Backreaction in the relativistic Zel'dovich approximation

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Outline

1. Inhomogeneous models
2. Relativistic Zel'dovich approximation (RZA)
3. Results
4. Conclusion

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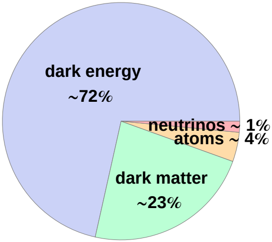
Cosmic energy budget 2011

- »Standard model« of cosmology: 95% dark physics
- Is this a measurement of the energy content of the universe?
- Yes, if you believe in the homogeneous isotropic model behind



- »Standard model« parameters

$$\Omega_m := \frac{8\pi G}{3H^2}\,\varrho \qquad \Omega_\Lambda := \frac{\Lambda}{3H^2}$$

$$\Omega_k := -\frac{k}{a^2 H^2}$$

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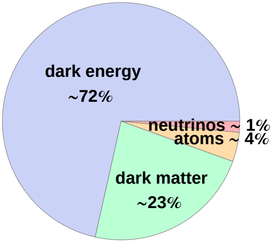
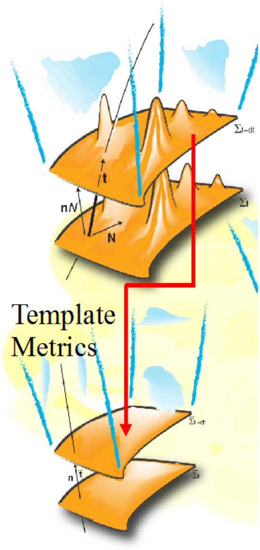


- »Standard model« parameters, complemented by at least one additional parameter for an inhomogeneous model

$$\Omega_m^{\mathcal{D}} := \frac{8\pi G}{3H_{\mathcal{D}}^2}\langle\varrho\rangle_{\mathcal{D}} \quad \Omega_\Lambda^{\mathcal{D}} := \frac{\Lambda}{3H_{\mathcal{D}}^2}$$

$$\Omega_{\mathcal{R}}^{\mathcal{D}} := -\frac{\langle\mathcal{R}\rangle_{\mathcal{D}}}{6H_{\mathcal{D}}^2} \quad \Omega_{\mathcal{Q}}^{\mathcal{D}} := -\frac{\mathcal{Q}_{\mathcal{D}}}{6H_{\mathcal{D}}^2}$$

Dynamic curvature



(Picture by Mauro Carfora)

- Modified dynamics

$$
\begin{aligned}
3H_{\mathcal{D}}^2 &= 8\pi G\langle\varrho\rangle_{\mathcal{D}} - \frac{1}{2}\langle\mathcal{R}\rangle_{\mathcal{D}} - \frac{1}{2}\mathcal{Q}_{\mathcal{D}} + \Lambda \\
3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} &= -4\pi G\langle\varrho\rangle_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}} + \Lambda \\
\mathcal{Q}_{\mathcal{D}} &= \frac{2}{3}\left(\langle\theta^2\rangle_{\mathcal{D}} - \langle\theta\rangle_{\mathcal{D}}^2\right) - 2\langle\sigma^2\rangle_{\mathcal{D}}
\end{aligned}
$$

- Curvature fluctuation coupling

$$a_{\mathcal{D}}^{-2}\partial_t\left(a_{\mathcal{D}}^2\langle\mathcal{R}\rangle_{\mathcal{D}}\right) = -a_{\mathcal{D}}^{-6}\partial_t\left(a_{\mathcal{D}}^6\mathcal{Q}_{\mathcal{D}}\right)$$

- Use of the RZA:
  - Quantitative estimate of the importance of inhomogeneities
  - Initialization of N–body simulations

