Effects of the modification of gravity on the production of primordial black holes

Antonio Enea Romano
UDEA/CERN

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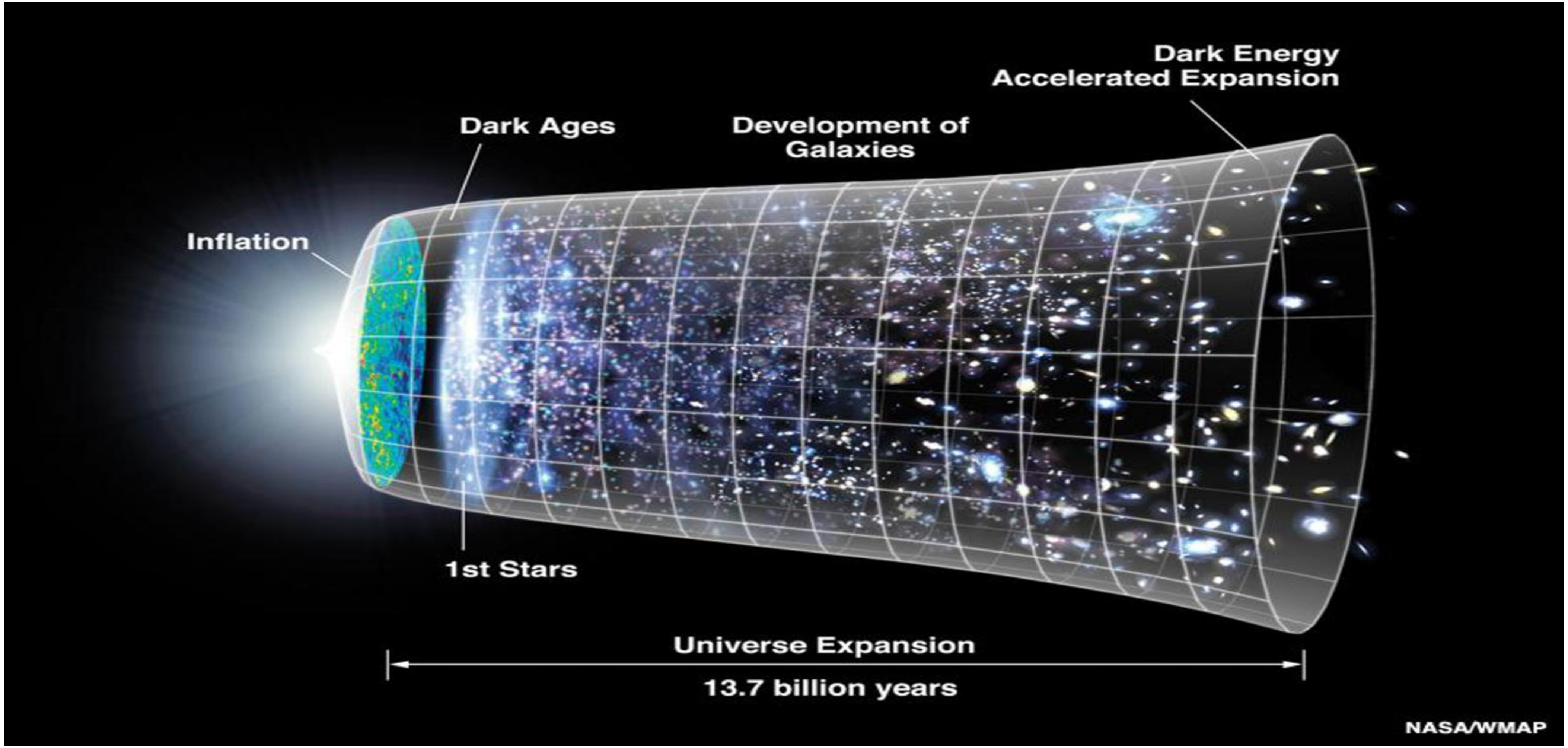
