The Flavour Puzzle as a Vacuum Problem

Heidelberg July 26, 2022 Ferruccio Feruglio INFN Padova

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based on collaborations with

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The flavour puzzle

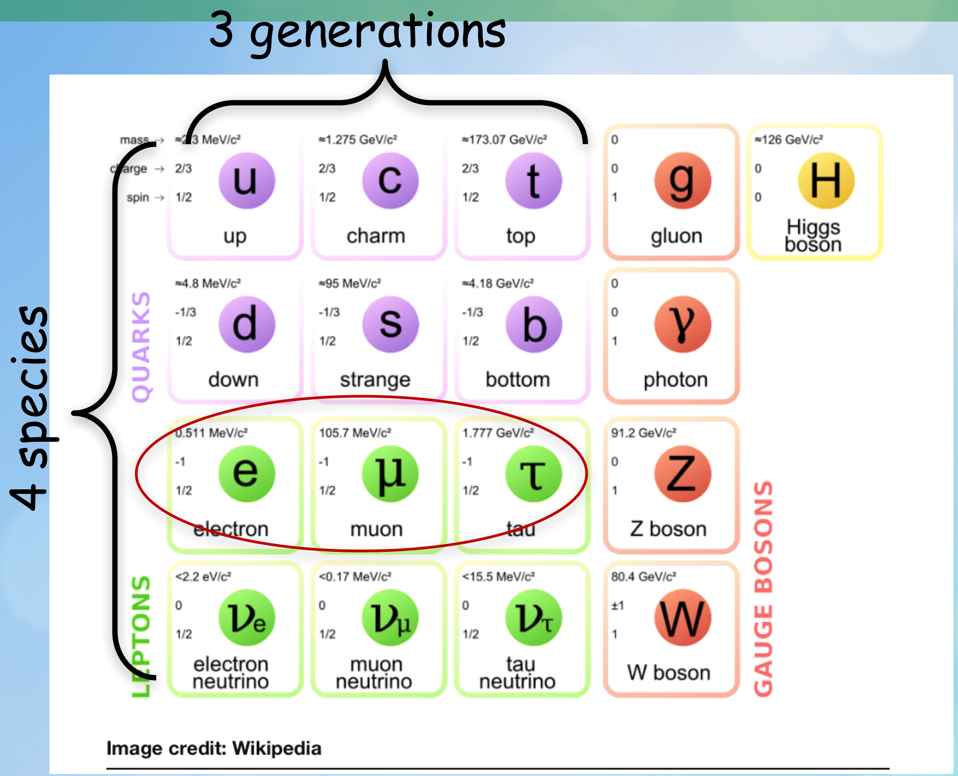


Image credit: Wikipedia

- gauge and Higgs boson masses
- all gauge interactions of 3 families of quark and lepton masses (15 x 3 Weyl spinors)

- 3 gauge couplings
- 2 masses (G_F, m_H)

fermion masses and mixing angles require (up to) 22 additional parameters [fermion bilinears]

$$\mathcal{L}_Y = -\overline{\Psi}\,\mathcal{Y}\,\Phi\,\Psi$$

$$\mathcal{L}_\nu = -\frac{1}{\Lambda}(\Phi\,\Psi)\mathcal{W}\,(\Phi\,\Psi)$$

- 6+6 masses
- 3+3 mixing angles
- 1+3 phases

$$m_{ij} = \mathcal{Y}_{ij}\,v$$
$$m_{\nu ij} = \mathcal{W}_{ij} v^2/\Lambda$$

loosely constrained by gauge symmetry

Flavour Symmetries

"traditional" Flavour Symmetries

$g \, \epsilon \, G_{fl}$

$$\Psi \xrightarrow{g} \varrho(g)\,\Psi$$

$$\Psi = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}$$

⬇

$$\varrho(g)^{+}\mathcal{Y}\,\varrho(g) = \mathcal{Y}$$

can be combined with CP

$$\Psi \xrightarrow{CP} X_{CP}\Psi^{*}$$

example: Isospin SU(2) in strong interactions $m_p = m_n$

- flavour symmetries of this type are necessarily broken

e.g.
largest
flavour symmetry
of the quark
sector

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix}$$

$$u^c \;\; c^c \;\; t^c$$

$$d^c \;\; s^c \;\; b^c$$

$$U(3)^3$$

[up to anomalies]

broken down
to $U(1)_B \times U(1)_Y$

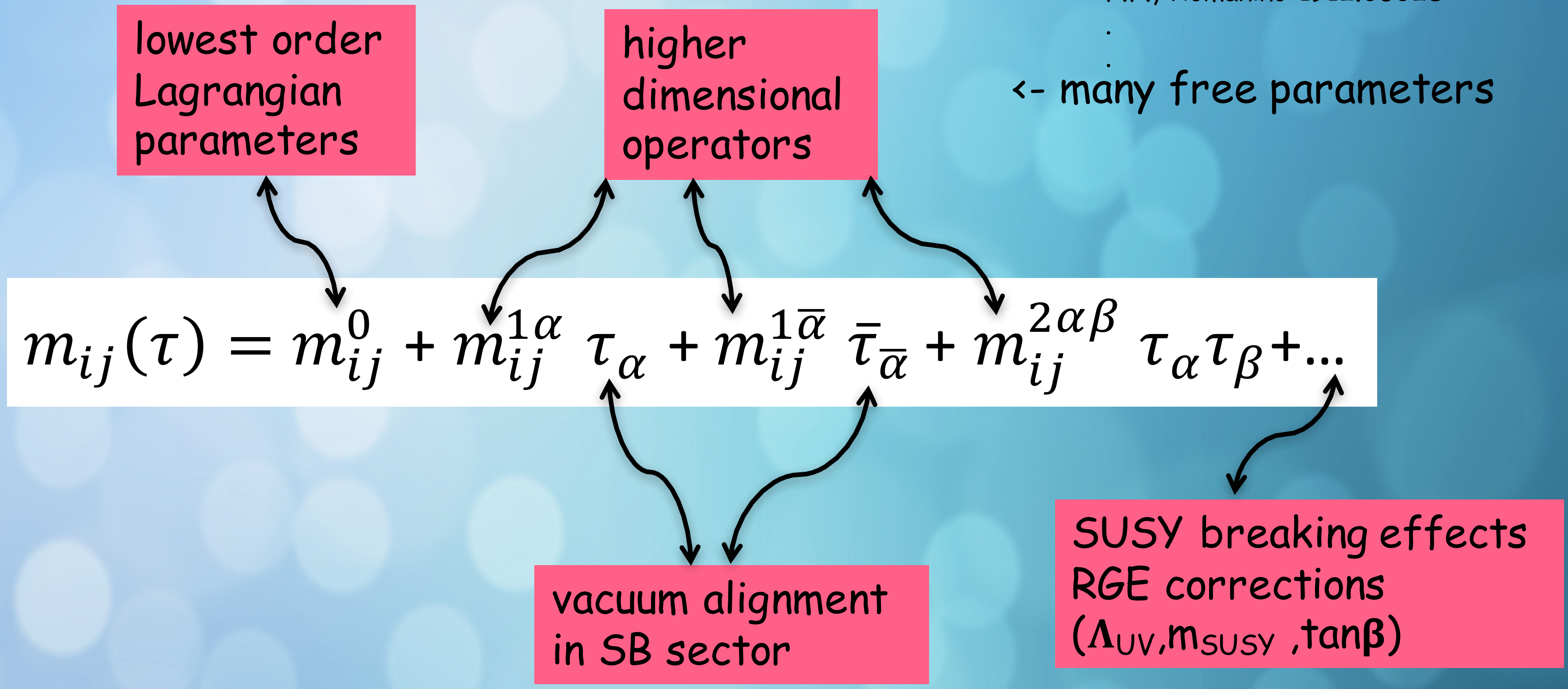
no constraint on
quark masses/mixing angles

Symmetry Breaking

- τ_α symmetry breaking sector:
  set of dimensionless, gauge invariant
  scalar fields, charged under G_{fl}

[τ_α stands for $\langle\tau_\alpha\rangle/\Lambda_F$ where the scale Λ_F has been set to 1]

reviews:
Ishimori, Kobayashi, Ohki, Shimizu, Okada, Tanimoto , 1003.3552;
King, Luhn, 1301.1340;
King, Merle, Morisi, Shimizu, Tanimoto, 1402.4271;
King, 1701.04413
Hagedorn, 1705.00684;
F.F., Romanino 1912.06028
.
.



lowest order Lagrangian parameters

higher dimensional operators

<- many free parameters

$$m_{ij}(\tau) = m^{0}_{ij} + m^{1\alpha}_{ij}\;\tau_\alpha + m^{1\bar{\alpha}}_{ij}\;\bar{\tau}_{\bar{\alpha}} + m^{2\alpha\beta}_{ij}\;\tau_\alpha\tau_\beta + \ldots$$

vacuum alignment in SB sector

SUSY breaking effects
RGE corrections
(Λ_{UV}, m_{SUSY} , $\tan\beta$)

- huge number of models: G_{fl} continuous/discrete, global/local,......
  no baseline model in bottom-up approach

usual path:

choose G_f → assign f to G_f multiplets → look for some SB sector and adjust $\langle\tau_\alpha\rangle$

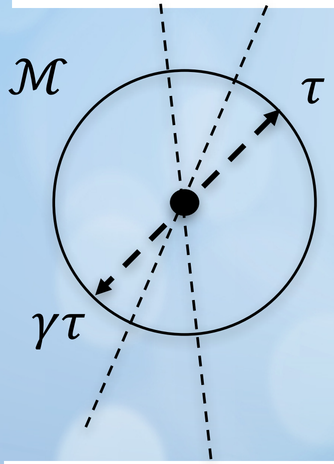
the crucial sector relegated to the last step

can we reverse the logic?

look for physically and/or mathematically motivated SB sector

$\tau \in \mathcal{S}$ = moduli space parametrizes inequivalent vacua

$\tau \in \mathcal{S}$ = {lines of the plane passing through the origin}



$$\mathcal{S} = \frac{\mathcal{M}}{\mathbb{Z}_2}$$

where scalar fields take their values

discrete gauge symmetry: candidate flavour symmetry

$\mathcal{L}_{IR}(\mathcal{Y}(\gamma\tau),\varphi') = \mathcal{L}_{IR}(\mathcal{Y}(\tau),\varphi)$ is a gauge symmetry

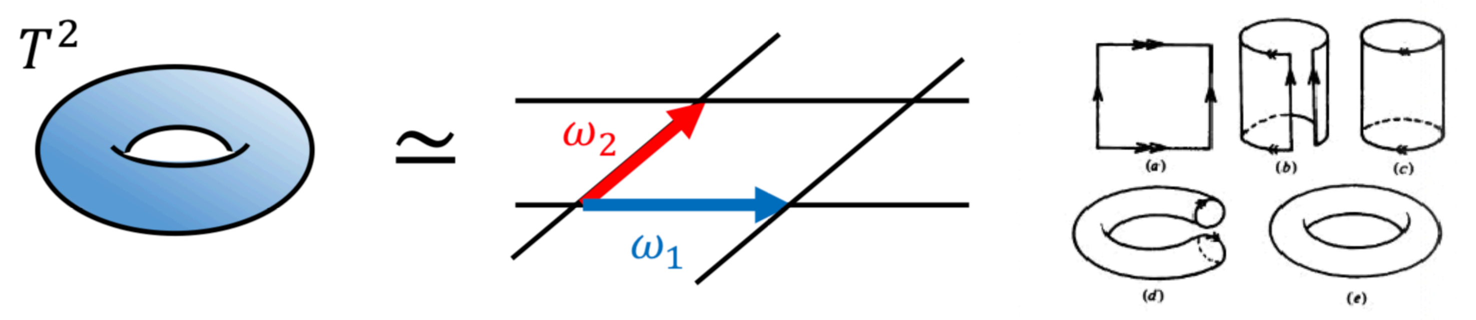
$$\begin{cases} \tau \to \gamma\tau \\ \varphi'=\xi(\gamma)\,\varphi \end{cases}$$

flavour symmetry G_f

action on matter fields

$\mathcal{S}$ = {tori up to rotations and dilations}

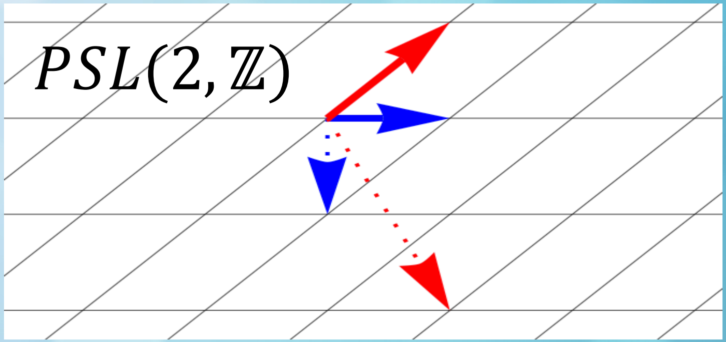
torus ~ lattice $\Lambda = n_1\,\omega_1 + n_2\,\omega_2$



up to rotations and dilations tori parametrized by

$$\mathcal{M} = \mathcal{H} = \left\{ \tau = \frac{\omega_2}{\omega_1} \quad Im(\tau) > 0 \right\}$$

lattice left invariant by modular transformations:



$$\begin{pmatrix}\omega_2\\ \omega_1\end{pmatrix} \to \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_2\\ \omega_1\end{pmatrix}$$

a, b, c, d integers

$$ad - bc = 1$$

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d}$$

➡

$$\mathcal{S} = \frac{\mathcal{M}}{\Gamma} = \frac{\mathcal{H}}{PSL(2,\mathbb{Z})}$$