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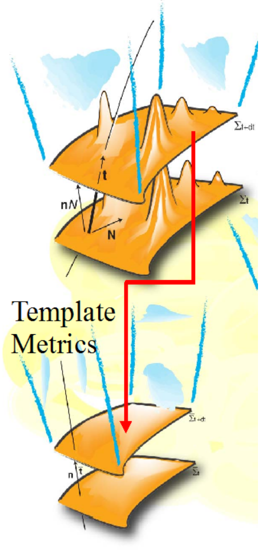

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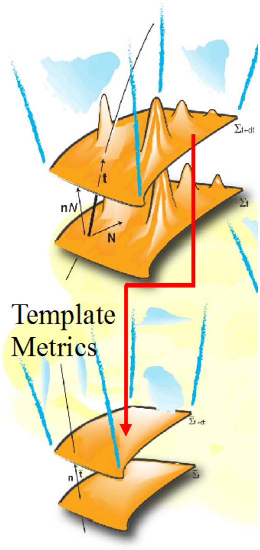

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Outline

Perturbative approach

- The coframe decomposition

$$g_{ij} = G_{ab}\eta^a{}_i\eta^b{}_j$$

- allows for a generalization of the Zel'dovich approximation to GR [4]

$${}^{\mathrm{RZA}}\eta^a{}_i(t,X^k) := a(t)\left(\delta^a{}_i + \xi(t)\,\dot{\mathscr{P}}^a{}_i\right)$$

$$\dot{\mathscr{P}}^a{}_i = \dot{\mathscr{P}}^a{}_i\left(t_{\mathbf{i}},X^k\right)\ ;\ \xi\left(t_{\mathbf{i}}\right) = 0\ ;\ a\left(t_{\mathbf{i}}\right) = 1$$

- As in the Newtonian case, the volume deformation is central

$$a_{\mathcal{D}}^3 = a^3\Sigma(t,X^k) := a^3\left(1 + \xi(t)\mathrm{I}_{\mathbf{i}} + \xi^2(t)\mathrm{II}_{\mathbf{i}} + \xi^3(t)\mathrm{III}_{\mathbf{i}}\right)$$

$$\mathrm{I}_{\mathbf{i}} := \mathrm{I}\left(\dot{\mathscr{P}}^a{}_i\right);\ \mathrm{II}_{\mathbf{i}} := \mathrm{II}\left(\dot{\mathscr{P}}^a{}_i\right);\ \mathrm{III}_{\mathbf{i}} := \mathrm{III}\left(\dot{\mathscr{P}}^a{}_i\right)$$

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Backreaction in the RZA

- Result for the backreaction term in the RZA

$$ {}^{\mathrm{RZA}}\mathcal{Q}_{\mathcal{D}} = \frac{\dot{\xi}^2\left(\gamma_1 + \xi\gamma_2 + \xi^2\gamma_3\right)}{\left(1 + \xi\langle \mathrm{I_i}\rangle_{\mathcal{C_D}} + \xi^2\langle \mathrm{II_i}\rangle_{\mathcal{C_D}} + \xi^3\langle \mathrm{III_i}\rangle_{\mathcal{C_D}}\right)^2} $$

$$ \begin{cases} \gamma_1 := 2\langle \mathrm{II_i}\rangle_{\mathcal{C_D}} - \frac{2}{3}\langle \mathrm{I_i}\rangle_{\mathcal{C_D}}^2 \\ \gamma_2 := 6\langle \mathrm{III_i}\rangle_{\mathcal{C_D}} - \frac{2}{3}\langle \mathrm{II_i}\rangle_{\mathcal{C_D}}\langle \mathrm{I_i}\rangle_{\mathcal{C_D}} \\ \gamma_3 := 2\langle \mathrm{I_i}\rangle_{\mathcal{C_D}}\langle \mathrm{III_i}\rangle_{\mathcal{C_D}} - \frac{2}{3}\langle \mathrm{II_i}\rangle_{\mathcal{C_D}}^2 \end{cases} $$