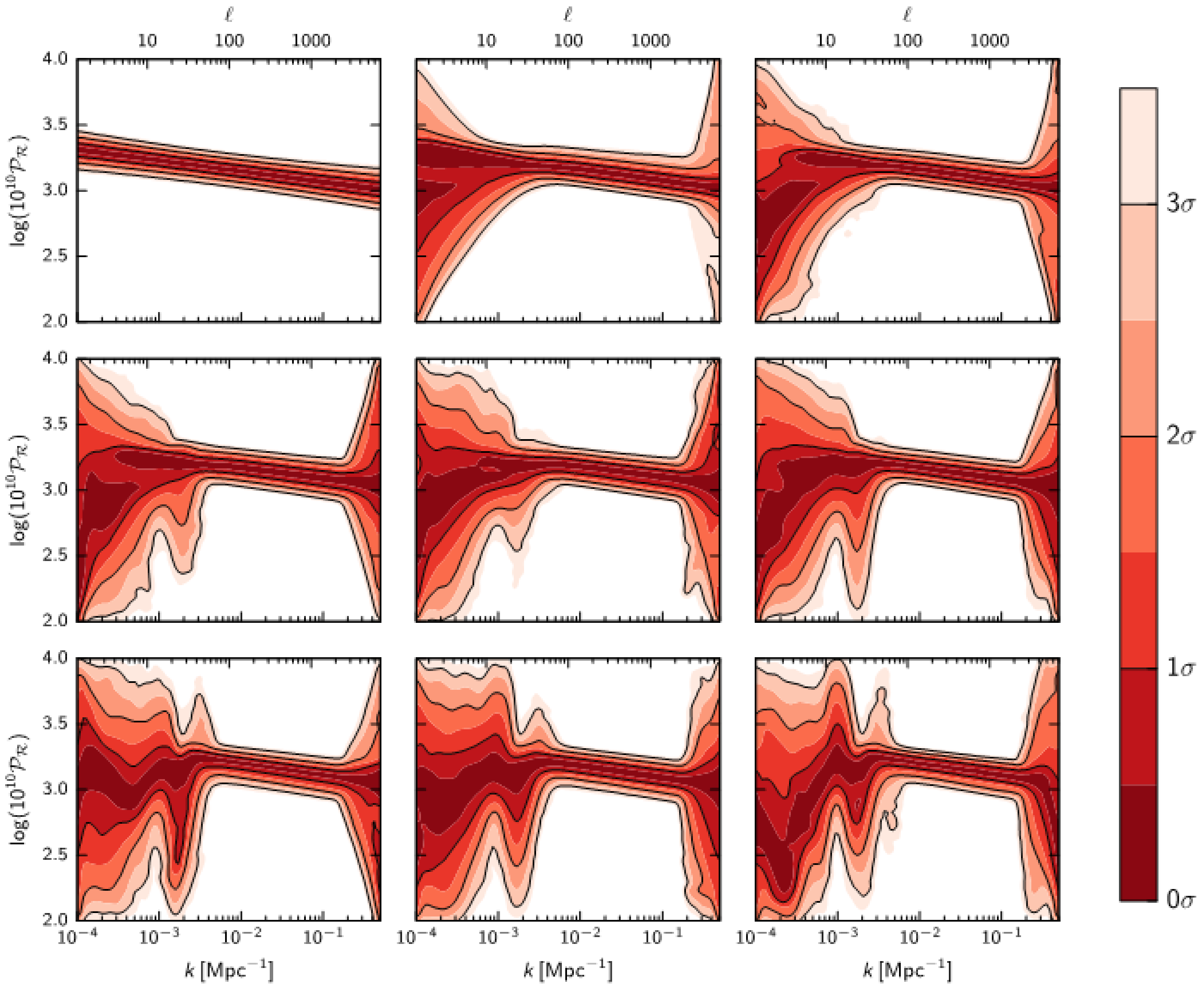
- Cosmological perturbations are fundamental to establish any predictive model of the Universe
- They provide the seeds for cosmic background radiation anisotropies and for large scale structure formation