Model Building

- $\mathcal{L}(\tau,\varphi\,) = \mathcal{L}_{Kin}(\tau,\varphi\,) + \mathcal{L}_{Yuk}(\tau,\varphi;Y\,)$

invariant under $PSL(2,\mathbb{Z})$

$$\tau \to \gamma\tau \equiv \frac{a\tau+b}{c\tau+d}$$

- field transformations:

automorphy factor $j(\gamma,\tau)$

$$j(\gamma_1\gamma_2,\tau) = j(\gamma_1,\gamma_2\tau)\,j(\gamma_2,\tau)$$

$$\varphi' = (c\,\tau+d)^{-k_\varphi} \qquad \varphi$$

the weight,
an integer

Model Building

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$$\varphi' = (c\,\tau+d)^{-k_\varphi}\,\rho_\varphi(\gamma)\,\varphi$$

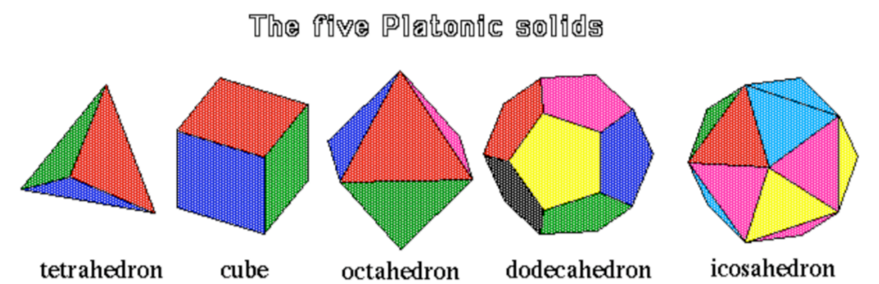
[F.F. 1706.08749]

unitary representation
of the finite modular group

N	2	3	4	5	...
Γ_N	S_3	A_4	S_4	A_5	...

$$\Gamma_N = PSL(2,\mathbb{Z}_N)$$

N = level



$\mathcal{N}$=1 SUSY modular invariant theories

Yukawa interactions in $\mathcal{N}$=1 global SUSY [extension to $\mathcal{N}$=1 SUGRA straightforward]

$$S = \int d^4x d^2\theta \, w(\tau,\varphi) + h.c + \int d^4x d^2\theta d^2\bar{\theta} \, K(\tau,\varphi,\bar{\tau},\bar{\varphi})$$

superpotential =
Yukawa interactions

Kahler potential =
kinetic terms

invariance
satisfied by
"minimal"
Kahler potential

$$w(\tau,\varphi) = \sum_n Y_{I_1 \dots I_n}(\tau) \varphi^{(I_1)} \dots \varphi^{(I_n)}$$

field-dependent
Yukawa couplings

invariance of $w(\Phi)$ guaranteed by an holomorphic $Y_{I_1 \dots I_n}(\tau)$ such that

$$Y_{I_1 \dots I_n}(\gamma\tau) = (c\tau+d)^{k_Y(n)} \rho(\gamma) \, Y_{I_1 \dots I_n}(\tau)$$

1. $-k_Y(n) + k_{I_1} + \cdots + k_{I_n} = 0$
2. $\rho \times \rho^{I_1} \times \dots \times \rho^{I_n} \supset 1$

modular forms
of level N and weight k_Y
form a linear space $\mathcal{M}_k(\Gamma_N))$
of finite dimension

Example

$\Gamma_3 \approx A_4$

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} \to (c\tau+d)^{-1}\rho(\gamma)\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}$$

$k_\nu = +1$

~ 3 of Γ_3

$$w(\tau,\nu) = m_0\, \nu\, Y(\tau)\, \nu + h.c.$$

modular form of level 3
$k = +2$ and $\rho \subset 3+1+1'+1''$

$$d(\mathcal{M}_2(\Gamma_3)) = 3$$
$$\rho = 3$$

$Y_i(\tau)$ explicitly known functions

$$m(\tau) = m_0 \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau)\\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau)\\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau)\end{pmatrix}$$

[F.F. 1706.08749]

mass matrix completely determined in terms of τ up to an overall constant

no corrections from higher order operators in the exact SUSY limit

Modular forms of level 3

[F.F. 1706.08749]

dimension of linear space $\mathcal{M}_k\left(\Gamma(3)\right)$ is (k+1) , k > 0 even integer

- 3 linearly independent modular forms of level 3 and minimal weight $k_I = 2$

$$\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right] \\
Y_2(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] \\
Y_2(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] .
\end{aligned}$$

can be expressed in terms of the Dedekind eta function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q \equiv e^{i2\pi\tau}$$

- they transform in a triplet 3 of Γ_3

$$Y(-1/\tau) = \tau^2\,\rho(S)Y(\tau) \qquad Y(\tau+1) = \rho(T)Y(\tau)$$

$$\rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \qquad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

- they generate the whole ring $\mathcal{M}(\Gamma(3))$

any modular form of level 3 and weight 2k can be written as an homogeneous polynomial in Y_i of degree k

a realistic model for leptons

$\Gamma_4 \approx S_4$

no additional flavons beyond τ

CP spontaneously broken by $\langle\tau\rangle$

[P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, 1905.11970]

- neutrino masses from see-saw
- flavour universal kinetic terms
- charged lepton masses fitted, not predicted

4 parameters describe 9 observables

best fit values

$\mathrm{Re}\,\tau$	± 0.09922
$\mathrm{Im}\,\tau$	1.016
g'/g	-0.02093
$v_u^2\, g^2/\Lambda$ [eV]	0.0135

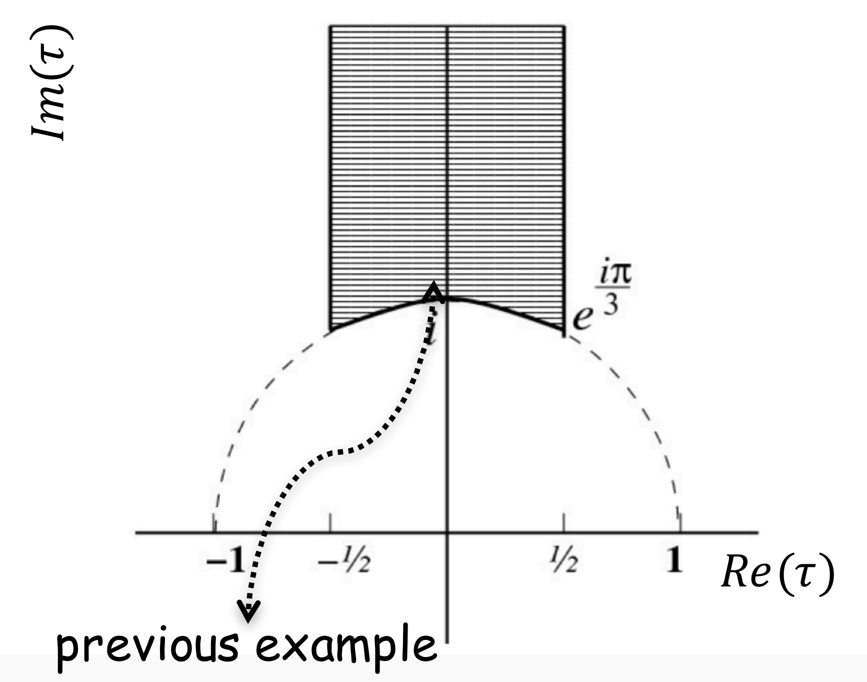
r	0.02981
δm^2 [10^{-5} eV^2]	7.326
$\lvert\Delta m^2\rvert$ [10^{-3} eV^2]	2.457
$\sin^2\theta_{12}$	0.305
$\sin^2\theta_{13}$	0.02136
$\sin^2\theta_{23}$	0.4862

$N\sigma$	1.012

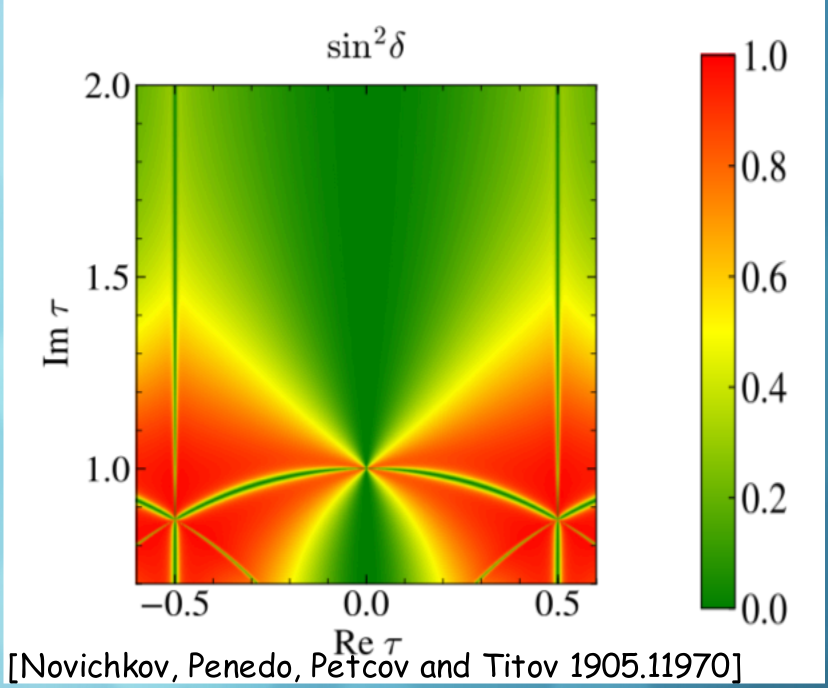
Ordering	NO
m_1 [eV]	0.01211
m_2 [eV]	0.01483
m_3 [eV]	0.05139
$\sum_i m_i$ [eV]	0.07833
$\lvert\langle m\rangle\rvert$ [eV]	0.01201
δ/π	± 1.641
α_{21}/π	± 0.3464
α_{31}/π	± 1.254

- value of τ maximizing the agreement with data is closed to fixed points, such as $\tau \approx i$ where $\tau \to -1/\tau$ is preserved

$$\mathcal{S} = \frac{\mathcal{H}}{SL(2,\mathbb{Z})}$$



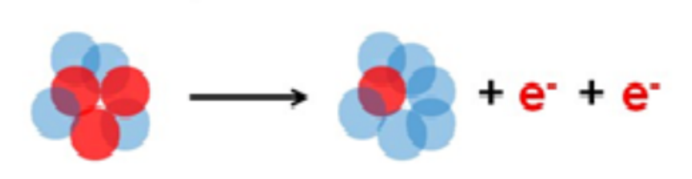
previous example



[Novichkov, Penedo, Petcov and Titov 1905.11970]

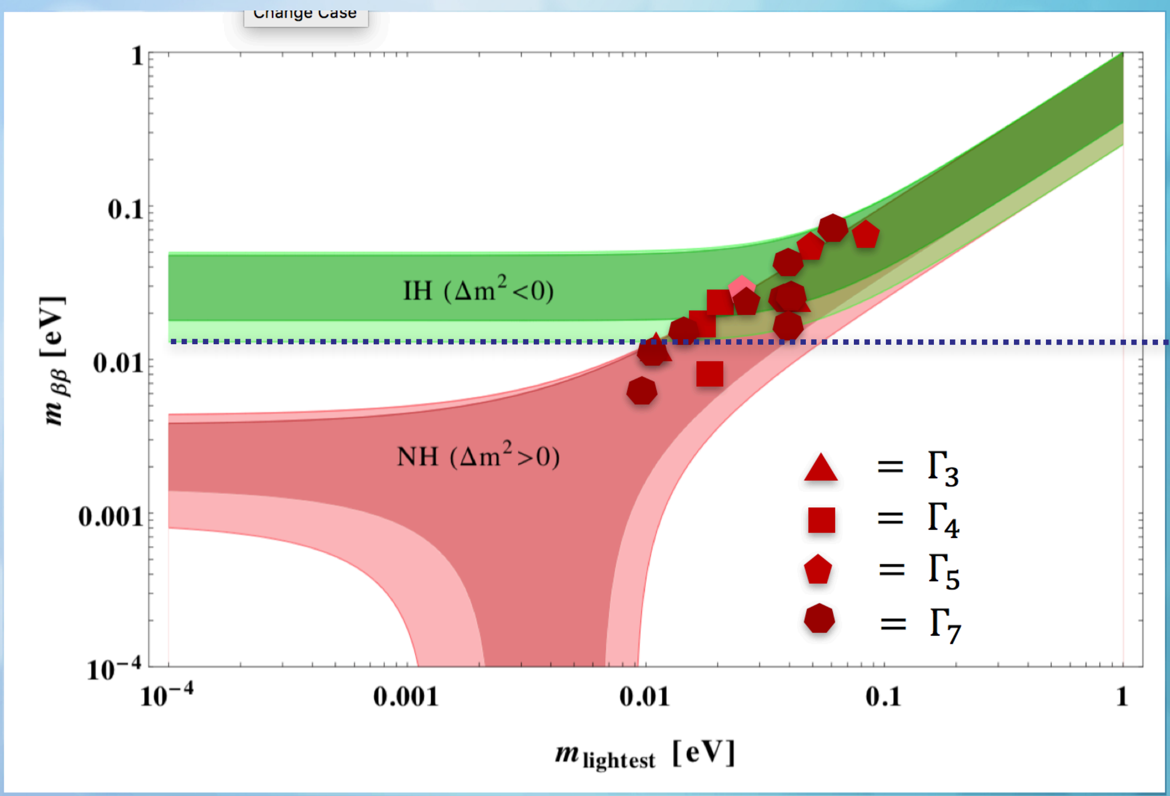
Some trend

- neutrinoless double beta decay



$|m_{ee}|$=$|\sum_i U_{ei}\, m_i\,|$

- most of the solutions with NO prefer a nearly degenerate spectrum $m_1 > \;10\, meV$ and $|m_{ee}|$ on the high side of allowed range



sensitivity of next experiments

[$0\nu\beta\beta$ region from: S. Dell'Oro, S. Marcocci and F. Vissani, 1404.2616]

generalization

- Ding, F. Liu 2010.07952
- Ding, F. Liu 2102.06716

- $\mathcal{H} = SL(2,\mathbb{R})/SO(2)$ → $\mathcal{M} = G/K$
- $SL(2,\mathbb{R})$ → G Lie group
- $SO(2)$ → K compact subgroup of G
- $SL(2,\mathbb{Z})$ → $\Gamma \subseteq G$ candidate (discrete, gauge) flavour group
- $SL(2,\mathbb{Z}_N) = \dfrac{SL(2,\mathbb{Z})}{\Gamma'(N)}$ → $\Gamma_{finite} = \dfrac{\Gamma}{\Gamma_{norm}}$