- Expression has the same form as in Newtonian theory $\Rightarrow$If one starts with flat initial conditions, the evolution of $\mathcal{Q}_{\mathcal{D}}$ is basically Newtonian
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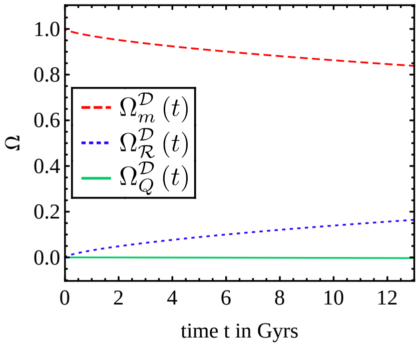
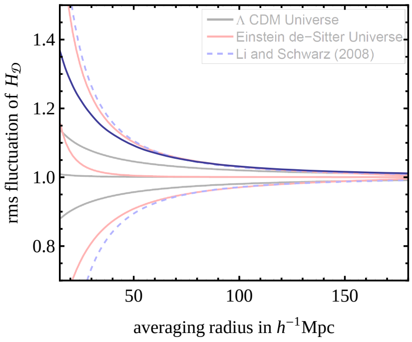
Outline

Deviations from background parameters





arXiv:1011.3959

- Time evolution of cosmic parameters on 100 Mpc already for a $1-\sigma$ deviation strongly affected
- Magnitude consistent with other calculations in standard perturbation theory[1]

Expected expansion fluctuations

- The constituents of $\mathcal{Q}_{\mathcal{D}}$ have an important domain independent component

$$\mathbb{E}\left[\left\langle\theta^2\right\rangle_{\mathcal{D}} - \langle\theta\rangle_{\mathcal{D}}^2\right] = \frac{1}{a}H_{\mathbf{i}}^2\left(\int_{\mathbb{R}^3} \mathrm{d}^3k P_{\mathbf{i}}(k) - (2\pi)^6 \int_{\mathbb{R}^3} \mathrm{d}^3k P_{\mathbf{i}}(k)\, \widetilde{W}_{\mathcal{D}}(k)^2\right)$$

$$\mathbb{E}\left[\left\langle\sigma^2\right\rangle_{\mathcal{D}}\right] = \frac{1}{3a}H_{\mathbf{i}}^2 \int_{\mathbb{R}^3} \mathrm{d}^3k P_{\mathbf{i}}(k)$$

- that cancels out in the complete expression and reduces $\mathcal{Q}_{\mathcal{D}}$ to the domain dependent contribution

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Conclusion

- RZA in coframe formulation allows a direct comparison with Newtonian case
- Perturbative results the same as in Newtonian cosmology
- GR effects?

- Directions to explore
  - Higher orders also Newtonian?
  - Averages on the lightcone? Effects?

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Backup slides

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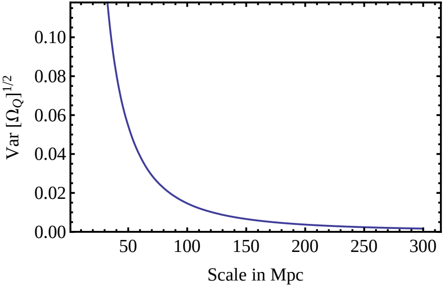

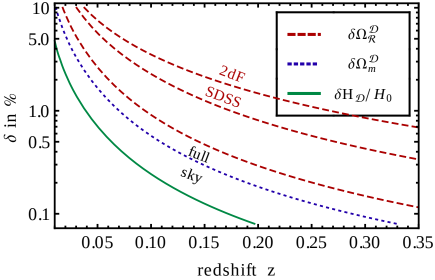

- Fluctuations of $\mathcal{Q}_{\mathcal{D}}$ only interesting below 100 Mpc
- Other parameters fluctuate more strongly for even bigger domains

Evolution of the fluctuations

- Equation for the time evolution of the perturbations

$$\ddot{\xi}(t) + 2\frac{\dot{a}(t)}{a(t)}\dot{\xi}(t) + \left(3\frac{\ddot{a}(t)}{a(t)} - \Lambda\right)(\xi(t)+1) = 0$$

