Adiabatic Renormalization with an IR cut off

Pietro Conzinu

C. Animali, P.C, G. Marozzi, JCAP 05 (2022) 05, 026

Università di Pisa

Istituto Nazionale di Fisica Nucleare

Outline

- Introduction
- Motivation: a problematic example
- How to fix univocally the renormalization scheme: conformal anomaly
- Conclusions

- Introduction

- Motivation: a problematic example

- How to fix univocally the renormalization scheme: conformal anomaly

- Conclusions

Introduction to the adiabatic Renormalization

- **UV divergences:** As in the case of flat space time, observables are characterized by divergences in the deep UV.
- **new divergences:** The presence of gravity led to new divergences that are not matched by the Minkowski ones.

$$\langle T_{\mu\nu}\rangle \equiv \langle \text{out,0}|\, T_{\mu\nu}\, |\text{in,0}\rangle$$

In flat-space time

$$\langle T_{\mu\nu}\rangle = \langle T_{\mu\nu}\rangle - \langle 0_{\text{MINK}}|\, T_{\mu\nu}\, |0_{\text{MINK}}\rangle$$

In curved space it is again divergent in UV.

- **Vacuum choice:** There is not a preferred choice of the vacuum.

Let us consider the minimal energy vacuum at time t_0 defined by

$$\phi(x) = \sum_k \{\mathbf{A}_k f_k(x) + \mathbf{A}_k^\dagger f_k^*(x)\}\,, \qquad \mathbf{A}_k\,|0\rangle = 0$$

At $t > t_0$ the presence of a non trivial (gravity) background mixes positives and negatives modes.

$$g(x) = \alpha_k f(x) + \beta_k f^*(x)\;, \qquad |\alpha_k^2 - \beta_k^2| = 1$$

the field can be represented as a new combination of mode functions such that:

$$\phi(x) = \sum_k \{\mathbf{B}_x g_k(x) + \mathbf{B}_k^\dagger g_k^*(x)\}, \qquad \mathbf{A}_k = \alpha_k \mathbf{B}_k + \beta_k {\mathbf{B}^\dagger}_{-k}$$

the "vacuum"-state is not anymore empty

$$\langle 0|\,\mathbf{A}_k^\dagger \mathbf{A}_k\,|0\rangle \neq 0 = N_k = |\beta_k|^2$$

Physical request:
particles should not be created in the limit when the energy of a single particle is larger w.r.t the energy scale of the spacetime.
For our cosmological model:

$$\frac{k^2}{a^2(t)} + m^2 > \left(\frac{\dot{a}}{a}\right)^2, \ \frac{\ddot{a}}{a} \qquad \Rightarrow \quad N_k \sim \text{const.}$$

the particle content should not change if the change rate of $a(t)$ is adiabatic.

- **Adiabatic vacuum:**
  the vacuum that minimizes the creation of particle due to the presence of a time-dependent metric.

Adiabatic renormalization prescription:

- evaluate expectation values w.r.t. the adiabatic vacuum
- mode functions are given in terms of WKB ansatz.
- expand up to the adiabatic order that matches the energy dimensions of the operator.
- subtract the adiabatic term from the bare quantity.

For example the expectation value of $T_{\mu\nu}$

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = \langle T_{\mu\nu}\rangle_{\text{bare}} - \langle T_{\mu\nu}\rangle_{\text{ad}} \,,$$

For this particular case of the energy-momentum tensor one should consider the adiabatic expansion up to fourth order.

- Introduction
- Motivation: a problematic example
- How to fix univocally the renormalization scheme: conformal anomaly
- Conclusions

Axion-gauge fields

Let us consider a pseudo-scalar inflaton field ϕ coupled to $U(1)$ gauge field A_μ.

The Lagrangian of the model is given by

$$\mathcal{L} = -\frac{1}{2}(\nabla\phi)^2 - V(\phi) - \frac{1}{4}(F^{\mu\nu})^2 - \frac{g\phi}{4}F^{\mu\nu}\tilde{F}_{\mu\nu}\,,$$

where $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}/2 = \epsilon^{\mu\nu\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)/2$.

The background is fixed by the homogeneous inflaton field $\phi(t)$

[**Ballardini et al '19**]

- Due to the coupling with the inflaton field ϕ, quantum fluctuations of the gauge field A_μ are amplified.

