t-channel singularity in dark matter considerations: the problem and its solution

Michał Iglicki

approximate pronunciation:
ME-HOW EAGLE-EATS-KEY

University of Warsaw

based on:

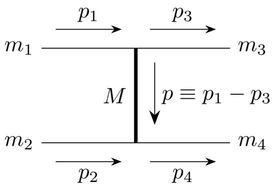
- B. Grządkowski, M. Iglicki, S. Mrówczyński
  t-channel singularities in cosmology and particle physics
  (2108.01757, currently under review)
- M. Iglicki
  Thermal regularization of t-channel singularities in cosmology and particle physics: the general case
  (2208.xxxxx, in preparation)

PASCOS 2022
Heidelberg, 28 July 2022

PART I
THE PROBLEM

Introduction: the t-channel singularity

- definition:



$$\mathcal{M} \sim \frac{1}{t - M^2}\,, \qquad t \equiv p^2$$

$t = M^2 \quad \Rightarrow \quad$ singular matrix element

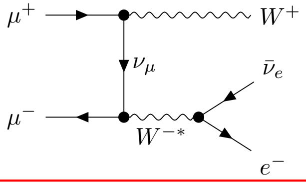
$\qquad\quad\;\; \Rightarrow \quad$ infinite cross section

⋆ IR regularization not applicable if $M > 0$

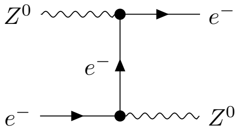
⋆ BW resummation not applicable if $\Gamma = 0$

- SM and BSM examples:

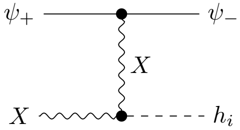
muon colliders



weak Compton scattering

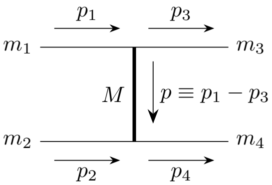


dark matter in early Universe



(model: A. Ahmed et al., Eur.Phys.J.C 78 (2018) 11, 905)

$2 \leftrightarrow 2$ process: when does the t-channel singularity occur?



- Mandelstam variables:

$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2$$
$$t \equiv p^2 = (p_1 - p_3)^2$$

- matrix element:

$$\mathcal{M} \sim \frac{1}{t - M^2}$$

- cross section

$$\sigma(s) \leftarrow \int_{t_{\min}(s)}^{t_{\max}(s)} \frac{dt}{(t - M^2)^2}$$

- thermally av. cross section

$$\langle \sigma v \rangle(T) \leftarrow \int \sigma(s)\, f(E_1, E_2, T)\, ds$$

- singularity condition:

$$t_{\min}(s) < M^2 < t_{\max}(s) \quad \Rightarrow \quad \text{singularity}$$

$$t_{\min} = m_1^2 + m_3^2 - 2E_1E_3 - 2|\mathbf{p}_1||\mathbf{p}_3| \qquad t_{\max} = m_1^2 + m_3^2 - 2E_1E_3 + 2|\mathbf{p}_1||\mathbf{p}_3|$$

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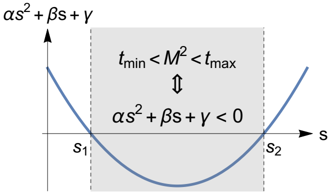
- in terms of the CMS energy ($\sqrt{s}$):

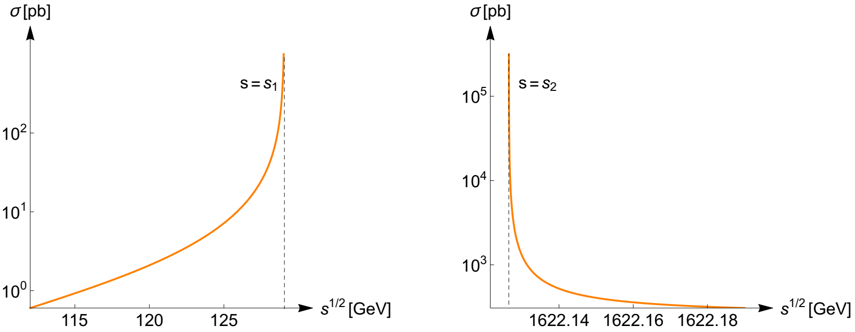
$$t_{\min}(s) < M^2 < t_{\max}(s)$$
$$\Leftrightarrow \quad s_1 < s < s_2$$