  $\Gamma'(N) \subseteq SL(2,\mathbb{Z})$

  Γ_{norm} normal subgroup of Γ
- $(c\tau + d)^{-k}$ → $j(\gamma,\tau)$ automorphy factor

example: symplectic modular invariance

- Ding, F. Liu 2010.07952
- Ding, F. Liu 2102.06716

- $\mathcal{H} = SL(2,\mathbb{R})/SO(2)$ → $\mathcal{M} = Sp(2g,\mathbb{R})/U(g)$
  Siegel upper half plane
- $SL(2,\mathbb{R})$ → $G = Sp(2g,\mathbb{R})$
- $SO(2)$ → $K = U(g)$
- $SL(2,\mathbb{Z})$ → $\Gamma = Sp(2g,\mathbb{Z})$
  Siegel modular group
- $SL(2,\mathbb{Z}_N) = \dfrac{SL(2,\mathbb{Z})}{\Gamma'(N)}$
  $\Gamma'(N) \subseteq SL(2,\mathbb{Z})$
  → $\Gamma_{g,n}$ finite Siegel modular group
  genus level
- $(c\tau + d)^{-k}$ → $j(\gamma,\tau) = \det(C\tau + D)$
  automorphy factor

relation to string theory

- modular properties of Yukawa couplings in heterotic string theory compactification and orientifold compactifications of Type II strings established long ago

Hamidi, Vafa 1987
Dixon, Friedan, Martinec and Shenker 1987
Lauer, Mas and Nilles 1989, 1991
Cremades, Ibanez and Marchesano 2003
Blumenhagen, Cvetic, Langacker and Shiu 2005
…

Yukawa coupling derived from overlap of zero mode wave functions and recognized as functions with modular properties

- here: bottom-up approach
  modular invariance is postulated, and the most general EFT is constructed
  level N, weights, representations are unconstrained

- key feature: Yukawa couplings are level N modular forms of given weight; they are finitely many and can be explicitly built without reference to a UV complete theory.

- model building and pheno analysis made easier

- can be combined with TD approach

Baur, Nilles, Trautner and Vaudrevange, 1901.03251
Nilles, Ramos-Sanchez and Vaudrevange 2001.01736
Nilles, Ramos-Sanchez and Vaudrevange 2004.05200
Almumin, Chen, Knapp-Perez, Ramos-Sanchez, Ratz and Shukla, 2102.11286

Conclusions

- flavour puzzle as a vacuum problem
- Flavour symmetry and representations of matter fields from Moduli Space
- Flavour symmetry is a (discrete) gauge symmetry ⟺ redundancy of description in moduli space
- moduli space as $(G/K)/\Gamma$
- Γ flavour (gauge) symmetry is a discrete group of G
- Yukawa couplings $\mathcal{Y}(\tau)$ are modular forms
- Open questions:
  - model building
  - vacuum selection
  - flavour dependence of Kahler potential

THANK YOU!

back-up slides

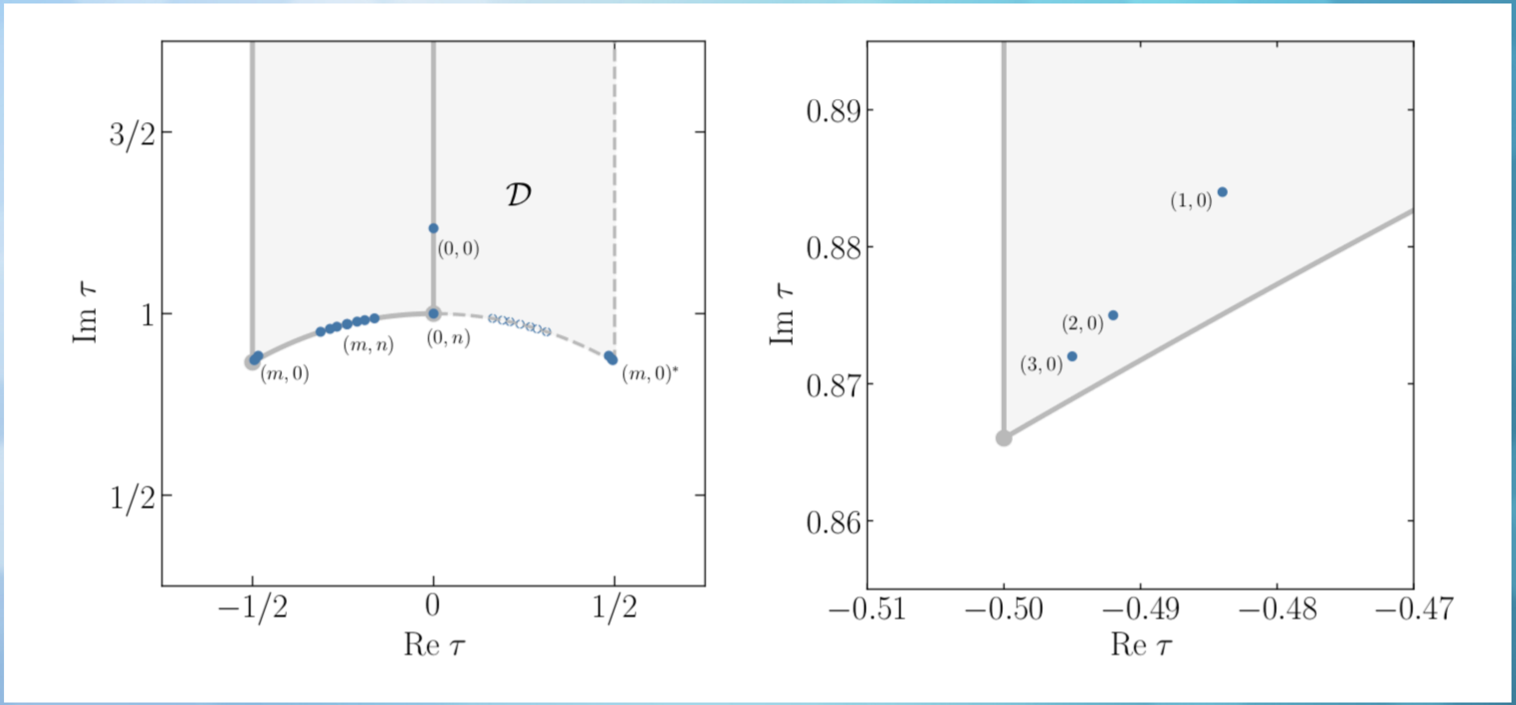
moduli stabilization

[Cvetic, Font, Ibanez, Lust and Quevedo, Nucl.Phys.B 361 (1991) 194
Kobayashi, Shimizu, Takagi, Tanimoto, Tatsuishi and Uchida, 1910.11553
Abe, Kobayashi, Uemura and Yamamoto, 2003.03512
Ishiguro, Kobayashi and Otsuka, 2011.09154]

what determines the value of τ ?

- anthropic selection
- cosmological evolution
- extrema of $V(\tau)$

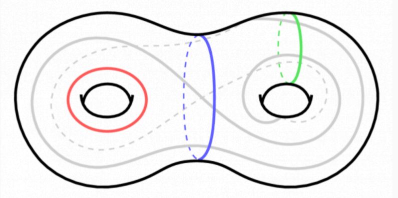
extrema of $V(\tau)$ at the border of the fundamental region and along the $Im(\tau)$ axis ?



[Novichkov, Penedo and Petcov, 2201.02020]

$G = Sp(4,\mathbb{R})$ $K = U(2)$ $\Gamma = Sp(4,\mathbb{Z})$ $N = 2$

$$\mathcal{M} = \left\{ \tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix} \middle| \ \det(Im(\tau)) > 0, tr\big(Im(\tau)\big) > 0 \right\}$$



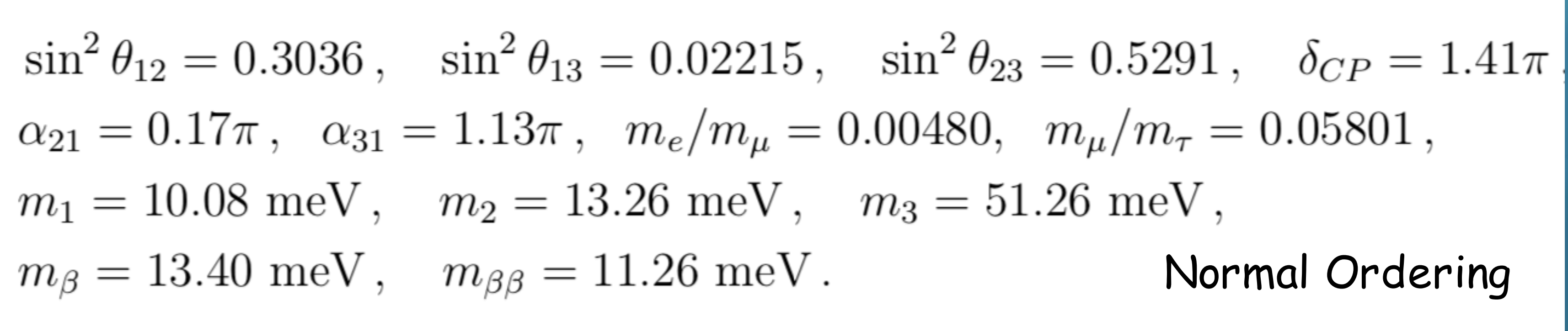
$\mathcal{M}$ restricted to $\tau_1 = \tau_2$

$\Gamma_{2,2} \rightarrow S_4 \times Z_2$ CP

	E_D^c	E_3^c	L	H_u	H_d
k_φ	-3	-1	-1	0	0
$S_4 \times Z_2$	2	1	$3'$	1	1

τ optimized at $\tau = \begin{pmatrix} \tau_1 & \tau_1/2 \\ \tau_1/2 & \tau_1 \end{pmatrix}$

5 real parameters + 1 complex τ 12 observables → 5 predictions



$$\sin^2\theta_{12} = 0.3036\,, \quad \sin^2\theta_{13} = 0.02215\,, \quad \sin^2\theta_{23} = 0.5291\,, \quad \delta_{CP} = 1.41\pi$$

$$\alpha_{21} = 0.17\pi\,, \quad \alpha_{31} = 1.13\pi\,, \quad m_e/m_\mu = 0.00480, \quad m_\mu/m_\tau = 0.05801\,,$$

$$m_1 = 10.08\text{ meV}\,, \quad m_2 = 13.26\text{ meV}\,, \quad m_3 = 51.26\text{ meV}\,,$$

$$m_\beta = 13.40\text{ meV}\,, \quad m_{\beta\beta} = 11.26\text{ meV}\,.$$

Normal Ordering

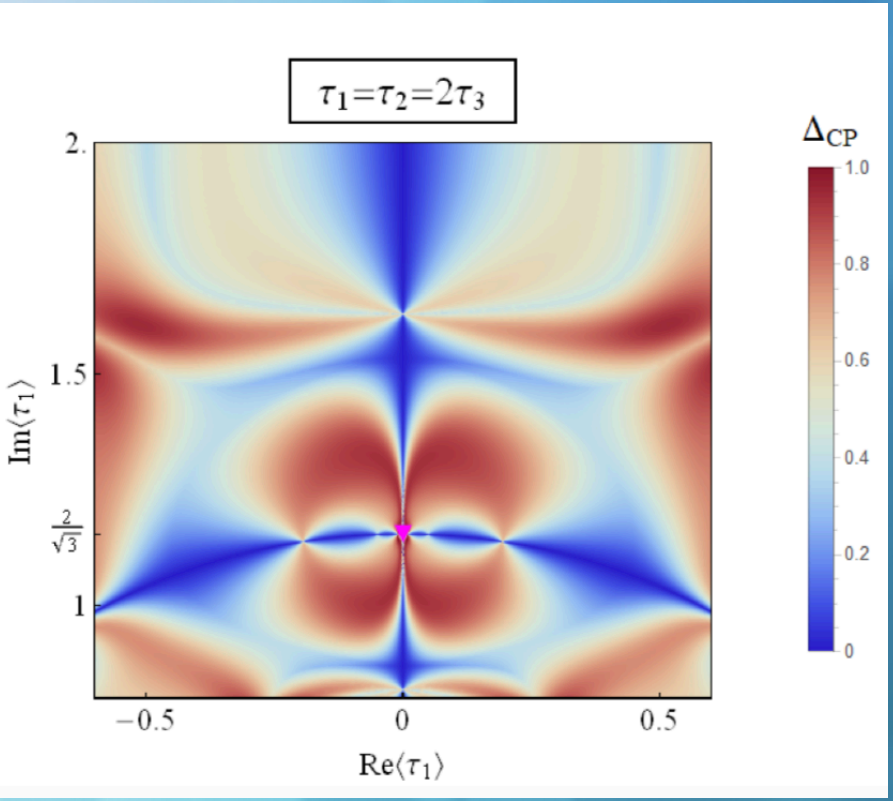
$\tau_1 = \tau_2 = 2\tau_3 =$
$-0.0283 + i\, 1.1761$