- Explicit form of the invariants

$$\begin{aligned}
\mathrm{I}\left(\dot{\mathscr{P}}^a{}_i\right) &:= \frac{1}{2}\epsilon_{abc}\epsilon^{ijk}\,\dot{\mathscr{P}}^a{}_i\delta^b{}_j\delta^c{}_k \\
\mathrm{II}\left(\dot{\mathscr{P}}^a{}_i\right) &:= \frac{1}{2}\epsilon_{abc}\epsilon^{ijk}\,\dot{\mathscr{P}}^a{}_i\dot{\mathscr{P}}^b{}_j\delta^c{}_k \\
\mathrm{III}\left(\dot{\mathscr{P}}^a{}_i\right) &:= \frac{1}{6}\epsilon_{abc}\epsilon^{ijk}\,\dot{\mathscr{P}}^a{}_i\dot{\mathscr{P}}^b{}_j\dot{\mathscr{P}}^c{}_k
\end{aligned}$$

Newtonian limit

- Sending the coframes to integrable ones

$$\eta^a_i \to {}^N\eta^a_i = f^a_{|i}$$

- leads to the Newtonian equivalent of the perturbation one forms which are second derivatives of the Newtonian potential

$${}^N\dot{P}^a_i = \psi^{|a}_{|i}$$

- and therefore the invariants become

$$\mathbf{I_i} := \mathbf{I}(\psi^{|i}_{|j}), \ \mathbf{II_i} := \mathbf{II}(\psi^{|i}_{|j}), \ \mathbf{III_i} := \mathbf{III}(\psi^{|i}_{|j})$$

Newtonian backreaction

- Backreaction in terms of kinematical variables

$$Q_{\mathcal{D}} = \tfrac{2}{3}\left(\left\langle\theta^2\right\rangle_{\mathcal{D}} - \langle\theta\rangle_{\mathcal{D}}^2\right) + 2\left\langle\omega^2 - \sigma^2\right\rangle_{\mathcal{D}}$$

- becomes dependent on derivatives of the peculiar velocity field

$$Q_{\mathcal{D}} = 2\,\langle\mathbf{II}(v_{i,j})\rangle_{\mathcal{D}} - \tfrac{2}{3}\,\langle\mathbf{I}(v_{i,j})\rangle_{\mathcal{D}}^2$$

- which means that it is a surface term

$$Q_{\mathcal{D}} = \frac{1}{a^2}\left[2\,\frac{1}{V_q}\int_{\partial\mathcal{D}_q} \mathrm{d}\mathbf{S}\cdot(\mathbf{u}(\nabla_q\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla_q)\mathbf{u}) - \frac{2}{3}\,\left(\frac{1}{V_q}\int_{\partial\mathcal{D}_q}\mathrm{d}\mathbf{S}\cdot\mathbf{u}\right)^2\right]$$

Breakdown of perturbation theory

- General form of corrections to the average expansion rate

$$\langle\theta\rangle = 3H_\tau \left(1 + \frac{1}{(aH)^2} \langle\partial_i\Phi\partial_i\Phi\rangle_0 \sum_{n=0}^{\infty} \lambda_n \left\langle\delta^2\right\rangle_0^n + \dots\right)$$

- If perturbations become of order unity on any scale the sum $\sum_{n=0}^{\infty}\lambda_n$ plays the important role.
- May be large in LTB models even if Φ is small and one would expect the perturbed FRW metric to apply

References

- Our value for $\mathrm{Var}(\delta_H)^{1/2}$ in the rhs figure of slide 10 has been multiplied by a factor of $5/2$ to match the wrong results of Li and Schwarz and Umeh et al. However, a correct calculation shows the consistency of the three results, but on a lower level. The qualitative picture however is the same. The upgraded Li and Schwarz result may be found in: Wiegand and Schwarz *in preparation*.

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