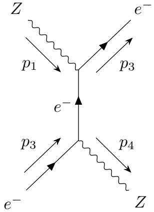
$$s_{1,2} \equiv \frac{-\beta \mp \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

α, β, γ – functions of m_1, m_2, m_3, m_4 and M

example: weak Compton scattering

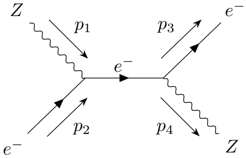


$$Ze^- \to Ze^- =$$



singular

$+$



regular

- singularity condition:

$$t_{\min}(s) < M^2 < t_{\max}(s) \qquad \Rightarrow \qquad \text{singularity}$$

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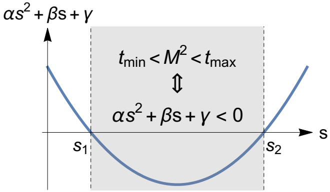


$$s_{1,2} \equiv \frac{-\beta \mp \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

α, β, γ – functions of m_1, m_2, m_3, m_4 and M

- thermally averaged cross section $\leftarrow$ integration over $s \in [s_{\min}, \infty)$
  (weighted by thermal distribution functions)

- conclusion for the cosmological case:

  if $s_2 > s_{\min} \equiv \max\{(m_1+m_2)^2, (m_3+m_4)^2\}$,
  singularity in the allowed range $\quad \Rightarrow \quad \langle \sigma v \rangle = \infty$

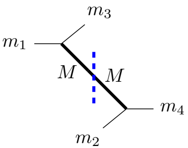
$$s_2 \equiv \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

α, β, γ – functions of m_1, m_2, m_3, m_4 and M

if $\underline{s_2 > s_{\min} \equiv \max\{(m_1 + m_2)^2, (m_3 + m_4)^2\}}$, singularity in the allowed range

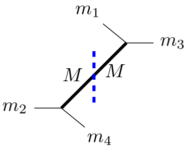
$\Updownarrow$

$m_1 > M + m_3$ and $m_4 > M + m_2$



or

$m_2 > M + m_4$ and $m_3 > M + m_1$



◇ Coleman-Norton theorem

S. Coleman & R. E. Norton, Nuovo Cim 38, 438–442 (1965)

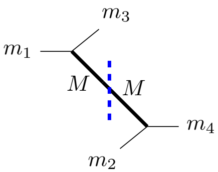
"It is shown that a Feynman amplitude has singularities on the physical boundary if and only if the relevant Feynman diagram can be interpreted as a picture of an energy- and momentum-conserving process occurring in space-time, with all internal particles real, on the mass shell, and moving forward in time"

PART II

NON-SOLUTIONS

Unsatisfactory ideas
$\longrightarrow$ complex mass of unstable particles

I. Ginzburg, Nucl.Phys.B Proc.Suppl. 51 (1996) 85-89



idea: finite lifetime should affect the wavefunction

- at rest: $$e^{im_1 t} \rightarrow e^{im_1 t} e^{-\Gamma_1 t}$$ $$= e^{i\widetilde{m}_1 t}\,, \qquad \widetilde{m}_1 \equiv m_1\left(1 + i\frac{\Gamma_1}{m_1}\right)$$
- after Lorentz boost: $$p_1 \rightarrow \widetilde{p}_1 \equiv p_1\left(1 + i\frac{\Gamma_1}{m_1}\right)$$

$\rightarrow$ problem: $(\widetilde{p}_1 - \widetilde{p}_3)^2 \neq (\widetilde{p}_4 - \widetilde{p}_2)^2 \Rightarrow$ lack of symmetry

(momentum conservation...)