best fit point of τ
close to

$$\hat{\tau} = \frac{i}{\sqrt{3}} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$2/\sqrt{3} = 1.1547$

enhanced symmetry point:
CP & S_3 x Z_2



CP violation
masses/mixing pattern

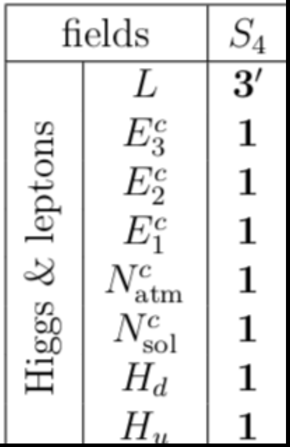
↔ small departure from $\hat{\tau}$

Orbifolds from Sp(4,Z) and
their modular symmetries

Nilles, Ramos-Sanchez, Trautner and Vaudrevange,
2105.08078

Example

matter fields

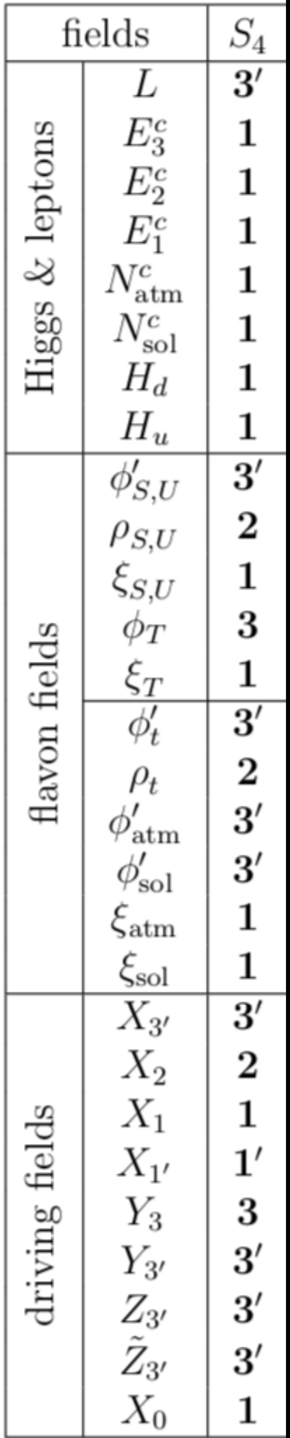


fields		S_4
Higgs & leptons	L	$\mathbf{3'}$
	E_3^c	$\mathbf{1}$
	E_2^c	$\mathbf{1}$
	E_1^c	$\mathbf{1}$
	N_{atm}^c	$\mathbf{1}$
	N_{sol}^c	$\mathbf{1}$
	H_d	$\mathbf{1}$
	H_u	$\mathbf{1}$

Example

matter fields

symmetry breaking sector

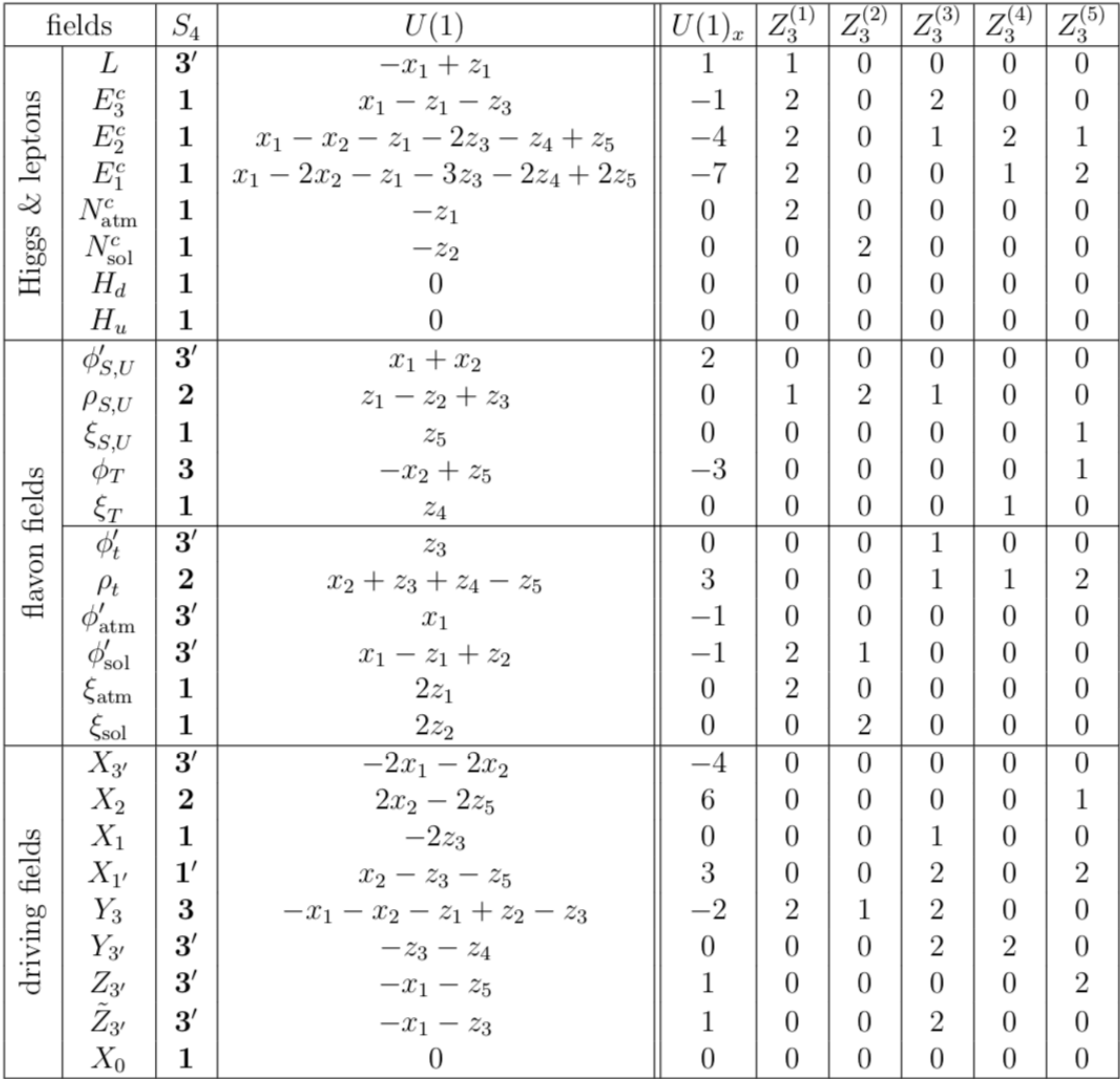


fields		S_4
Higgs & leptons	L	$\mathbf{3'}$
	E_3^c	$\mathbf{1}$
	E_2^c	$\mathbf{1}$
	E_1^c	$\mathbf{1}$
	$N^c_{\rm atm}$	$\mathbf{1}$
	$N^c_{\rm sol}$	$\mathbf{1}$
	H_d	$\mathbf{1}$
	H_u	$\mathbf{1}$
flavon fields	$\phi'_{S,U}$	$\mathbf{3'}$
	$\rho_{S,U}$	$\mathbf{2}$
	$\xi_{S,U}$	$\mathbf{1}$
	ϕ_T	$\mathbf{3}$
	ξ_T	$\mathbf{1}$
	ϕ'_t	$\mathbf{3'}$
	ρ_t	$\mathbf{2}$
	$\phi'_{\rm atm}$	$\mathbf{3'}$
	$\phi'_{\rm sol}$	$\mathbf{3'}$
	$\xi_{\rm atm}$	$\mathbf{1}$
	$\xi_{\rm sol}$	$\mathbf{1}$
driving fields	$X_{3'}$	$\mathbf{3'}$
	X_2	$\mathbf{2}$
	X_1	$\mathbf{1}$
	$X_{1'}$	$\mathbf{1'}$
	Y_3	$\mathbf{3}$
	$Y_{3'}$	$\mathbf{3'}$
	$Z_{3'}$	$\mathbf{3'}$
	$\tilde{Z}_{3'}$	$\mathbf{3'}$
	X_0	$\mathbf{1}$

Example

matter fields

symmetry breaking sector



fields		S_4	$U(1)$	$U(1)_x$	$Z_3^{(1)}$	$Z_3^{(2)}$	$Z_3^{(3)}$	$Z_3^{(4)}$	$Z_3^{(5)}$
Higgs & leptons	L	$\mathbf{3'}$	$-x_1+z_1$	1	1	0	0	0	0
	E_3^c	$\mathbf{1}$	$x_1-z_1-z_3$	-1	2	0	2	0	0
	E_2^c	$\mathbf{1}$	$x_1-x_2-z_1-2z_3-z_4+z_5$	-4	2	0	1	2	1
	E_1^c	$\mathbf{1}$	$x_1-2x_2-z_1-3z_3-2z_4+2z_5$	-7	2	0	0	1	2
	N_{atm}^c	$\mathbf{1}$	$-z_1$	0	2	0	0	0	0
	N_{sol}^c	$\mathbf{1}$	$-z_2$	0	0	2	0	0	0
	H_d	$\mathbf{1}$	0	0	0	0	0	0	0
	H_u	$\mathbf{1}$	0	0	0	0	0	0	0
flavon fields	$\phi'_{S,U}$	$\mathbf{3'}$	x_1+x_2	2	0	0	0	0	0
	$\rho_{S,U}$	$\mathbf{2}$	$z_1-z_2+z_3$	0	1	2	1	0	0
	$\xi_{S,U}$	$\mathbf{1}$	z_5	0	0	0	0	0	1
	ϕ_T	$\mathbf{3}$	$-x_2+z_5$	-3	0	0	0	0	1
	ξ_T	$\mathbf{1}$	z_4	0	0	0	0	1	0
	ϕ'_t	$\mathbf{3'}$	z_3	0	0	0	1	0	0
	ρ_t	$\mathbf{2}$	$x_2+z_3+z_4-z_5$	3	0	0	1	1	2
	ϕ'_{atm}	$\mathbf{3'}$	x_1	-1	0	0	0	0	0
	ϕ'_{sol}	$\mathbf{3'}$	$x_1-z_1+z_2$	-1	2	1	0	0	0
	ξ_{atm}	$\mathbf{1}$	$2z_1$	0	2	0	0	0	0
	ξ_{sol}	$\mathbf{1}$	$2z_2$	0	0	2	0	0	0
driving fields	$X_{3'}$	$\mathbf{3'}$	$-2x_1-2x_2$	-4	0	0	0	0	0
	X_2	$\mathbf{2}$	$2x_2-2z_5$	6	0	0	0	0	1
	X_1	$\mathbf{1}$	$-2z_3$	0	0	0	1	0	0
	$X_{1'}$	$\mathbf{1'}$	$x_2-z_3-z_5$	3	0	0	2	0	2
	Y_3	$\mathbf{3}$	$-x_1-x_2-z_1+z_2-z_3$	-2	2	1	2	0	0
	$Y_{3'}$	$\mathbf{3'}$	$-z_3-z_4$	0	0	0	2	2	0
	$Z_{3'}$	$\mathbf{3'}$	$-x_1-z_5$	1	0	0	0	0	2
	$\tilde{Z}_{3'}$	$\mathbf{3'}$	$-x_1-z_3$	1	0	0	2	0	0
	X_0	$\mathbf{1}$	0	0	0	0	0	0	0