- backreaction of gauge field

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = g \langle \mathbf{E} \cdot \mathbf{B} \rangle$$

$$H^2 = \frac{1}{3M_p^2}\left[\frac{\dot{\phi}^2}{2} + V(\phi) + \frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2}\right]$$

$$\dot{H} = -\frac{1}{2M_p^2}\left[\dot{\phi}^2 + \frac{2}{3}\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle\right]$$

energy density

$$\frac{\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle}{2} = \int \frac{\mathrm{d}k}{(2\pi)^2 a(\tau)^4} k^2 \left[|A'_+|^2 + |A'_-|^2 + k^2 \left(|A_+|^2 + |A_-|^2\right)\right] .$$

helicity integral

$$\langle \mathbf{E} \cdot \mathbf{B} \rangle = -\int \frac{\mathrm{d}k}{(2\pi)^2 a(\tau)^4} k^3 \frac{\partial}{\partial \tau} \left(|A_+|^2 - |A_-|^2\right) ,$$

We assume de Sitter expansion for the background:

$$a(\tau) = -1/(H\tau), \quad \tau < 0, \quad H = \text{const.}, \quad \dot{\phi} = \text{const.}$$

The Fourier mode functions $A_\pm$ of the gauge fields satisfy the EOM

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} A_\pm(\tau, k) + \left(k^2 \mp k g \phi'\right) A_\pm(\tau, k) = 0$$

with analytical solution

$$\boxed{A_\pm(\tau, k) = \frac{1}{\sqrt{2k}} e^{\pm\pi\xi/2} W_{\pm i\xi, \frac{1}{2}}(-2ik\tau)}$$

where W Whittaker function and

$$\xi \equiv g\phi'/(2a(\tau)H) = g\dot{\phi}/(2H)$$

$$\frac{1}{2}\left\langle \mathbf{E}^2+\mathbf{B}^2\right\rangle_{\mathsf{bare}} = \frac{\Lambda^4}{8\pi^2}+\frac{H^2\Lambda^2\xi^2}{8\pi^2}+\frac{3H^4\xi^2(5\xi^2-1)\log\left(2\Lambda/H\right)}{16\pi^2} + \frac{H^4\xi^2(-79\xi^4+22\xi^2+29)}{64\pi^2(1+\xi^2)}+\frac{H^4\xi(30\xi^2-11)\sinh\left(2\pi\xi\right)}{64\pi^3} + \frac{3iH^4\xi^2(5\xi^2-1)(\psi^{(1)}(1-i\xi)-\psi^{(1)}(1+i\xi))\sinh\left(2\pi\xi\right)}{64\pi^3} - \frac{3H^4\xi^2(5\xi^2-1)(\psi(-1-i\xi)+\psi(-1+i\xi))}{32\pi^2},$$

$$\left\langle \mathbf{E}\cdot\mathbf{B}\right\rangle_{\mathsf{bare}} = -\frac{H^2\Lambda^2\xi}{8\pi^2}-\frac{3H^4\xi\left(5\xi^2-1\right)\log\left(2\Lambda/H\right)}{8\pi^2} + \frac{H^4\xi(47\xi^2-22)}{16\pi^2}-\frac{H^4(30\xi^2-11)\sinh\left(2\pi\xi\right)}{32\pi^3} - \frac{3iH^4\xi\left(5\xi^2-1\right)\left(\psi^{(1)}(1-i\xi)-\psi^{(1)}(1+i\xi)\right)\sinh(2\pi\xi)}{32\pi^3} + \frac{3H^4\xi\left(5\xi^2-1\right)\left(\psi(1-i\xi)+\psi(1+i\xi)\right)}{16\pi^2},$$

- $\Lambda^4,\ \Lambda^2$ and $\log[\Lambda]$ UV divergences for the energy density.
- Λ^2 and $\log[\Lambda]$ UV divergences for the helicity integral.
- well-behaved in the infrared, not exhibiting IR divergences.