The CMB leaves room from deviations from a power law spectrum,

Planck 2015 results. XX. Constraints on inflation



Enhancement mechanisms of the primordial curvature spectrum

Features in the spectrum of primordial curvature perturbations could also produce **primordial black holes (PBH)**

There are scenarios which can enhance the spectrum:

- Multi-fields
- Slow-roll violation in single field
- Modification of gravity
- A combination of the above

In presence of anisotropy the equation using SESS takes the form

$$\ddot{\mathcal{R}} + \frac{\partial_t(Z^2)}{Z^2}\dot{\mathcal{R}} - \frac{v_s^2}{a^2}\overset{(3)}{\Delta}\mathcal{R} + \frac{v_s^2}{\epsilon}\overset{(3)}{\Delta}\Pi + \frac{1}{3Z^2}\partial_t\left(\frac{Z^2}{H\epsilon}\overset{(3)}{\Delta}\Pi\right) = 0\,, \quad Z^2 \equiv \frac{\epsilon a^3}{v_s^2}$$

while using the standard approach we obtained

$$\ddot{\mathcal{R}} + \frac{\partial_t(z^2)}{z^2}\dot{\mathcal{R}} - \frac{c_s^2}{a^2}\overset{(3)}{\Delta}\mathcal{R} + \frac{c_s^2}{\epsilon}\overset{(3)}{\Delta}\Pi + \frac{1}{2z^2}\partial_t\left[\frac{a^3}{c_s^2 H}\left(\Gamma + \frac{2}{3}\overset{(3)}{\Delta}\Pi\right)\right] = 0\,,$$

$$z^2 \equiv \frac{a^3\epsilon}{c_s^2}$$

The equations we have obtained are *completely general*.

They can be applied to modified gravity theories, multi-fields systems, or any combination of these models!

The momentum dependent effective sound speed (MESS) is defined as

$$\tilde{v}_k^2(t) \equiv \frac{\delta \tilde{P}_c(t)}{\delta \tilde{\rho}_c(t)}$$

where $\delta \tilde{P}_c$ and $\delta \tilde{\rho}_c$ are the Fourier transforms of the comoving pressure and energy density respectively.

In absence of anisotropy, using MESS and manipulating the Einstein's equations in Fourier space we get

$$\ddot{\mathcal{R}}_k + \frac{\partial_t(\tilde{Z}_k^2)}{\tilde{Z}_k^2}\dot{\mathcal{R}}_k + \frac{\tilde{v}_k^2}{a^2}k^2\mathcal{R}_k = 0\,, \qquad \tilde{Z}_k^2 \equiv \frac{\epsilon a^3}{\tilde{v}_k^2}$$

while in the standard approach there is a source term

$$\ddot{\mathcal{R}}_k + \frac{\partial_t(z^2)}{z^2}\dot{\mathcal{R}}_k + \frac{c_s^2}{a^2}k^2\mathcal{R}_k + \frac{1}{2z^2}\partial_t\left[\frac{a^3}{c_s^2 H}\Gamma_k\right] = 0\,, \quad z^2 \equiv \frac{a^3\epsilon}{c_s^2}$$

Important note:
The MESS $\tilde{v}_k$ is ***not*** simply the Fourier transform of the SESS v_s!

In presence of anisotropy, using MESS and manipulating the Einstein's equations in Fourier space we get

$$\ddot{\mathcal{R}}_k + \frac{\partial_t(\tilde{Z}_k^2)}{\tilde{Z}_k^2}\dot{\mathcal{R}}_k + \frac{\tilde{v}_k^2}{a^2}k^2\mathcal{R}_k - \frac{\tilde{v}_k^2}{\epsilon}k^2\Pi_k - \frac{1}{3\tilde{Z}_k^2}\partial_t\left(\frac{\tilde{Z}_k^2}{H\epsilon}k^2\Pi_k\right) = 0\,, \quad \tilde{Z}_k^2 \equiv \frac{\epsilon a^3}{\tilde{v}_k^2}$$

The equation with MESS is model independent.

MESS can be treated as an effective quantity in data analysis, without assuming any model.

Using the standard approach we get

$$\ddot{\mathcal{R}}_k + \frac{\partial_t(z^2)}{z^2}\dot{\mathcal{R}}_k + \frac{c_s^2}{a^2}k^2\mathcal{R}_k - \frac{c_s^2}{\epsilon}k^2\Pi_k + \frac{1}{2z^2}\partial_t\left[\frac{a^3}{c_s^2 H}\left(\Gamma_k - \frac{2}{3}k^2\Pi_k\right)\right] = 0\,,$$

$$z^2 \equiv \frac{a^3\epsilon}{c_s^2}$$

The equations we have obtained are *completely general*.