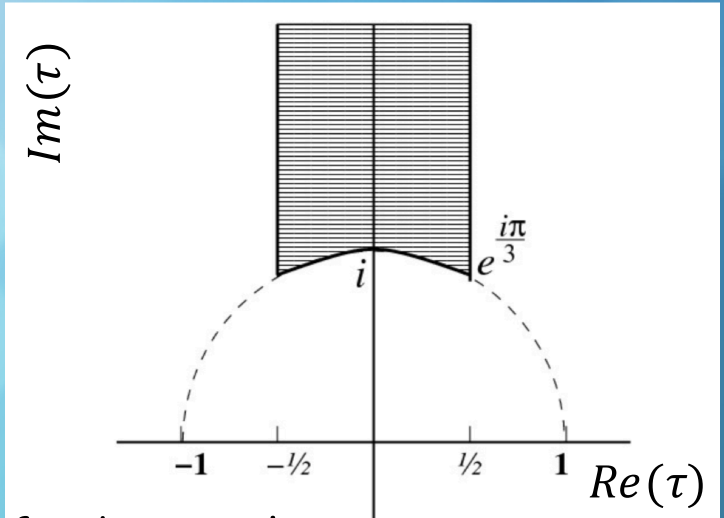
they form the (discrete, infinite) modular group $\overline{\Gamma}$ generated by

$$S:\tau\to-\frac{1}{\tau}\quad,\qquad T:\tau\to\tau+1$$

duality

discrete shift symmetry

$$S^2=1\quad,\quad(ST)^3=1$$



- can be thought of as a gauge symmetry
- with a "gauge choice" τ can be restricted to a fundamental region

most general transformation on a set of $\mathcal{N}$=1 SUSY chiral multiplets $\varphi^{(I)}$

$$\begin{cases} \tau & \to \quad \gamma\tau\equiv\dfrac{a\tau+b}{c\tau+d} \\ \varphi^{(I)} & \to \quad (c\tau+d)^{k_I}\rho^{(I)}(\gamma)\varphi^{(I)} \end{cases}$$

e.g. $\varphi^{(I)}=\begin{pmatrix} e\\ \mu\\ \tau \end{pmatrix}$

[Ferrara, Lust, Shapere and Theisen, 1989]

the weight,
a real number

unitary representation
of the finite modular group $\Gamma_N\equiv\overline{\Gamma}/\overline{\Gamma}(N)$

$N=1,2,3,\ldots$

Modular Invariance and CP

[Novichkov, Penedo, Petcov and Titov 1905.11970
see also: Baur, Nilles, Trautner and Vaudrevange,
1901.03251]

- a unique CP law consistent with the modular group [Im(τ) > 0]

$$\tau \to -\tau^*$$

[up to modular transformations]

↕

outer automorphism of $\bar{\Gamma}$

$$S \to S \quad T \to T^{-1}$$

- CP on matter multiplets

$$\varphi^{(I)} \to X_{CP}\, \varphi^{(I)*}$$

$X_{CP} = \mathbb{I}$ not restrictive if
S and T symmetric matrices
[canonical CP basis]

[in such a basis]
CP invariance →

$$g_i^* = g_i$$

on Lagrangian parameters g_i

- CP conserved ↔ τ imaginary or at the border of the fundamental region

otherwise CP spontaneously broken by $\langle\tau\rangle$

Modular Forms

k > 0 even integer

	$d(\mathcal{M}_k(\Gamma_N))$	$k=2$	$k=4$	$k\geq 6$
$\Gamma_2 \approx S_3$	$k/2+1$	2	$1+2$	…
$\Gamma_3 \approx A_4$	$k+1$	3	$1+1'+3$	…
$\Gamma_4 \approx S_4$	$2k+1$	$2+3'$	$1+2+3$ $+3'$	…
$\Gamma_5 \approx A_5$	$5k+1$	$3+3'$ $+5$	$1+3+3'$ $+4+5+5$	…

[TTT]

[F]

[PP]

[NPPT DKL]

$$\left.\begin{array}{l}\Gamma_8 \supset \Delta(96)\\ \Gamma_{16} \supset \Delta(384)\end{array}\right\} \quad k=2 \quad \rho=3$$

[KT]

[TTT = T. Kobayashi, K. Tanaka and T. H. Tatsuishi,
1803.10391
F = F. Feruglio 1706.08749
PP = J. T. Penedo and S. T. Petcov 1806.11040
NPPT = P. P. Novichkov, J. T. Penedo,
S. T. Petcov and A. V. Titov 1812.02158
DKL = G. J. Ding, S. F. King and X. G. Liu 1903.12588
KT = T. Kobayashi and S. Tamba, 1811.11384]

built in terms of
Dedekind eta function
Klein forms
Jacobi theta functions

k > 0 odd/even integer

fall in representations of homogeneous finite modular groups Γ_N'
e.g. N=3 and k=1 gives a doublet of $\Gamma_3' = T'$
[Ding, Liu 1907.01488]

Corrections from SUSY breaking

- unknown breaking mechanism. Here:

F-component of a chiral supermultiplet, gauge and modular invariant

$$X = \vartheta^2 F$$

messenger scale M

SUSY-breaking scale

$$m_{SUSY} = \frac{F}{M}$$

- most general correction term to lepton masses and mixing angles

$$\delta\mathcal{S} = \frac{1}{M^2} \int d^4x d^2\theta d^2\bar{\theta}\; X^\dagger\; f(\Phi, \bar{\Phi}) + h.c.$$

$f(\Phi, \bar{\Phi})$ has dimension 3, determined by gauge invariance and lepton number conservation (treating Λ as spurion with L=+2)

$$\Rightarrow \quad \delta\mathcal{W}/\mathcal{W} \approx \delta\mathcal{Y}/\mathcal{Y} \approx \frac{m_{SUSY}}{M}$$

tiny, if sufficient gap between m_{SUSY} and M

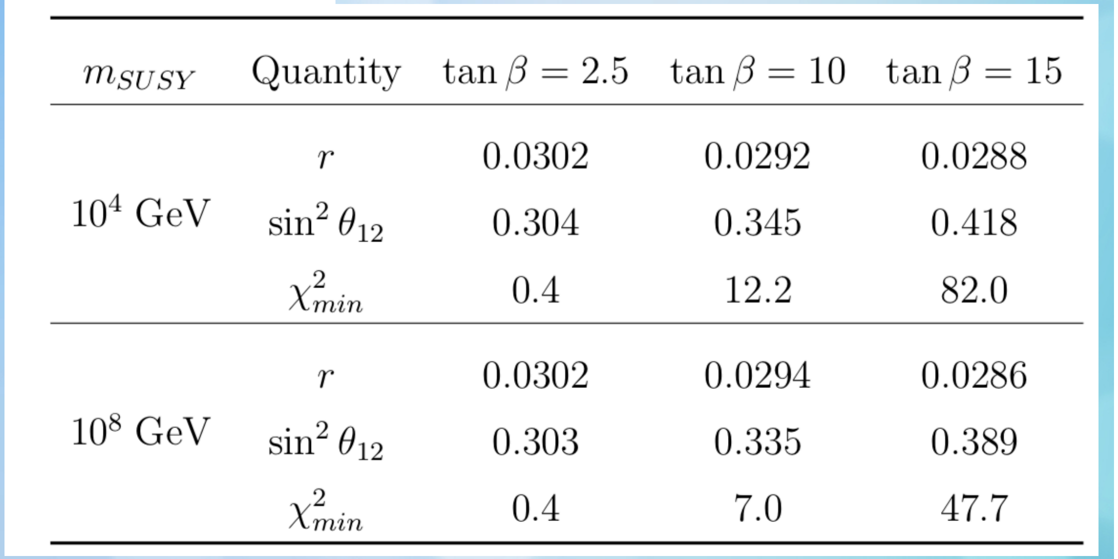
10^{-10} for m_{SUSY}=10^8 GeV
$M = 10^{18}$ GeV

Corrections from RGE

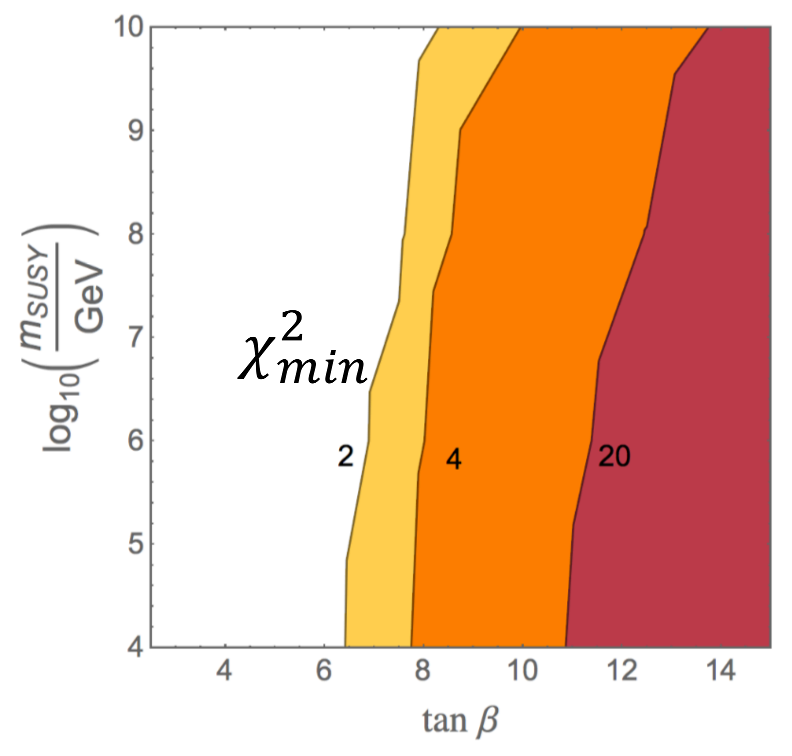
- **Model 1 (IO)**

  - r and $sin^2\vartheta_{12}$ mostly affected, at large $tan\beta$

$\Lambda = 10^{15}$ GeV



m_{SUSY}	Quantity	$\tan\beta = 2.5$	$\tan\beta = 10$	$\tan\beta = 15$
10^4 GeV	r	0.0302	0.0292	0.0288
	$\sin^2\theta_{12}$	0.304	0.345	0.418
	χ^2_{min}	0.4	12.2	82.0
10^8 GeV	r	0.0302	0.0294	0.0286
	$\sin^2\theta_{12}$	0.303	0.335	0.389
	χ^2_{min}	0.4	7.0	47.7



- **Model 2 (NO)**

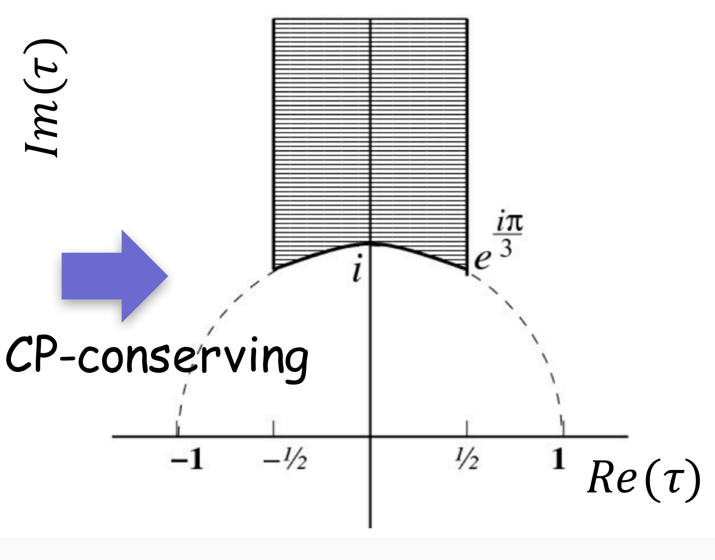
  negligible corrections for $tan\beta$ up to 25 and m_{SUSY} as low as 10^4 GeV

problems

vacuum selection

what determines the value of τ ?

extrema of $V(\tau)$ at the border of the fundamental region and along the $Im(\tau)$ axis ?
[Cvetic, Font, Ibanez, Lust and Quevedo, Nucl.Phys.B 361 (1991) 194]



CP-conserving

Novichkov, Penedo, Petcov and Titov
1905.11970

corrections from kinetic terms

previous models adopt minimal kinetic terms

$$K(\Phi,\bar{\Phi}) = -h\log(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\varphi^{(I)}|^2$$

no extra parameters

non-minimal kinetic terms
allowed by modular invariance

additional parameters
reduce predictability

[Chen, Ramon-Sanchez, Ratz 1909.06910]

for instance, in models 1 and 2

$$K \;=\; \alpha_0\,(-\mathrm{i}\,\tau+\mathrm{i}\,\bar{\tau})^{-1}\,(\bar{L}\,L)_{\mathbf{1}} + \sum_{k=1}^{7} \alpha_k\,(-\mathrm{i}\,\tau+\mathrm{i}\,\bar{\tau})\,(Y\,L\,\bar{Y}\,\bar{L})_{\mathbf{1},\,k} + \dots\;.$$

More symmetries ? Eclectic Flavour Groups

- less parameters in a more symmetric framework?
  which is the most general framework including Γ_N ?