Renormalization

$$\frac{1}{2}\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle_{\text{ren}} = \frac{1}{2}\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle_{\text{bare}} - \frac{1}{2}\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle_{\text{ad}}$$

$$\langle \mathbf{E} \cdot \mathbf{B} \rangle_{\text{ren}} = \langle \mathbf{E} \cdot \mathbf{B} \rangle_{\text{bare}} - \langle \mathbf{E} \cdot \mathbf{B} \rangle_{\text{ad}}$$

The adiabatic mode function of gauge fields for each polarization $\lambda = \pm$ is given by the WKB approximation

$$A_\lambda^{\mathsf{WKB}}(k,\tau) = \frac{1}{\sqrt{2\Omega_\lambda(k,\tau)}} e^{-i\int \Omega_\lambda(k,\tau')\mathrm{d}\tau'} .$$

Inserting into the equation of motion, where a mass regulator m is added to the equation of motion

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} A_\pm^{\mathsf{WKB}}(\tau,k) + \left(k^2 \mp gk\phi' + \frac{m^2}{H^2\tau^2}\right) A_\pm^{\mathsf{WKB}}(\tau,k) = 0$$

we obtain the exact equation for the WKB frequency

$$\Omega_\lambda^2(k,\tau) = \bar{\Omega}_\lambda^2(k,\tau) + \frac{3}{4}\left(\frac{\Omega_\lambda'(k,\tau)}{\Omega_\lambda(k,\tau)}\right)^2 - \frac{1}{2}\frac{\Omega_\lambda''(k,\tau)}{\Omega_\lambda(k,\tau)} ,$$

Adiabatic condition: slowly changes in time

$$\left|\frac{\dot{\Omega}}{\Omega^2}\right| \ll 1, \qquad\qquad \epsilon \ll 1 : \ \partial_t \to \epsilon \partial_t$$

$\Omega_k(t)$ is obtained as a power series in time derivatives

$$\Omega_k(t) = \Omega_k^{(0)}(t) + \epsilon\, \Omega_k^{(1)}(t) + \cdots + \epsilon^n\, \Omega_k^{(n)}(t)\,,$$

where $\Omega_k^{(n)}$ is given by iterating the recursive equation up to order n.

Standard adiabatic regularization

$$\frac{1}{2}\langle \mathbf{E}^2+\mathbf{B}^2\rangle_{\mathsf{ad}} = \int_0^{a(\tau)\Lambda} \frac{\mathrm{d}k}{(2\pi)^2 a(\tau)^4} k^2 \left[|A'_+|^2 + |A'_-|^2 + k^2\left(|A_+|^2 + |A_-|^2\right)\right]_{\mathsf{ad}}^{n=4}$$

$$\langle \mathbf{E}\cdot\mathbf{B}\rangle_{\mathsf{ad}} = -\int_0^{a(\tau)\Lambda} \frac{\mathrm{d}k}{(2\pi)^2 a(\tau)^4} k^3 \frac{\partial}{\partial\tau}\left(|A_+|^2 - |A_-|^2\right)_{\mathsf{ad}}^{n=4} ,$$

$$\frac{1}{2}\langle \mathbf{E}^2+\mathbf{B}^2\rangle_{\mathsf{ad}} = \frac{\Lambda^4}{8\pi^2}+\frac{H^2\Lambda^2\xi^2}{8\pi^2}+\frac{3H^4\xi^2(5\xi^2-1)\log\left(2\Lambda/H\right)}{16\pi^2} - \frac{H^4}{480\pi^2}-\frac{H^4\xi^2(23\xi^2-9)}{16\pi^2}-\frac{3H^4\xi^2(5\xi^2-1)\log\left(\frac{m}{H}\right)}{16\pi^2},$$

$$\langle \mathbf{E}\cdot\mathbf{B}\rangle_{\mathsf{ad}} = -\frac{H^2\Lambda^2\xi}{8\pi^2}-\frac{3H^4\xi(5\xi^2-1)\log\left(2\Lambda/H\right)}{8\pi^2} + \frac{H^4(19\xi-56\xi^3)}{16\pi^2}+\frac{3H^4\xi(5\xi^2-1)\log\left(\frac{m}{H}\right)}{8\pi^2}.$$

The standard adiabatic renormalization of these two quantities, despite correctly removes the divergences in the UV, also introduces **unphysical IR divergences**, leading to not well-defined final results.

[Ballardini et al '19]

Issues and Motivations:The needs of a IR cut off

- Adiabatic renormalization concerns the renormalization of UV divergences.
- WKB ansatz for the mode functions matches exactly the solution in the deep UV, where the space-time is well approximated by the Minkowski one.
- WKB is well defined for modes that feel small curvature (or slowly changing curvatures)
- In a cosmological fashion we should say that this is a good approximation only for those modes that are sub-horizon.