They can be applied to modified gravity theories, multi-fields systems, or any combination of these models!

Some notation for scalar perturbations : No gauge fixing

$$ds^2 = -(1+2A)dt^2 + 2a\partial_i B dx^i dt +$$
$$+ a^2 \{\delta_{ij}(1+2C) + 2\partial_i\partial_j E\} dx^i dx^j \; ,$$

TOTAL energy momentum tensor
Includes any matter, multi-fields,
Vector, scalar fields,
Modified gravity

$$T^0{}_0 = -(\rho + \delta\rho) \quad , \quad T^0{}_i = (\rho + P)\partial_i(v + B)$$

$$T^i{}_j = (P + \delta P)\delta^i{}_j + \delta^{ik}\partial_k\partial_j \Pi - \frac{1}{3}\delta^i{}_j \overset{(3)}{\Delta} \Pi \, .$$

Comoving slices gauge : $(T^0{}_i)_c = 0 \longrightarrow \alpha = \delta P_c, \beta = \delta\rho_c, \gamma = A_c, \mu = B_c, \zeta = C_c, \nu = E_c$

$$ds^2 = -(1+2\gamma)dt^2 + 2a\partial_i \mu \, dx^i dt +$$
$$+ a^2 \{\delta_{ij}(1+2\zeta) + 2\partial_i\partial_j \nu\} dx^i dx^j$$

$$(T^0{}_0)_c = -(\rho + \beta) \quad , \quad (T^i{}_j)_c = (P + \alpha)\delta^i{}_j$$

Standard definitions of entropy in the comoving gauge and uniform density gauge

$$\delta P_u = c_w(t)^2 \delta\rho + \delta P^{nad}$$

$c_w^2 = P'/\rho'$ Adiabatic sound speed

$$\delta P_c = c_s(t)^2 \delta\rho_c + \delta P_c^{nad}$$

Comoving curvature pertubation sound speed

$$\alpha(t, x^i) = c_s(t)^2 \beta(t, x^i) + \Gamma(t, x^i)$$

But ….the one in the comoving gauge it is not unique !

$$c_s^2 \to \tilde{c}_s(t)^2 = c_s(t)^2 + \Delta c_s(t)^2 ,$$

$$\Gamma \to \tilde{\Gamma}(t, x^i) = \Gamma(t, x^i) - \Delta c_s(t)^2 \beta(t, x^i)$$

Comparing it to the SESS we can get the relation between them

$$v_s^2(t, x^i) \equiv \frac{\alpha(t, x^i)}{\beta(t, x^i)}$$

$$\alpha(t, x^i) = c_s(t)^2 \beta(t, x^i) + \Gamma(t, x^i)$$

Relation of SESS with entropy and anisotropy

$$v_s^2 = c_s^2 \left[1 + \frac{\Gamma}{2H\epsilon \left(\dot{\zeta} + \frac{1}{3H\epsilon} \overset{(3)}{\Delta} \Pi \right)} \right]^{-1}$$

The SESS encodes the effects of entropy and anisotropy, but for anisotropy there is also an extra source term in the equations of motions.

Most general equation for any system including anisotropy and entropy effects

SESS

$$\dot{\zeta} = -\frac{v_s^2}{a^2 H\epsilon} \overset{(3)}{\Delta} \Psi_B - \frac{1}{3H\epsilon} \overset{(3)}{\Delta} \Pi$$

Without SESS

$$\dot{\zeta} = -\frac{c_s^2}{a^2 H\epsilon} \overset{(3)}{\Delta} \Phi_B - \frac{\Gamma}{2H\epsilon} - \frac{1}{3H\epsilon} \overset{(3)}{\Delta} \Pi$$

$$\ddot{\zeta} + \frac{\partial_t(Z^2)}{Z^2}\dot{\zeta} - \frac{v_s^2}{a^2} \overset{(3)}{\Delta} \zeta + \frac{v_s^2}{\epsilon} \overset{(3)}{\Delta} \Pi + \frac{1}{3Z^2}\partial_t\left(\frac{Z^2}{H\epsilon} \overset{(3)}{\Delta} \Pi\right) = 0$$

$$\ddot{\zeta} + \frac{\partial_t z^2}{z^2}\dot{\zeta} - \frac{c_s^2}{a^2} \overset{(3)}{\Delta} \zeta + \frac{c_s^2}{\epsilon} \overset{(3)}{\Delta} \Pi + \frac{1}{z^2}\partial_t\left[\frac{a^3}{c_s^2 H}\left(\Gamma + \frac{2}{3} \overset{(3)}{\Delta} \Pi\right)\right] = 0$$