[Nilles, Ramos-Sanchez and Vaudrevange 2001.01736]

look for G_{ecl} including new transformations leaving τ invariant

$$\tau \overset{\gamma}{\rightarrow} \gamma\tau \equiv \frac{a\,\tau+b}{c\,\tau+d} \qquad \varphi \overset{\gamma}{\rightarrow} (c\,\tau+d)^k\,\rho(\gamma)\,\varphi$$

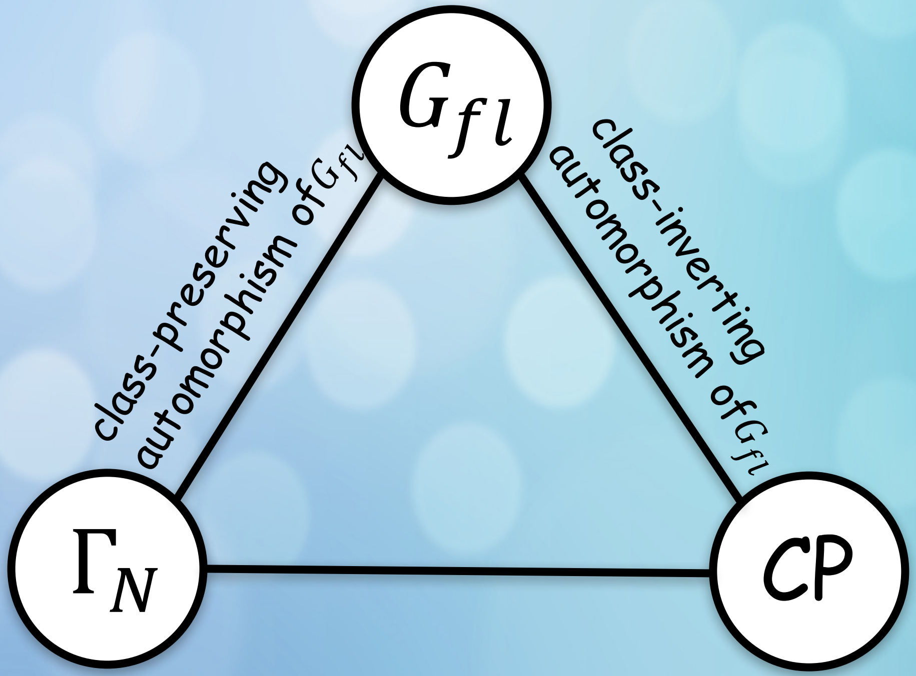
$$\tau \overset{g}{\rightarrow} \tau \qquad \varphi \overset{g}{\rightarrow} \varrho(g)\,\varphi \qquad g \in G_{fl}$$

- consistency condition

$$\rho(\gamma)\,\varrho(g)\,\rho(\gamma^{-1}) = \varrho(g')$$

G_{fl} is a normal subgroup in G_{ecl}

not all G_{fl} suitable to such an extension



unification of flavour, CP and modular symmetries

G_{ecl} shown to restrict the allowed Kahler potential

[Nilles, Ramos-Sanchez and Vaudrevange 2004.05200]

Top-down approach

- large freedom in BU approach: choice of level, weights, representations,…

- G_{ecl} naturally realized in orbifold string compactification

Baur, Nilles, Trautner and Vaudrevange, 1901.03251,

H. Ohki, S. Uemura and R. Watanabe, 2003.04174]

- a generalized lattice defines the consistent background for string propagation

- for 2 dimensions compactified on T^2: 4-dimensional Narain lattice $\Gamma_{2,2}$
  - even Lorentzian
  - self-dual
  - parametrized by $G_{11}, G_{12}, G_{22}, B_{12}$

$$\tau = \frac{G_{12}}{G_{11}} + i\,\frac{\sqrt{G}}{G_{11}}$$

complex structure

$$\rho = B_{12} + i\sqrt{G}$$

Kahler structure

- if $\tau = e^{i\frac{2\pi}{3}}$ an orbifold T^2/Z_3 can be defined, specified by a space group $S_N = \{\Theta, E\ N\}$

- the symmetries of the theory are transformations leaving S_N invariant

- they reproduce the eclectic group $G_{ecl} = \Omega(1)$ which combines G_{fl}=$\Delta(54)$ with the finite modular group Γ'_3

tests of modulus couplings

G-J. Ding, FF,
2003.13448

non standard neutrino interactions

$$\mathcal{L} = i \sum_{f=e,e^c,\nu} \bar{f}\,\bar{\sigma}^\mu \partial_\mu f + \frac{1}{2}\partial_\mu \varphi_\alpha \partial^\mu \varphi_\alpha - \frac{1}{2} M_\alpha^2 \varphi_\alpha^2 - (m_e + \mathcal{Z}^e_\alpha \varphi_\alpha) e^c e - \frac{1}{2}\nu(m_\nu + \mathcal{Z}^\nu_\alpha \varphi_\alpha)\nu + h.c. + \dots .$$

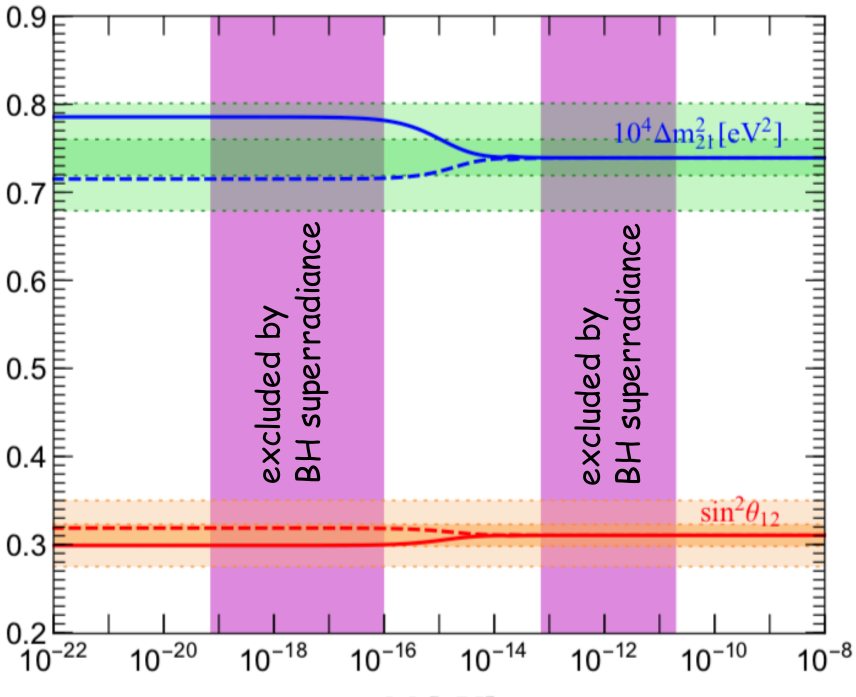
$$\tau = \langle \tau \rangle + \frac{\varphi_u + i\,\varphi_v}{\sqrt{2}}$$

in medium with non-zero electron number density

small, unless the modulus is very light

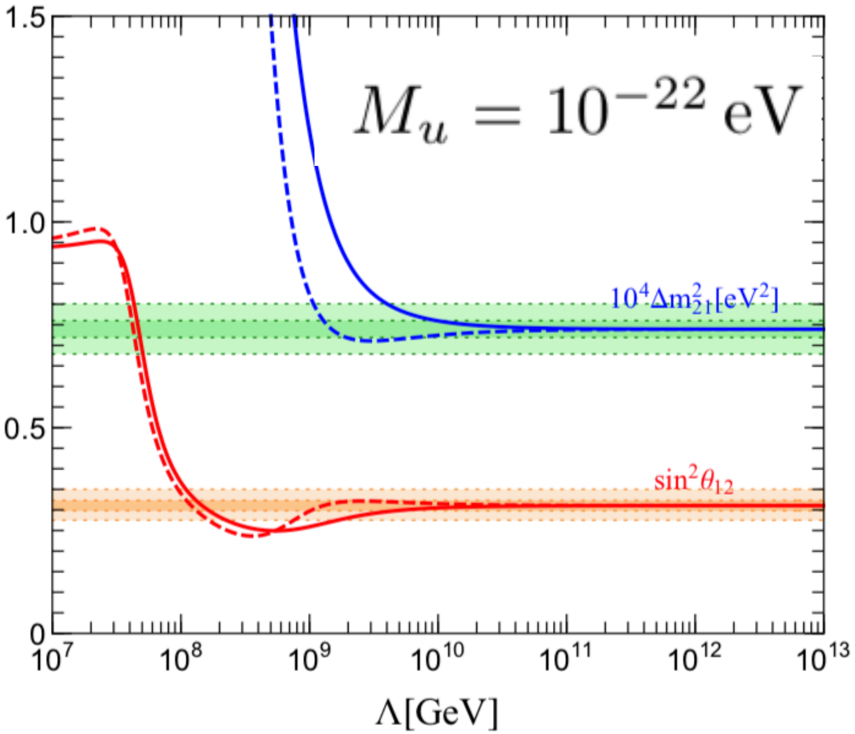
$$\delta m_\nu(0) = -n_e^0 \frac{\mathrm{Re}(\mathcal{Z}^e)\mathcal{Z}^\nu}{M^2(R)} ,$$

in the sun:



$\Lambda = 5 \times 10^9$ GeV

[modulus VEV]



$\mathcal{N}$=1 sugra invariant theories

This setup can be easily extended to the case of $N = 1$ local supersymmetry where Kahler potential and superpotential are not independent functions since the theory depends on the combination

$$\mathcal{G}(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) + \log w(\Phi) + \log w(\bar{\Phi}) \quad . \tag{20}$$

The modular invariance of the theory can be realized in two ways. Either $K(\Phi, \bar{\Phi})$ and $w(\Phi)$ are separately modular invariant or the transformation of $K(\Phi, \bar{\Phi})$ under the modular group is compensated by that of $w(\Phi)$. An example of this second possibility is given by the Kahler potential of eq. (14), with the superpotential $w(\Phi)$ transforming as

$$w(\Phi) \to e^{i\alpha(\gamma)}(c\tau + d)^{-h} w(\Phi) \tag{21}$$

In the expansion (17) the Yukawa couplings $Y_{I_1 \ldots I_n}(\tau)$ should now transform as

$$Y_{I_1 \ldots I_n}(\gamma\tau) = e^{i\alpha(\gamma)}(c\tau + d)^{k_Y(n)} \rho(\gamma)\ Y_{I_1 \ldots I_n}(\tau) \quad , \tag{22}$$

with $k_Y(n) = k_{I_1} + ... + k_{I_n} - h$ and the representation ρ subject to the requirement 2. When we have $k_{I_1} + ... + k_{I_n} = h$, we get $k_Y(n) = 0$ and the functions $Y_{I_1 \ldots I_n}(\tau)$ are τ-independent constants. This occurs for supermultiplets belonging to the untwisted sector in the orbifold compactification of the heterotic string.

choose the space $\mathcal{M}$ and guess the flavor symmetry G_f

interesting candidates for $\mathcal{M}$: Hermitian Symmetric Spaces [HSS]

- [Ding, F., Liu, 2010.07952]

- HSS are Kähler
- non-compact HSS as moduli space in sugra and compactified strings
- related to automorphic forms, building blocks of Yukawa couplings

connected Lie group

$$\mathcal{M} = G/K$$

compact subgroup

$$\mathcal{M} \begin{cases} \text{Riemannian metric} \\ \text{integrable complex structure} \end{cases}$$

Lie algebra

$$\mathcal{G} = \mathcal{V} \oplus \mathcal{A} \qquad [\mathcal{V},\mathcal{V}]\subset \mathcal{V} \qquad [\mathcal{V},\mathcal{A}]\subset \mathcal{A} \qquad [\mathcal{A},\mathcal{A}]\subset \mathcal{V}$$

automorphism	$\mathcal{V}+\mathcal{A} \leftrightarrow \mathcal{V}-\mathcal{A}$
compact	$B(\mathcal{A},\mathcal{A}) < 0$
noncompact	$B(\mathcal{A},\mathcal{A}) > 0$
euclidean	$B(\mathcal{A},\mathcal{A}) = 0$

$B(\mathcal{V},\mathcal{V}) < 0$

$B(X,Y)$ Killing form

[classification: Calabi, Visentini 1960, Wolf 1964]

an example at genus $g = 2$

$$\mathcal{M} = \left\{ \tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix} \middle| \ \det(Im(\tau)) > 0, tr\big(Im(\tau)\big) > 0 \right\}$$

fundamental domain $\mathcal{M}/\Gamma$: physically inequivalent vacua

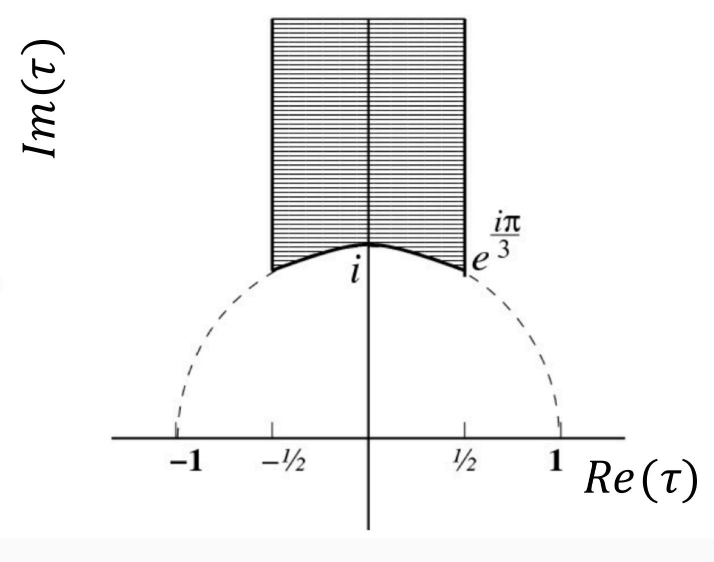
$$\mathcal{F}_2 = \left\{ \tau \in \mathcal{H}_2 \ \middle| \ \begin{cases} |\mathrm{Re}(\tau_1)| \leq 1/2, \quad |\mathrm{Re}(\tau_3)| \leq 1/2. \quad |\mathrm{Re}(\tau_2)| \leq 1/2, \\ \mathrm{Im}(\tau_2) \geq \mathrm{Im}(\tau_1) \geq 2\mathrm{Im}(\tau_3) \geq 0 \\ |\tau_1| \geq 1, \quad |\tau_2| \geq 1, \quad |\tau_1 + \tau_2 - 2\tau_3 \pm 1| \geq 1 \\ |\det(\tau + \mathcal{E}_i)| \geq 1 \end{cases} \right\}$$