G. L. Kotkin et al., Yad. Fiz. 42 (1982) 692
G. L. Kotkin et al., Int. Journ. Mod. Phys. A 7 (1992) 4707
K. Melnikov & V. G. Serbo, Nucl.Phys. B483 (1997) 67
C. Dams & R. Kleiss, Eur.Phys.J.C29 (2003) 11
C. Dams & R. Kleiss, Eur.Phys.J. C36 (2004) 177

idea: at colliders, the beams have finite size

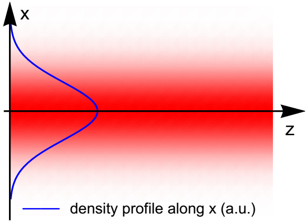
$\Downarrow$

they should not be treated as plain waves

example:

Gaussian beam moving along z axis

$$n(x,y) \sim e^{-\frac{x^2+y^2}{2a^2}} \qquad a \text{ – beam width}$$



$$\int \frac{dt}{|t-M^2+i\varepsilon|^2} \to \int \frac{a^3\, e^{-\frac{a^2\kappa^2}{2}}}{(2\pi)^{3/2}} \frac{d^3\kappa\, dt}{(t-M^2+i\varepsilon-\kappa\cdot\mathbf{q})(t-M^2-i\varepsilon+\kappa\cdot\mathbf{q})}$$

$$\sim \frac{\pi a}{|\mathbf{q}|}\,, \qquad \mathbf{q} \equiv \left[\frac{E_3}{E_1}\mathbf{p}_1 - \mathbf{p}_3\right]_{t=M^2}$$

$\longrightarrow$ problem: inapplicable in cosmological context

PART III
THE SOLUTION

Idea

- early Universe = hot gas
- every particle interacts with a thermal medium
- the mean life time cannot be infinite $\Rightarrow$ effective width
- QFT in a thermal medium: Matsubara (imaginary-time) or Keldysh-Schwinger (real-time) formalism

Matsubara (imaginary-time) →

H. A. Weldon
Phys. Rev. D 28 (1983) 2007

$\Downarrow$

G. F. Giudice et al.
Nucl. Phys. B 685 (2004) 89–149

Keldysh-Schwinger (real-time) →

our paper
2108.017579

Idea

- early Universe = hot gas
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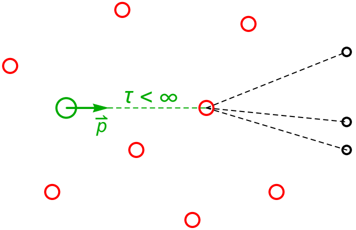
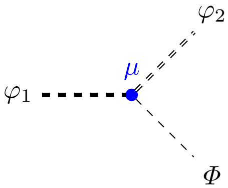


our paper
2108.017579

Toy model

- 3 real scalars: φ_1, φ_2, Φ
- Lagrangian:

$$\mathcal{L} = [\text{kinetic terms}] + \mu\, \varphi_1 \varphi_2 \Phi$$



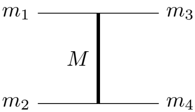
- discrete symmetries:

$$\mathbb{Z}_2 : \quad (\varphi_1\,,\ \varphi_2\,,\ \Phi) \to (-\varphi_1\,,\ \varphi_2\,,\ -\Phi)\ ,$$
$$\mathbb{Z}_2' : \quad (\varphi_1\,,\ \varphi_2\,,\ \Phi) \to (\varphi_1\,,\ -\varphi_2\,,\ -\Phi)\ ,$$

$\Rightarrow$ no power-3 terms except $\mu\varphi_1\varphi_2\Phi$

- power-4 terms (e.g. $\varphi_1^2\,\varphi_2^2$) dropped for simplicity

- general case:



singular if

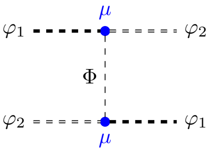
$m_1 > m_3 + M$ and $m_4 > m_2 + M$

or

$m_3 > m_1 + M$ and $m_2 > m_4 + M$

- toy model:

$$\mathcal{L} = [\text{kinetic terms}] + \mu\,\varphi_1\varphi_2\Phi$$



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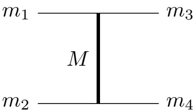
$m_1 > m_2 + M$

or

$m_2 > m_1 + M$

The singularity

- general case:



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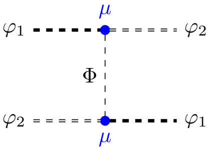
- toy model:

$$\mathcal{L} = \text{[kinetic terms]} + \mu\,\varphi_1\varphi_2\Phi$$



singular if

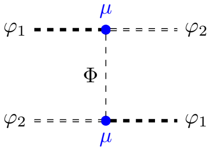
$\rightarrow \quad m_1 > m_2 + M$

or

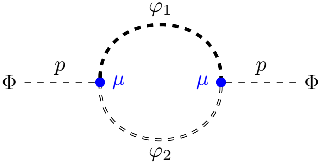
$m_2 > m_1 + M$

$$\mathcal{L} = [\text{kinetic terms}] + \mu\,\varphi_1\varphi_2\Phi\;, \qquad\qquad m_1 > m_2 + M$$

- singular process:



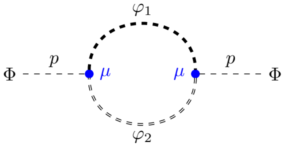
- one-loop contribution to mediator's self-energy:



$$i\Pi(x,y) = \mu^2\; i\Delta_1(x,y)\; i\Delta_2(y,x),$$

- non-zero imaginary part of self-energy acquired as a result of thermal interactions with the medium of particles (Keldysh-Schwinger formalism)

Calculation of one-loop self-energy



$$\mathcal{L} = [\text{kinetic terms}] + \mu\,\varphi_1\varphi_2\Phi$$
$$m_1 > m_2 + M$$

- one-loop contribution to the self-energy:

$$i\Pi(x,y) = \mu^2\; i\Delta_1(x,y)\; i\Delta_2(y,x),$$

- non-zero imaginary part of self-energy acquired as a result of thermal interactions with the medium of particles (Keldysh-Schwinger formalism)

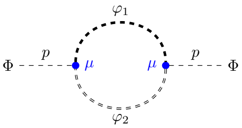
$$\boxed{\Pi^+(p,T) = \frac{i}{2}\mu^2 \int \frac{d^4k}{(2\pi)^4}\Big[\Delta_1^+(k+p)\,\Delta_2^{\mathsf{sym}}(k,T) + \Delta_1^{\mathsf{sym}}(k,T)\,\Delta_2^-(k-p)\Big]}$$

$$\Delta_l^{\mathsf{sym}}(k,T) \equiv -\frac{i\pi}{E_l}\left[\delta_{(E_l-k_0)} + \delta_{(E_l+k_0)}\right] \times \left[2f(E_l,T)+1\right], \quad \Delta_l^{\pm}(p) \equiv \frac{1}{p^2 - m_l^2 \pm i\,\mathsf{sgn}(p_0)\,\varepsilon}$$

$$f(\beta E_l) = (e^{\beta E_l}-1)^{-1}, \quad E_l \equiv \sqrt{\mathbf{k}^{\,2} + m_l^2}, \quad l = 1,2$$

- after tedious calculations:

$$\Sigma(|\mathbf{p}|,T) \equiv \Im\Pi^{+}(|\mathbf{p}|,T) = \frac{\mu^2}{16\pi}\frac{1}{\beta\,|\mathbf{p}|}\left[-\ln\frac{e^{\beta(b+a)}-1}{e^{\beta(b-a)}-1} + \ln\frac{e^{\beta(b+a)}e^{-\beta E_p}-1}{e^{\beta(b-a)}e^{-\beta E_p}-1}\right]$$