Regularization with IR-Cut off

We suggest that the procedure of adiabatic regularization should be always performed on a proper domain which excludes the IR tail of the spectrum.

- the adiabatic subtraction should be considered only up to a comoving IR cut-off $c = \beta a(t)H(t)$.
- This IR cut-off is associated to the scale at which the adiabatic solution is not anymore a good approximation for the mode functions, condition that happens when the modes start to feel the curvature of space-time.
- the coefficient β, should be determined by a proper physical prescription, fully in line with the spirit of each renormalization scheme.

New adiabatic regularization

$$\frac{1}{2}\left\langle \mathbf{E}^2+\mathbf{B}^2 \right\rangle_{\mathsf{ad}} = \int_{\beta a(\tau)H}^{a(\tau)\Lambda} \frac{\mathrm{d}k}{(2\pi)^2 a(\tau)^4} k^2 \left[|A_+'|^2 + |A_-'|^2 + k^2 \left(|A_+|^2 + |A_-|^2 \right) \right]_{\mathsf{ad}}$$

$$\left\langle \mathbf{E}\cdot\mathbf{B} \right\rangle_{\mathsf{ad}} = -\int_{\beta a(\tau)H}^{a(\tau)\Lambda} \frac{\mathrm{d}k}{(2\pi)^2 a(\tau)^4} k^3 \frac{\partial}{\partial\tau} \left(|A_+|^2 - |A_-|^2 \right)_{\mathsf{ad}} ,$$

$$
\frac{1}{2}\langle \mathbf{E}^2+\mathbf{B}^2\rangle_{\mathsf{ad}}^{c=\beta H a(\tau)} = \frac{\Lambda^4}{8\pi^2}+\frac{H^2\Lambda^2\xi^2}{8\pi^2}+\frac{3H^4\xi^2(5\xi^2-1)\log\left(2\Lambda/H\right)}{16\pi^2} - \frac{\beta^4H^4}{8\pi^2}-\frac{\beta^2H^4\xi^2}{8\pi^2}-\frac{3H^4\xi^2(5\xi^2-1)\log\left(2\beta\right)}{16\pi^2}\,,
$$

$$
\langle \mathbf{E}\cdot\mathbf{B}\rangle_{\mathsf{ad}}^{c=\beta H a(\tau)} = -\frac{H^2\Lambda^2\xi}{8\pi^2}-\frac{3H^4\xi(5\xi^2-1)\log\left(2\Lambda/H\right)}{8\pi^2} + \frac{\beta^2H^4\xi}{8\pi^2}+\frac{3H^4\xi(5\xi^2-1)\log\left(2\beta\right)}{8\pi^2}\,.
$$

This results are obtained taking properly the limit $m \to 0$

[**C.Animali**, **P.C**, **G. Marozzi**]

- Introduction
- Motivation: a problematic example
- How to fix univocally the renormalization scheme: conformal anomaly
- Conclusions

How to fix the scheme:

Conformal anomaly

In the conformal limit, a proper renormalization scheme should provide the conformal anomaly induced by quantum effects.

[Birrell '82]

when at the classical level $T^{\mu}{}_{\mu} = 0$

$$\langle T^{\mu}{}_{\mu} \rangle_{\mathsf{phys}} = -\langle T^{\mu}{}_{\mu} \rangle_{\mathsf{reg}} \,,$$

where $\langle T^{\mu}{}_{\mu} \rangle_{\mathsf{reg}}$ is the trace contribution to the energy-momentum tensor given by the particular renormalization method applied.

- The two helicities of the mode functions $A_\pm$ are equivalent to two conformally coupled massless scalar fields for $\xi = 0$.

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}A_\pm + \left(k^2 \pm \frac{2k\xi}{\tau} + \frac{m^2}{H^2\tau^2}\right)A_\pm = 0 \rightarrow \boxed{\left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + k^2\right)A_\pm = 0}$$

$$\lim_{\xi\to 0,\, m\to 0}\langle T^0{}_0\rangle_{\mathsf{ad}} = \lim_{\xi\to 0,\ m\to 0}\frac{\langle \mathbf{E}^2 + \mathbf{B}^2\rangle_{\mathsf{ad}}^{c=\beta H a(\tau)}}{2} = -\frac{\beta^4 H^4}{8\pi^2}\,,$$

this term should reproduce the expected value of the anomaly (twice the scalar case), i.e.

