t,\vec{x})$ (non-projectable)

$\delta N_i(t,x^k) = \partial_i B(t,x^k)$

$\delta g_{ij}(t,x^k) = a^2(t)\left[-2\,\psi(t,x^k)\,\delta_{ij} + 2\,E(t,x^k)_{|ij}\right]$

Under the gauge transformation

$$\tilde{t} = t + \chi^0(t), \quad \tilde{x}^k = x^k + \chi^k(t,x^i)$$

where

$$\chi^0 = \xi^0(\eta)\,, \quad \chi^i = \nabla^i \xi(\eta,x^k) + \xi^i(\eta,x^k)\,, \quad \nabla_i \xi^i = 0$$

scalar perturbations transform as follows:

$$\begin{aligned} \tilde{\nu} &= \nu - \mathcal{H}\xi^0 - \xi^{0\prime}, & \tilde{\psi} &= \psi + \mathcal{H}\xi^0, \\ \tilde{B} &= B + \xi^0 - \xi', & \tilde{E} &= E - \xi. \end{aligned}$$

(Wang and Maartens, 2010)

Gauge choice

projectable version	non-projectable version
$\nu = 0 = E$	$E = 0$

1st-order constraints

(here $\mathcal{K} = 0 \Rightarrow$ zero spatial curvature)

Hamiltonian constraint

$$\left(\int d^3x\right)\left[3H(3\lambda-1)(\dot{\psi}+H\nu)+(3\lambda-1)H\Delta B-2\Delta\psi+\frac{\delta\rho_M}{2\chi^2}\right]=0$$

with

$$\delta\rho_M \;=\; \dot{\varphi}_0\dot{\delta\varphi}-\nu\dot{\varphi}_0^2+V_{0,\varphi}(\varphi_0)\delta\varphi+V_4(\varphi_0)\Delta^2\delta\varphi\,.$$

Super-momentum constraint

$$\nabla_i\left[(\lambda-1)\Delta B+(3\lambda-1)\left(\dot{\psi}+H\nu\right)-\frac{1}{2\chi^2}q_M\right]=0\,,$$

with

$$q_M=\dot{\varphi}_0\delta\varphi\,.$$