$$\beta \equiv 1/T$$

$$a \equiv \frac{\lambda(m_1^2,m_2^2,M^2)^{1/2}}{2M^2}\,|\mathbf{p}|\,, \qquad b \equiv \frac{(m_1^2-m_2^2+M^2)}{2M^2}\,E_p$$

$$\lambda(m_1^2,m_2^2,M^2) \equiv \left[m_1^2-(m_2+M)^2\right]\left[m_1^2-(m_2-M)^2\right]\ .$$

effective width:

$$\Gamma_{\text{eff}}(|\mathbf{p}|,T) \equiv M^{-1}|\Sigma(|\mathbf{p}|,T)|$$

$$\Downarrow$$

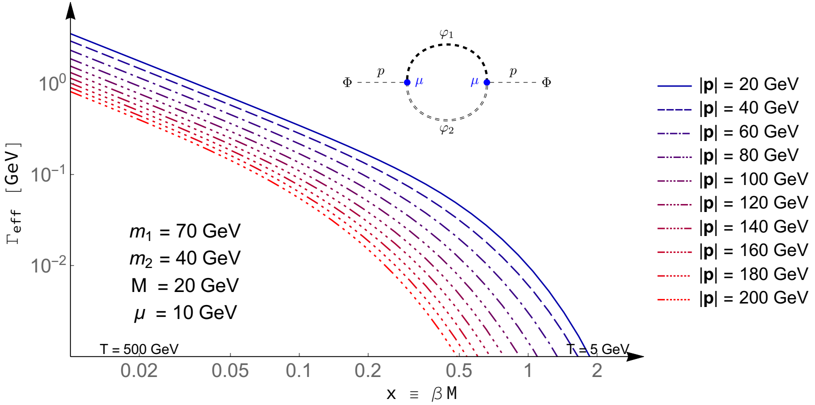
Breit-Wigner propagator:

$$\frac{1}{(t-M^2)^2} \to \frac{1}{(t-M^2)^2+M^2\Gamma_{\text{eff}}(|\mathbf{p}|,T)^2}$$

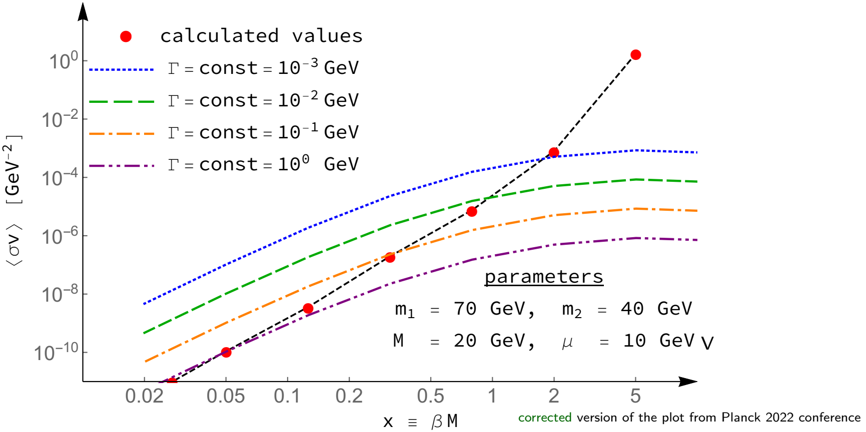
Results: effective width

$\beta \equiv 1/T$

$$\Gamma_{\text{eff}}(|\mathbf{p}|, T) \simeq \begin{cases} \frac{\mu^2}{16\pi M} \frac{1}{\beta\,|\mathbf{p}|} \times \left[-\ln \frac{b+a}{b-a} + \ln \frac{b+a-E_p}{b-a-E_p} \right] & \text{for high } T \\ \frac{\mu^2}{16\pi M} \frac{1}{\beta\,|\mathbf{p}|} \times e^{-\beta(b-a-E_p)}(1-e^{-\beta E_p})(1-e^{-2\beta a}) & \text{for low } T \end{cases}$$