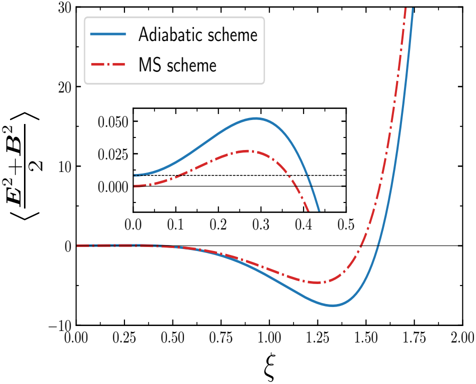
$$\frac{\beta^4 H^4}{8\pi^2} = \frac{H^4}{480\pi^2} \implies \boxed{\beta = \frac{1}{\sqrt{2}\times 15^{1/4}} \approx 0.359}$$

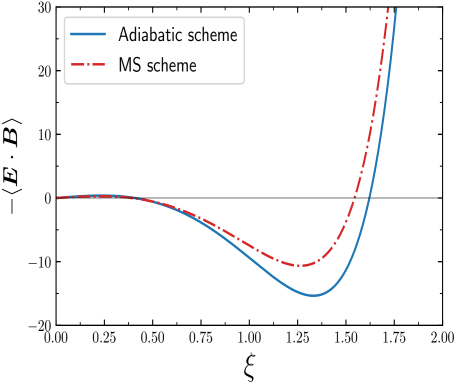
We can now perform the proper renormalization procedure

$$\frac{1}{2}\langle \mathbf{E}^2+\mathbf{B}^2\rangle_\beta = \frac{2H^4}{960\pi^2} + \frac{H^4\xi^2\left(-1185\xi^4+(330+4\sqrt{15})\xi^2+435+4\sqrt{15}\right)}{960\pi^2\left(1+\xi^2\right)}$$
$$-\frac{3H^4\xi^2\left(5\xi^2-1\right)\log\left(15/4\right)}{64\pi^2} + \frac{H^4\xi\left(30\xi^2-11\right)\sinh(2\pi\xi)}{64\pi^3}$$
$$-\frac{3H^4\xi^2\left(5\xi^2-1\right)\left(\psi^{(0)}(-1-i\xi)+\psi^{(0)}(-1+i\xi)\right)}{32\pi^2}$$
$$+\frac{3iH^4\xi^2\left(5\xi^2-1\right)\left(\psi^{(1)}(1-i\xi)-\psi^{(1)}(1+i\xi)\right)\sinh(2\pi\xi)}{64\pi^3},$$

$$\langle \mathbf{E}\cdot\mathbf{B}\rangle_\beta = \frac{H^4\xi\left(705\xi^2-330-\sqrt{15}\right)}{240\pi^2} + \frac{3H^4\xi\left(5\xi^2-1\right)\log\left(15/4\right)}{32\pi^2}$$
$$+\frac{3H^4\xi\left(5\xi^2-1\right)\left(\psi^{(0)}(1-i\xi)+\psi^{(0)}(1+i\xi)\right)}{16\pi^2}$$
$$+\frac{3iH^4\xi\left(5\xi^2-1\right)\left(-\psi^{(1)}(1-i\xi)+\psi^{(1)}(1+i\xi)\right)\sinh(2\pi\xi)}{32\pi^3}$$
$$+\frac{H^4\left(11-30\xi^2\right)\sinh(2\pi\xi)}{32\pi^3}.$$

- This is a physically motivated prescription that is able to fix univocally the renormalization scheme.
- we are able to obtain univocal finite results for the averaged energy density and helicity of gauge fields.
- adiabatic renormalization method succeeds in providing the conformal anomaly in the proper limit.





Comparison between the new adiabatic scheme and the minimal subtraction scheme (MS).

Next:

- Study of phenomenological implication: evolution of inflaton field, number of e-folds.
- Application of such method to other pathological scenarios such as the model of a pseudo-scalar field coupled to the gravitational Chern-Simons term [**Kamada et al '20**].

- Introduction
- Motivation: a problematic example
- How to fix univocally the renormalization scheme: conformal anomaly
- Conclusions

Conclusion

- Adiabatic renormalization is a powerful renormalization scheme to regularize UV divergences.
- Should be truncated up to an IR cut-off proportional to the horizon size.
- This cut-off should be fixed by a proper physical prescription.

Thank you for the attention