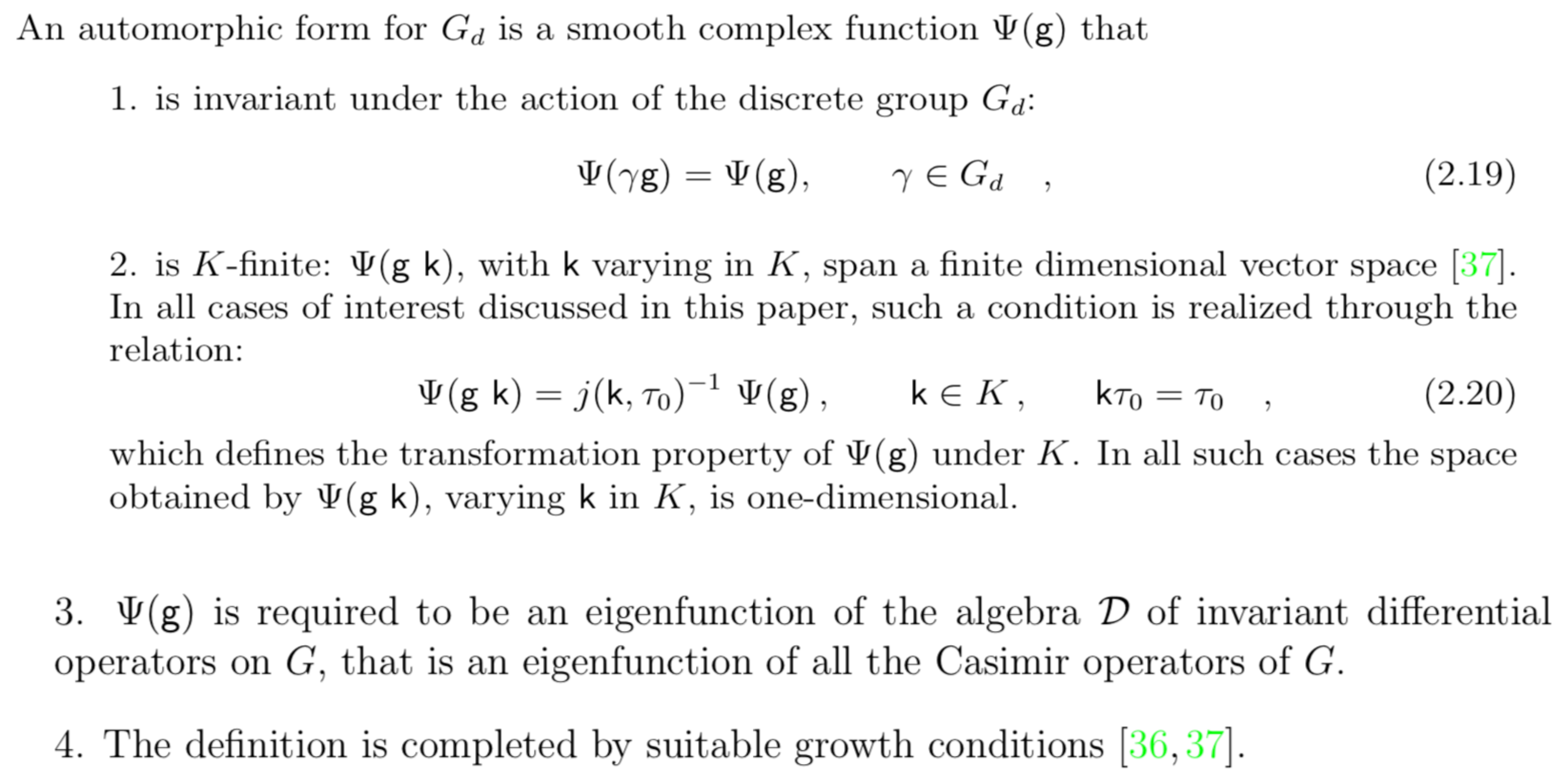
$\{\mathcal{E}_i\}$ includes the following 15 matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \pm1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & \pm1 \end{pmatrix}, \quad \begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix},$$

$$\begin{pmatrix} \pm1 & 0 \\ 0 & \mp1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \pm1 \\ \pm1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \pm1 & \pm1 \\ \pm1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \pm1 \\ \pm1 & \pm1 \end{pmatrix}$$

[cfr. g=1]





An automorphic form for G_d is a smooth complex function $\Psi(\mathsf{g})$ that

1. is invariant under the action of the discrete group G_d:

$$\Psi(\gamma \mathsf{g}) = \Psi(\mathsf{g}), \qquad \gamma \in G_d \quad , \tag{2.19}$$

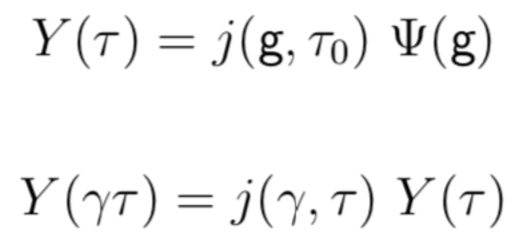
2. is K-finite: $\Psi(\mathsf{g}\ \mathsf{k})$, with k varying in K, span a finite dimensional vector space [37]. In all cases of interest discussed in this paper, such a condition is realized through the relation:

$$\Psi(\mathsf{g}\ \mathsf{k}) = j(\mathsf{k}, \tau_0)^{-1}\ \Psi(\mathsf{g})\,, \qquad \mathsf{k} \in K\,, \qquad \mathsf{k}\tau_0 = \tau_0 \quad , \tag{2.20}$$

which defines the transformation property of $\Psi(\mathsf{g})$ under K. In all such cases the space obtained by $\Psi(\mathsf{g}\ \mathsf{k})$, varying k in K, is one-dimensional.

3. $\Psi(\mathsf{g})$ is required to be an eigenfunction of the algebra $\mathcal{D}$ of invariant differential operators on G, that is an eigenfunction of all the Casimir operators of G.

4. The definition is completed by suitable growth conditions [36, 37].



$$Y(\tau) = j(\mathsf{g}, \tau_0)\ \Psi(\mathsf{g})$$

$$Y(\gamma\tau) = j(\gamma, \tau)\ Y(\tau)$$

the finite Siegel modular groups $\Gamma_{2,n}$ are too big: $|\Gamma_{2,2}| = 720, |\Gamma_{2,3}| = 51840, \ldots$

choose an invariant subspace of $G/K = Sp(4,\mathbb{R})/U(2)$

$$G/K \to \Omega = \{\tau \in G/K|\ H\tau = \tau\}$$

$$H \subset \Gamma$$

here the flavour group can be restrict to the normalizer $N(H)$

$$\Gamma \to N(H)=\{\gamma \in \Gamma|\gamma^{-1}H\gamma = H\}$$

for instance $\tau_1 = \tau_2$

$$\Omega=\left\{\tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_1 \end{pmatrix} \middle| \tau \in G/K\right\}$$

$$H=Z_2\times Z_2$$

working here $\Gamma_{2,2}$
is projected into the smaller group $S_4\times Z_2$

automorphic forms

- real periodic functions

$Y(t+1) = Y(t)$ $\quad t \in \mathbb{R}$ $\quad$ basis: $\begin{cases} \sin p\, 2\pi\, t \\ \cos p\, 2\pi\, t \end{cases}$

- an equivalent description

G continuous translation group $\quad g\, t = t + a$

generator of G: $-i\,\frac{d}{dt}$ $\quad$ Casimir: $\Delta = -\frac{d^2}{dt^2}$

no points fixed under G or subgroups

G_{dis} discrete subgroup $\mathbb{Z}$ $\quad \gamma\, t = t + n$

$\mathbb{R} \approx G$ $\quad t = g\, t_0$ $\quad t_0 = 0 \quad a = t$

- $\Psi(g) \equiv Y(t)$ $\quad \Psi(\gamma g) = \Psi(g)$ $\quad \mathbb{Z}$ invariance

- $\Delta\, \Psi_p(g) = \; p^2\, \Psi_p(g)$ $\quad$ basis elements are eigenvalues of Δ

classical modular forms

holomorphic

- $Y(\gamma\tau) = j(\gamma,\tau)Y(\tau)$ $\qquad$ $\mathcal{H} = \{Im\,\tau > 0\}$

$$\gamma\tau = \frac{a\tau+b}{c\tau+d}$$

$\gamma \in G_{dis} = \mathrm{SL}(2,\mathbb{Z})$ (or one of its normal subgroups)

$j(\gamma,\tau) = (c\tau+d)^{-k}$ automorphy factor

- can $Y(\tau)$ can be seen as a function $\Psi(g)$ acting on a group $G \supseteq G_{dis}$?

candidate $G = SL(2,\mathbb{R})$

$$g\tau = \frac{a\tau+b}{c\tau+d}$$

$ad - bc = 1$

generators

$[E,F] = H$

$[H,E] = +2E$

$[H,F] = -2F$

Casimir

$\Delta = H^2 - 2H + 4EF$

mismatch: $\dim(\mathcal{H}) = 2$ $\quad$ $\dim(G) = 3$

$t_0 = i$ fixed point under $K = SO(2) \subseteq SL(2,\mathbb{R})$

$$ki \equiv \frac{\cos\theta\, i - \sin\theta}{\sin\theta\, i + \cos\theta} = i$$

$$\mathcal{H} \approx \frac{SL(2,\mathbb{R})}{SO(2)} = G/K$$

$$\tau = g\,\tau_0 = gk\,\tau_0$$

define

$$\Psi(g) = j(g,\tau_0)^{-1}\,Y(g\,\tau_0)$$

- $$\Psi(\gamma g) = \Psi(g)$$

  $\Psi(g)$ is G_{dis}-invariant

- $$\Psi(gk) = j(k,\tau_0)^{-1}\,\Psi(g)$$

- holomorphy of $Y(\tau)$

  ⬇

  $$\Delta\Psi(g) = k(k-2)\Psi(g)$$

$$\mathcal{H} \approx \frac{SL(2,\mathbb{R})}{SO(2)} = G/K$$

$$\tau = g\,\tau_0 = gk\,\tau_0$$

define

$$\Psi(g) = j(g,\tau_0)^{-1}\, Y(g\,\tau_0)$$

🟥 $\Psi(\gamma g) = \Psi(g)$

$\Psi(g)$ is G_{dis}-invariant

🟦 $Y(\gamma\tau) = j(\gamma,\tau) Y(\tau)$

🟥 $\Psi(gk) = j(k,\tau_0)^{-1}\, \Psi(g)$

🟦 $Y(\tau)$ is K-invariant

🟥 holomorphy of $Y(\tau)$

⇓

$$\Delta\Psi(g) = k(k-2)\Psi(g)$$

🟦 $Y(\tau)$ holomorphic

$\mathcal{M} = \mathcal{H}$ is a quotient space G/K

- $G = SL(2,\mathbb{R})$ $K = SO(2)$ maximal compact subgroup of $SL(2,\mathbb{R})$

$$g = \begin{pmatrix} \sqrt{y} & x\sqrt{y^{-1}} \\ 0 & \sqrt{y^{-1}} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$y > 0$

unique decomposition as $g = g_{coset}\, k$

$\tau_0 \equiv i$ fixed point under $K = SO(2)$

$$ki \equiv \frac{\cos\theta\, i - \sin\theta}{\sin\theta\, i + \cos\theta} = i$$

$g\tau_0 = x + i\, y \equiv \tau$

$\mathcal{H} = SL(2,\mathbb{R})/SO(2)$

$\mathcal{M} = G/K$

- $\Gamma = SL(2,\mathbb{Z})$ and $\Gamma'(N)$ are discrete subgroups of $G = SL(2,\mathbb{R})$

Physics:

in the limit of vanishing Yukawa couplings, the modulus τ is the Goldstone boson of the breaking G down to K. When Yukawas are turned on, a residual discrete subroup Γ of G is gauged.