The first and second order equations are obtained using the following important relations, obtained from Manipulating the Einstein's equations in the comoving gauge. The Poisson eq. Is more used in the modified gravity theories literature

$$\frac{1}{a^2} \overset{(3)}{\Delta} \Psi_B = \frac{1}{2}\beta \qquad \zeta = -\Psi_B + \frac{H^2}{\dot{H}}\left(\Phi_B + H^{-1}\dot{\Psi}_B\right) \qquad \dot{\zeta} = -\frac{1}{2H\epsilon}\left(\alpha + \frac{2}{3} \overset{(3)}{\Delta} \Pi\right)$$

G-inflation

T. Kobayashi, M. Yamaguchi, and J. Yokoyama
Phys. Rev. Lett. **105**, 231302 (2010).

$$L(\phi, \chi) = K(\phi, \chi) + G(\phi, \chi)\Box\phi$$

$$\delta T^0{}_i = -\left(K_\chi + 2G_\phi - \frac{3\mathcal{H}\phi'}{a^2}G_\chi\right)\frac{\phi'^2}{a^2}\partial_i\delta\phi + \\ - \frac{\phi'^2}{a^4}G_\chi\partial_i\left(\delta\phi' - \phi' A\right)\,,$$

The unitary and comoving gauge do NOT coincide

Gauge transformations

$$\delta\phi \to \delta\phi - \phi'\delta\tau\,,$$
$$A \to A - \mathcal{H}\delta\tau - \delta\tau'\,,$$
$$B \to B + \delta\tau - \delta x'\,,$$
$$C \to C + \mathcal{H}\delta\tau\,,$$
$$E \to E - \delta x\,.$$

Unitary gauge

T. Kobayashi, M. Yamaguchi, and J. Yokoyama, Prog. Theor. Phys. **126**, 511 (2011), arXiv:1105.5723.

$$\delta\tau_u = \frac{\delta\phi}{\phi'}$$

$$S_\zeta^{(2)} = \int dt d^3 x a^3 \left[\mathcal{G}_S \dot{\zeta}^2 - \frac{\mathcal{F}_S}{a^2} (\partial_i \zeta)^2\right]$$

$$\zeta_k'' + \left(2\mathcal{H} + \frac{\mathcal{G}_S'}{\mathcal{G}_S}\right)\zeta_k' + c_s^2 k^2 \zeta_k = 0$$

Comoving gauge

$$\delta T^0{}_i \to \delta T^0{}_i + \partial_i \left(\frac{\phi'^2}{a^4} D \delta\tau \right)$$

$$\delta\tau_c = \frac{1}{\phi' D} \Big[-\phi' G_\chi (3\mathcal{H}\delta\phi + \phi' A - \delta\phi') + a^2 (2G_\phi + K_\chi)\delta\phi \Big] .$$

$$D = a^2(2G_\phi + K_\chi) + G_\chi(-4\mathcal{H}\phi' + \phi'')$$

$$G(\phi,\chi) = G(\phi) \quad \rightarrow \quad \delta\tau_c = \frac{\delta\phi}{\phi'}$$

$$\mathcal{R}_k'' + \alpha_k(\tau)\mathcal{R}_k' + \beta_k(\tau)k^2\mathcal{R}_k = 0$$