Results: thermally averaged cross section



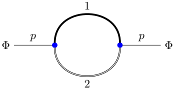
corrected version of the plot from Planck 2022 conference

$$\langle\sigma v\rangle_{12\to 34}(T) = \mu^4 \int d\Pi_1\, d\Pi_2\, f(E_1, E_2, T) \int d\Pi_3\, d\Pi_4\, \frac{(2\pi)^4\, \delta^{(4)}(p_1+p_2-p_3-p_4)}{(t-M^2)^2 + M^2\, \Gamma_{\text{eff}}(|\mathbf{p}|, T)^2}$$

$$d\Pi_k \equiv \frac{d^3 p_k}{(2\pi)^3\, 2E_k}$$ – phase-space element $(k = 1, 2, 3, 4)$

Results: general spin case

$$\Sigma(|\mathbf{p}|, T) = \frac{1}{2}\int d\Pi_1\, d\Pi_2 \times |\mathcal{M}_{1\to 2,M}|^2 \times [\pm f_1(\beta E_1) + f_2(\beta E_2)] \times (2\pi)^4 \delta^{(4)}(p_1 - p_2 - p)$$



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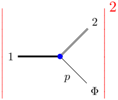

$\begin{cases} \text{„+"} & \text{1 \& 2 are both fermions (bosons)} \\ \text{„−"} & \text{otherwise} \end{cases}$

H. A. Weldon
Phys. Rev. D 28 (1983) 2007

(1-B, 2-B) $$\Sigma(|\mathbf{p}|, T) = -\frac{1}{16\pi}\frac{|\mathcal{M}_{1\to 2,M}|^2}{\beta\,|\mathbf{p}|}\left[\ln\frac{e^{\beta(b+a)}-1}{e^{\beta(b-a)}-1} - \ln\frac{e^{\beta(b+a)}e^{-\beta E_p}-1}{e^{\beta(b-a)}e^{-\beta E_p}-1}\right]$$

(1-B, 2-F) $$\Sigma(|\mathbf{p}|, T) = \frac{1}{16\pi}\frac{|\mathcal{M}_{1\to 2,M}|^2}{\beta\,|\mathbf{p}|}\left[\ln\frac{e^{\beta(b+a)}-1}{e^{\beta(b-a)}-1} - \ln\frac{e^{\beta(b+a)}e^{-\beta E_p}+1}{e^{\beta(b-a)}e^{-\beta E_p}+1}\right]$$

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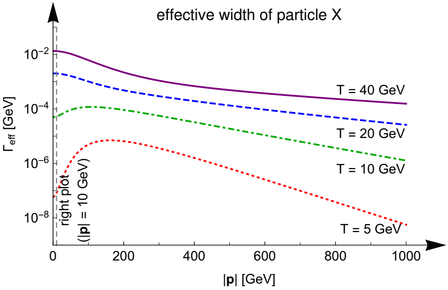
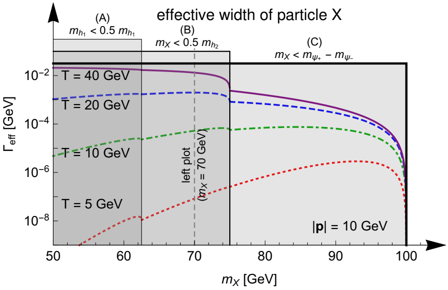
$$\beta \equiv 1/T$$

$$a \equiv \frac{\lambda(m_1^2, m_2^2, M^2)^{1/2}}{2M^2}\,|\mathbf{p}|\,, \qquad b \equiv \frac{(m_1^2 - m_2^2 + M^2)}{2M^2}\,E_p$$

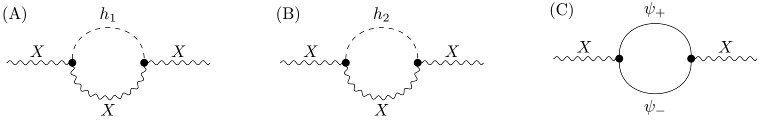
$$\lambda(m_1^2, m_2^2, M^2) \equiv \left[m_1^2 - (m_2 + M)^2\right]\left[m_1^2 - (m_2 - M)^2\right]\,.$$