$\mathcal{N}$=1 SUSY invariant theories

- Yukawa interactions in $\mathcal{N}$=1 global SUSY [extension to $\mathcal{N}$=1 SUGRA straightforward]

$$S = \int d^4x d^2\theta \, w(\tau,\varphi) + h.c + \int d^4x d^2\theta d^2\bar{\theta} \, K(\tau,\varphi,\bar{\tau},\bar{\varphi})$$

superpotential =
Yukawa interactions

Kahler potential =
kinetic terms

- a minimal Kahler potential

$$K = -h \log Z(\tau,\bar{\tau}) + \sum_I Z(\tau,\bar{\tau})^{k_I} \left|\varphi^{(I)}\right|^2$$

for general
G, K, Γ

$$Z(\tau,\bar{\tau}) = [j^+(\gamma,\tau_0) j(\gamma,\tau_0)]^{-1}$$

$$\mathcal{M} \ni \tau = \gamma \cdot \tau_0 \qquad \gamma \in G$$

$$K \cdot \tau_0 = \tau_0$$

in our previous example:

$$K = -h \log \det(-i\tau + i\tau^+) \; + \sum_I [\det(-i\tau + i\tau^+)]^{k_I} \left|\varphi^{(I)}\right|^2$$

$h > 0$

action on matter fields

- automorphy factor $j(\gamma,\tau)$ $\quad j(\gamma_1\gamma_2,\tau)=j(\gamma_1,\gamma_2\,\tau)\,j(\gamma_2,\tau)$

cocycle condition

- unitary representation $\rho^I(\gamma)$ of (a finite copy of) Γ

finite copy of Γ: $\Gamma_{finite}=\Gamma/\Gamma_{norm}$

where $\Gamma_{norm}\subset\Gamma$ normal subgroup of finite index

- nonlinear realization of Γ

$$\varphi^{(I)}\rightarrow j(\gamma,\tau)^{k_I}\,\rho^I(\gamma)\,\varphi^{(I)} \qquad \gamma\in\Gamma$$

weight

unitary representation of Γ_{finite}

in the limit of vanishing Yukawa couplings the theory describes the SSB of $G = SL(2,\mathbb{R})$ into $K = SO(2)$

$$\mathcal{L}_{Kin}(\tau,\varphi\ ) = \frac{h}{[-i(\tau-\bar{\tau})]^2}\partial_\mu\bar{\tau}\,\partial^\mu\tau$$

invariant under $SL(2,\mathbb{R})$

$$g\tau = \frac{a\tau+b}{c\tau+d} \qquad ad-bc=1$$

in the vacuum $\tau_0 \equiv i$

the subgroup $K = SO(2)$ is unbroken

$$ki \equiv \frac{\cos\theta\, i - \sin\theta}{\sin\theta\, i + \cos\theta} = i$$

τ describes the 2 Goldstone bosons of the breaking $SL(2,\mathbb{R}) \rightarrow SO(2)$

$$\mathcal{H} \ = SL(2,\mathbb{R})/SO(2)$$

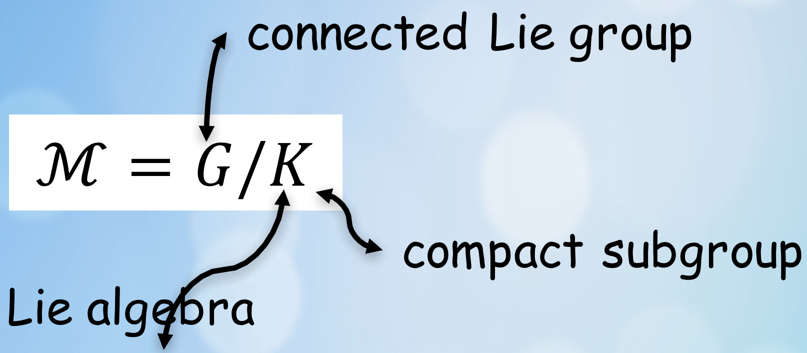
$\Gamma = SL(2,\mathbb{Z})$ is a discrete subgroup of $G = SL(2,\mathbb{R})$

when Yukawas are turned on, a residual discrete subroup Γ of G is gauged.

$$\mathcal{S} = \frac{\mathcal{M}}{\Gamma} = \frac{SL(2,\mathbb{R})/SO(2)}{SL(2,\mathbb{Z})}$$

candidates for $\mathcal{M}$: Hermitian Symmetric Spaces [HSS]

\- [Ding, F., Liu, 2010.07952]



$$\mathcal{G} = \mathcal{V} \oplus \mathcal{A} \qquad [\mathcal{V},\mathcal{V}] \subset \mathcal{V} \qquad [\mathcal{V},\mathcal{A}] \subset \mathcal{A} \qquad [\mathcal{A},\mathcal{A}] \subset \mathcal{V}$$

automorphism $\quad \mathcal{V} + \mathcal{A} \leftrightarrow \mathcal{V} - \mathcal{A}$

- HSS are Kähler
- non-compact HSS as moduli space in sugra and compactified strings
- related to automorphic forms, building blocks of Yukawa couplings

- flavour symmetry: a discrete subgroup Γ of G

$$\mathcal{M} = \mathcal{M}_c \times \mathcal{M}_{nc} \times \mathcal{M}_e$$

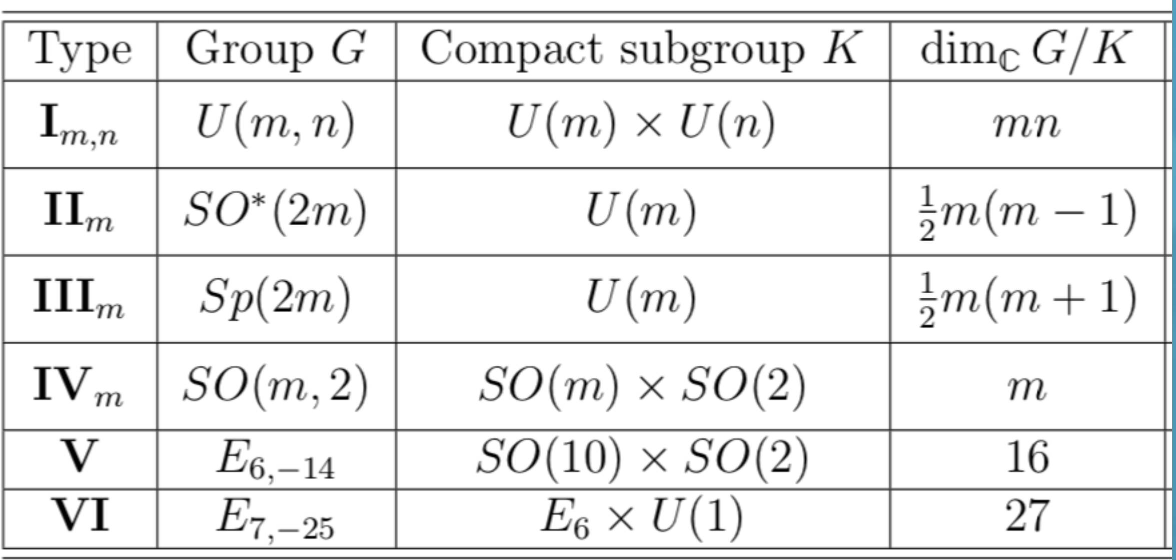
classification of irreducible HHS [Calabi, Visentini 1960, Wolf 1964]

$$\mathcal{M}_{irr} = G/K$$

simply connected Lie group

maximal compact subgroup

for noncompact



Type	Group G	Compact subgroup K	$\dim_{\mathbb{C}} G/K$
$\mathbf{I}_{m,n}$	$U(m,n)$	$U(m) \times U(n)$	mn
$\mathbf{II}_m$	$SO^*(2m)$	$U(m)$	$\frac{1}{2}m(m-1)$
$\mathbf{III}_m$	$Sp(2m)$	$U(m)$	$\frac{1}{2}m(m+1)$
$\mathbf{IV}_m$	$SO(m,2)$	$SO(m) \times SO(2)$	m
$\mathbf{V}$	$E_{6,-14}$	$SO(10) \times SO(2)$	16
$\mathbf{VI}$	$E_{7,-25}$	$E_6 \times U(1)$	27

compact case reproduced through $(\mathcal{V}, \mathcal{A}) \rightarrow (\mathcal{V}, i\,\mathcal{A})$

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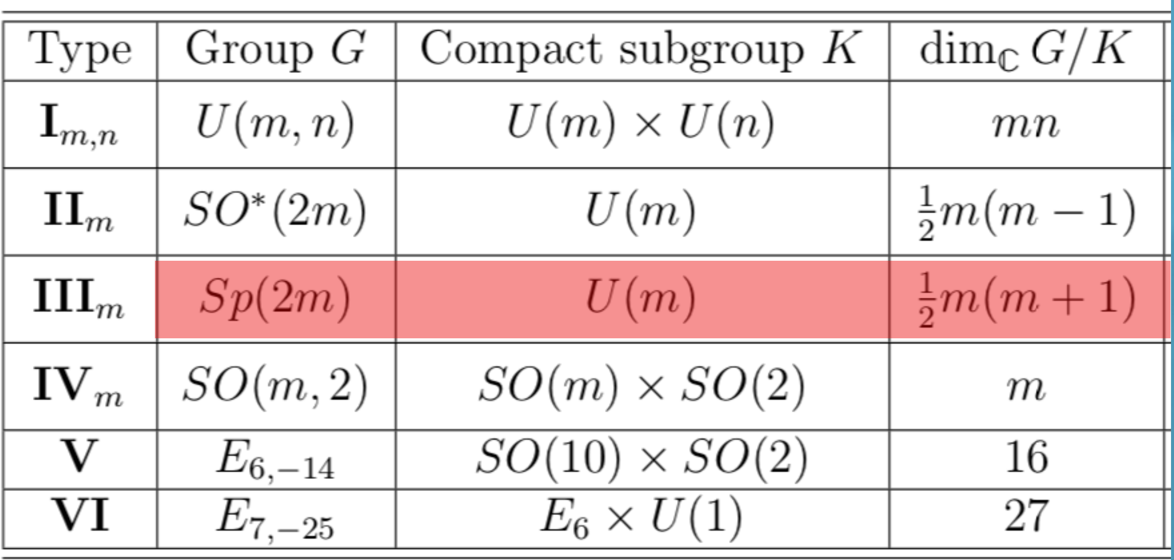

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compact case reproduced through $(\mathcal{V},\mathcal{A}) \to (\mathcal{V}, i\,\mathcal{A})$

$G = Sp(2g,\mathbb{R}) \qquad K = U(g)$

- $G = Sp(2g,\mathbb{R}) \qquad K = U(g)$ $\qquad \mathcal{M} = G/K$ noncompact

  genus

  $$\mathcal{M}=\{\tau \in GL(g,\mathbb{C}) | \tau^t = \tau, Im(\tau) > 0\}$$

  Siegel upper half plane complex dimension $g(g+1)/2$

- action of G on τ

  $$\tau \to \gamma\tau = (A\tau + B)(C\tau + D)^{-1}$$

  $$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \gamma^t J \gamma = J \qquad J = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix}$$

- candidate flavour group

  $$\Gamma = Sp(2g,\mathbb{Z})$$ Siegel modular group

$G = Sp(2g,\mathbb{R}) \qquad K = U(g)$

transformation law of matter fields

- automorphy factor

$$j(\gamma,\tau) = \det(C\tau + D) \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

- finite copies of $Sp(2g,\mathbb{Z})$

$$\Gamma/\Gamma_{norm}(n) = \Gamma_{g,n}$$

finite Siegel modular group

genus level

$$\Gamma_{norm}(n) = \{\gamma \in \Gamma \,|\, \gamma = \mathbb{I}_{2g} \; mod \; n\}$$

principal congruence subgroup

$$\varphi^{(I)} \rightarrow \det(C\tau + D)^{k_I} \rho^I(\gamma) \, \varphi^{(I)}$$

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field-dependent
Yukawa couplings

$$w(\tau,\varphi)=\sum_p \xrightarrow{} Y_{I_1\dots I_p}(\tau)\varphi^{(I_1)}\dots\varphi^{(I_p)}$$

invariance of $w(\Phi)$ guaranteed by an holomorphic $Y_{I_1\dots I_p}(\tau)$ such that

$$Y_{I_1\dots I_p}(\gamma\tau)=j(\gamma,\tau)^{k_Y(p)}\rho(\gamma)\,Y_{I_1\dots I_p}(\tau) \qquad \gamma\in\Gamma$$

1. $k_Y(n)+k_{I_1}+\cdots+k_{I_p}=0$
2. $\rho\times\rho^{I_1}\times\ \dots\ \times\rho^{I_p}\supset 1$

modular forms
For $\Gamma_{norm}(n)$ and weight k_Y

form a linear space $\mathcal{M}_k(\Gamma_{norm}(n)))$
of finite dimension

special case of automorphic forms for G, K, Γ

$$\Psi(g) = j(g, \tau_0)^{-1} \, Y(g \, \tau_0)$$

$$K \, \tau_0 = \tau_0$$

- $\Psi(\gamma g) = \Psi(g)$

  $\Psi(g)$ is Γ-invariant
- $\Psi(gk) = j(k, \tau_0)^{-1} \, \Psi(g)$
- $\Psi(g)$ eigenfunction of G-Casimir operator
- suitable growth condition

- $Y(\gamma\tau) = j(\gamma, \tau) Y(\tau)$
- $Y(\tau)$ is K-invariant
- $Y(\tau)$ holomorphic

CP invariance in $Sp(2g,\mathbb{Z})$ [Ding, F., Liu 2102.06716]

CP belongs to Out(Γ)

$$\mathcal{CP}\;\gamma\;\mathcal{CP}^{-1} = u(\gamma)$$

$$\gamma \in \Gamma = Sp(2g,\mathbb{Z})$$

$$\tau \to \tau_{CP} = -\tau^*$$

up to In(Γ)

[moduli]

$$\begin{cases} \varphi^{(I)} \to \det(C\tau + D)^{\,k_I}\,\rho^I(\gamma)\,\varphi^{(I)} \\ \varphi^{(I)} \xrightarrow{CP} X_I\,\bar{\varphi}^{(I)}(x_P) \end{cases}$$

[matter fields]

$$\begin{cases} Y(\gamma\tau) = \det(C\tau + D)^{\,k_Y}\,\rho_Y(\gamma)\,Y(\tau) \\ Y^a(\tau) \xrightarrow{CP} Y^a(-\tau^*) = \lambda^a_b\,X_Y\,Y^{b*}(\tau) \end{cases}$$

[modular forms]

$$X_I\,\rho^{I*}(\gamma)\,X_I^{-1} = \chi(\gamma)^{g\,k_I}\rho^I u(\gamma))$$

[Nilles, Ramos-Sanchez and Vaudrevange 2001.01736]
Baur, Nilles, Trautner and Vaudrevange, 1901.03251,
H. Ohki, S. Uemura and R. Watanabe, 2003.04174]

CP violation as a property of the vacuum

[Novichkov, Penedo, Petcov and Titov 1905.11970]

[Baur, Kade, Nilles, Ramos-Sanchez, Vaudrevange 2012.09586]