As predicted by the MESS formalism the sound speed is momentum dependent

Relation between the curvature in the unitary and comoving gauge

$$\mathcal{R} \equiv -C_c = -C - \mathcal{H}\delta\tau_c$$

$$\delta\tau_{uc} = -\frac{\phi' G_\chi}{D} A_u \qquad \mathcal{R} = \zeta + \mathcal{H}\frac{\phi' G_\chi}{D} A_u$$

$$\mathcal{R} = \zeta + \mathcal{H}\frac{\phi' G_\chi}{D}\left(\frac{\phi'^3 G_\chi}{2M_{Pl}^2 a^2} + \mathcal{H}\right)^{-1}\zeta'$$

$$= \zeta + \mathcal{E}(\tau)\zeta' .$$

Enhancement function

Spectrum enhancement

$$P_{\mathcal{R}} = \frac{k^3}{2\pi^2}|\mathcal{R}_k|^2 = P_\zeta + \frac{k^3}{2\pi^2}\Delta$$

$$\Delta = \left[\mathcal{E}\zeta^*\zeta' + \mathcal{E}^*\zeta'^*(\zeta + \mathcal{E}\zeta')\right]$$

Effects on PBH formation

$$\zeta = const \Rightarrow \zeta = \mathcal{R} = const \qquad \mathcal{R} = const \not\Rightarrow \zeta = const$$

$$\beta \equiv \left.\frac{\overline{\rho}_{PBH}}{\overline{\rho}}\right|_F \qquad \beta(M) = \gamma \int_{\delta_t}^{1} P(\delta)\mathrm{d}\delta \qquad \beta(M) \approx \frac{\gamma}{\sqrt{2\pi}\nu(M)} \exp\left[-\frac{\nu(M)^2}{2}\right]$$

$$\nu(M) \equiv \delta_t/\sigma(M)$$

$$\begin{aligned}\sigma^2(M) &= \int \mathrm{d}\ln k W^2(kR)\mathcal{P}_\delta(k) \\ &= \int \mathrm{d}\ln k W^2(kR)\left(\frac{16}{81}\right)(kR)^4\mathcal{P}_\mathcal{R}(k)\end{aligned}$$

$$R(M) = \left.(a^2\mathcal{H})^{-1}\right|_F = 2GM/a_F\gamma^{-1}$$

Conclusions

- The unitary and commoving gauge do not coincide in Horndesky's theory, as verified in a specific case
- The time evolution of curvature perturbations in the two gauges can differ before they freeze
- This can be important for small scale perturbations which re-enter the horizon before freezing, specifically for models which could produce PBH by modifying the curvature spectrum on small scales not probed by CMB
- MGT effects on PBH formation ?

Effects of a local momentum variation of the MESS:

$$\tilde{v}_k = 1 + A_c \exp\left[-\left(\frac{k-k_0}{\sigma}\right)^2\right]$$

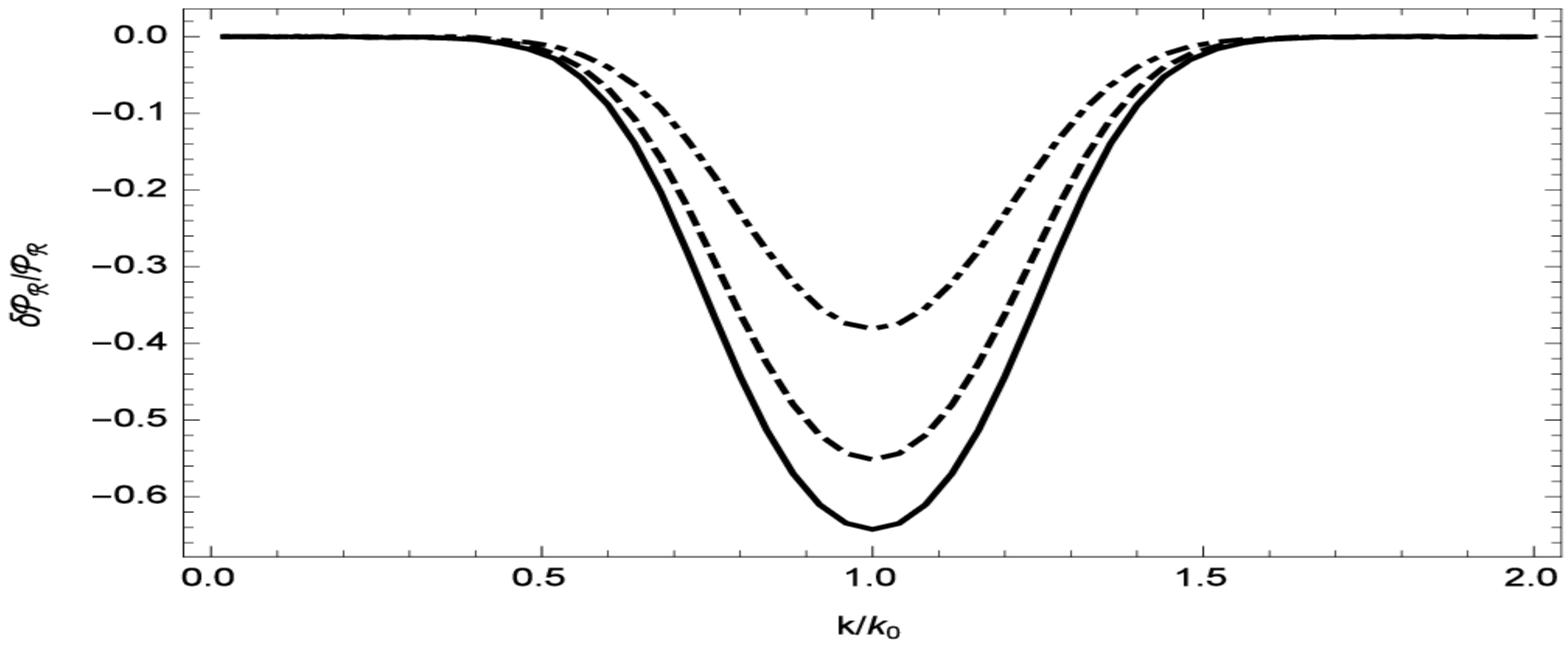


FIG. 1: The relative difference $\Delta\mathcal{P}_\zeta/\mathcal{P}_\zeta$ is plotted as a function of k/k_0. The solid, dashed and dot-dashed lines correspond $\sigma = 2.5\times10^{-1}k_0$ and $A_c = 4\times10^{-1}$, $A_c = 3\times10^{-1}$ and $A_c = 1.7\times10^{-1}$ respectively.

The scale k0 could have different origins: turning point in multi-fields modes, particle production, modification of gravity, etc.

Einstein's equations in the comoving gauge and derivations of EOM in terms of MESS

$$\frac{1}{a^2}\overset{(3)}{\Delta}\left[-\zeta + aH\sigma\right] = \frac{\beta}{2}, \qquad (11)$$

$$\gamma = \frac{\dot{\zeta}}{H}, \qquad (12)$$

$$-\ddot{\zeta} - 3H\dot{\zeta} + H\dot{\gamma} + (2\dot{H} + 3H^2)\gamma = \frac{\alpha}{2}, \qquad (13)$$

$$\dot{\sigma} + 2H\sigma - \frac{\gamma + \zeta}{a} = 0, \qquad (14)$$

12 ,13 >

$$\dot{\zeta} = -\frac{1}{2H\epsilon}\alpha, \qquad (15)$$

$$\sigma = \frac{\Phi_B + \zeta}{aH} \quad , \quad \gamma = \Phi_B + \partial_t(a\sigma). \qquad (16)$$

16,11 ->

$$\frac{1}{a^2}\overset{(3)}{\Delta}\Phi_B = \frac{1}{2}\beta. \qquad (17)$$

$$v_s^2(t, x^i) \equiv \frac{\alpha(t, x^i)}{\beta(t, x^i)}, \qquad (18)$$

15,17,18 ->

$$\dot{\zeta} = -\frac{v_s^2}{a^2 H\epsilon}\overset{(3)}{\Delta}\Phi_B. \qquad (19)$$

16,12->

$$\zeta = -\Phi_B + \frac{H^2}{\dot{H}}\left(\Phi_B + H^{-1}\dot{\Phi}_B\right) = \frac{H^2}{a\dot{H}}\partial_t\left(\frac{a\Phi_B}{H}\right) \qquad (20)$$

20,d/dt 19->

$$\partial_t\left(\frac{a^3\epsilon}{v_s^2}\dot{\zeta}\right) - a\epsilon\overset{(3)}{\Delta}\zeta = 0$$