Results: general spin case





contributing diagrams



values of model's parameters

$m_X = 70$ GeV (left plot) $\quad g_x = 0.1 \quad \sin\alpha = 0.3$

$m_{\psi_+} = 120$ GeV $\quad m_{h_1} = 125$ GeV $\quad m_{\psi_-} = 20$ GeV $\quad m_{h_2} = 150$ GeV

model: $G_{SM} \times U(1)_X$ with gauge boson X, two Higgs-like states $h_{1,2}$ and two Majorana fermions $\psi_\pm$

(see A. Ahmed et al., Eur.Phys.J.C 78 (2018) 11, 905 for details)

PART IV
BUT. . .

Is thermal width a „true” solution?

Thermal width

- well-motivated
- works for $\langle \sigma v \rangle$
- works for σ

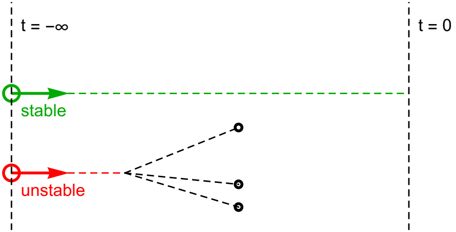
Is thermal width a „true” solution?

Thermal width

- well-motivated
- works for $\langle \sigma v \rangle$
- works for σ ... but what is σ for unstable initial-state particles?

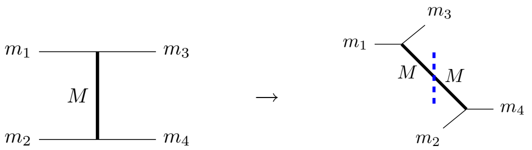
$\longrightarrow$ no asymptotic states

$\longrightarrow$ consistent field-theoretical approach to be found

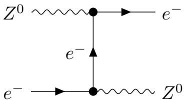


Summary

- t-channel singularity of $\langle \sigma v \rangle$ occurs if
  - the process can be seen as a sequence of decay and fusion



  - the mediator has no width (is stable)
- the singularity is present both in SM and BSM physics



- apart thermal FT, known approaches are either unsatisfactory or inapplicable
- interaction with the medium results in a non-zero effective width (obtained within thermal FT) that regulates the singularity
- the effective width depends on temperature and mediator's momentum (momentum transfer) and behaves in an expected, natural way

$$\Gamma_{\text{eff}} = \Gamma_{\text{eff}}(T, |\mathbf{p}|)$$

- systematic QFT approach to unstable particles still to be found

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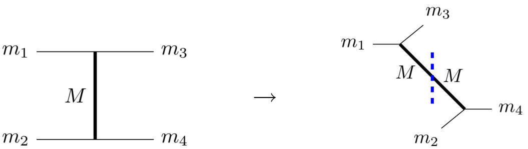

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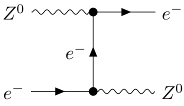

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thank you!

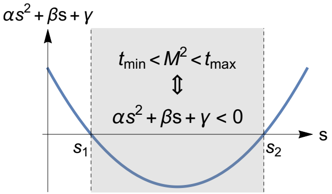
BACKUP SLIDES

Values of s_1, s_2 in terms of masses

- in terms of the CMS energy ($\sqrt{s}$):

$$t_{\min}(s) < M^2 < t_{\max}(s)$$
$$\Leftrightarrow \quad s_1 < s < s_2$$

$$s_{1,2} \equiv \frac{-\beta \mp \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$



$$\alpha \equiv M^2$$
$$\beta \equiv M^4 - M^2(m_1^2 + m_2^2 + m_3^2 + m_4^2) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)$$
$$\gamma \equiv M^2(m_1^2 - m_2^2)(m_3^2 - m_4^2) + (m_1^2 m_4^2 - m_2^2 m_3^2)(m_1^2 - m_2^2 - m_3^2 + m_4^2)$$