The difference between the uniform density field and the comoving gauge

The uniform density field (aka “unitary”) is in general different from the comoving gauge
They coincide for K(X) – inflation, but not for Horndesky theory or multi-fields systems

$$v + B \to v + B - \delta t \quad \longrightarrow \quad \delta t_c = v + B$$

We can now define explicitly gauge invariant quantities:

comoving pressure perturbation α
comoving density perturbation β
comoving curvature perturbation ζ

$$\begin{aligned}
\alpha &= \delta P + \dot{P}\delta t_c \quad , \quad \beta = \delta\rho + \dot{\rho}\delta t_c \, , \\
\gamma &= A + \dot{\delta t}_c \quad , \quad \mu = B - a^{-1}\delta t_c \, , \\
\sigma &= a\dot{E} - B + a^{-1}\delta t_c = a\dot{\nu} - \mu \, , \\
\zeta &= C - H\delta t_c \, .
\end{aligned}$$

Einstein's equations in the comoving

$$\frac{1}{a^2} \overset{(3)}{\Delta} \left[-\zeta + aH\sigma\right] = \frac{\beta}{2} \, ,$$

$$\gamma = \frac{\dot{\zeta}}{H} \, ,$$

$$-\ddot{\zeta} - 3H\dot{\zeta} + H\dot{\gamma} + (2\dot{H} + 3H^2)\gamma = \frac{\alpha}{2} \, ,$$

$$\dot{\sigma} + 2H\sigma - \frac{\gamma + \zeta}{a} = 0 \, ,$$

How general is this equation?

- SESS reduces to the standard definition of sound speed for single field K(X) theories
- It is a space dependent quantity which effectively reproduces the effects of the source terms
- in the EOM which in the standard formulation are associated to entropy perturbations
- Given the generality of the assumptions this formulation is valid for any system for which
- an energy momentum tensor can be defined, including multi-fields systems or modified
- gravity theories (MGT)
- It is also valid for MGT, after writing the MGT field equations as Einstein's equations with
- an appropriate definition of an effective energy momentum tensor

How useful SESS and MESS are?

- These equations are completely general and can be applied to any physical system for example:
- Multi-fields , scalar or vector fields (scalar part)
- Modified gravity, e.g. Horndesky theory, in terms of an effective EM tensor : $G_{\mu\nu}=T_{\mu\nu}^{eff}$
- Non-Gaussianity can be studied in terms of MESS and SESS
- The anisotropy stress term is easy to add and does not modify the definition of MESS and SESS
- Mess is not simply the Fourier transform of SESS !

After substituting SESS we get the "standard " source term

$$\dot{\zeta} = \frac{H}{a^2 \dot{H}} \overset{(3)}{\Delta} \Phi_B - \frac{1}{2} H \Theta \,,$$

$$\ddot{\zeta} + \frac{\partial_t (z^2)}{z^2} \dot{\zeta} - \frac{1}{a^2} \overset{(3)}{\Delta} \zeta + \frac{1}{z^2} \partial_t \left(\frac{z^2 H \Theta}{2} \right) = 0$$

Generalization to multi-fields

$$\theta_{ij} = \left(\frac{\delta\phi_i}{\dot{\phi}_i} - \frac{\delta\phi_j}{\dot{\phi}_j} \right) \frac{\partial}{\partial t} \left(\frac{\dot{\phi}_i^{\,2} - \dot{\phi}_j^{\,2}}{\sum_i^n \dot{\phi}_i^{\,2}} \right) \,, \; \Theta = \chi_N \sum_{i>j}^{N} \theta_{ij}$$

Single field KGB : intrinsic entropy

$$L_{KG}(\Phi, X) = K(\Phi, X) + G(\Phi, X) \Box \Phi$$

$$\alpha = c_s^2(t)\beta + \Gamma^{int}$$

$$v_{KG}^2 = c_s^2 \left(1 + \frac{\Gamma^{int}}{2\epsilon H \dot{\zeta}} \right)^{-1}$$

Multi-field Horndeski : nKGB

$$\Gamma_{NKG} = \sum_i^N \Gamma_i^{int} + \chi_N \sum_{i>j}^N \Gamma_{ij}$$

$$v_{NKG}^2 = c_s^2 \left(1 + \frac{\Gamma_{NKG}}{2\epsilon H \dot{\zeta}} \right)^{-1}$$