Comparison of Viscosities from Chapman-Enskog & Relaxation Time Methods

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Topics

- Objectives
- Shear viscosities in the Chapman-Enskog approximation
- Shear viscosities in the Relaxation time approximation
- Test cases & comparisons
- Summary

Objectives

- ❑ **Transport coefficients (shear & bulk viscosities) are important inputs to viscous hydrodynamic simulations of relativistic heavy-ion collisions.**
- ❑ Shear viscosities affect the elliptic flow versus transverse momentum observed in heavy ion collisions.
- ❑ Comparison of different approximation schemes (Chapman-Enskog & Relaxation time methods) for shear viscosities.
- ❑ Test cases with different elastic cross sections:

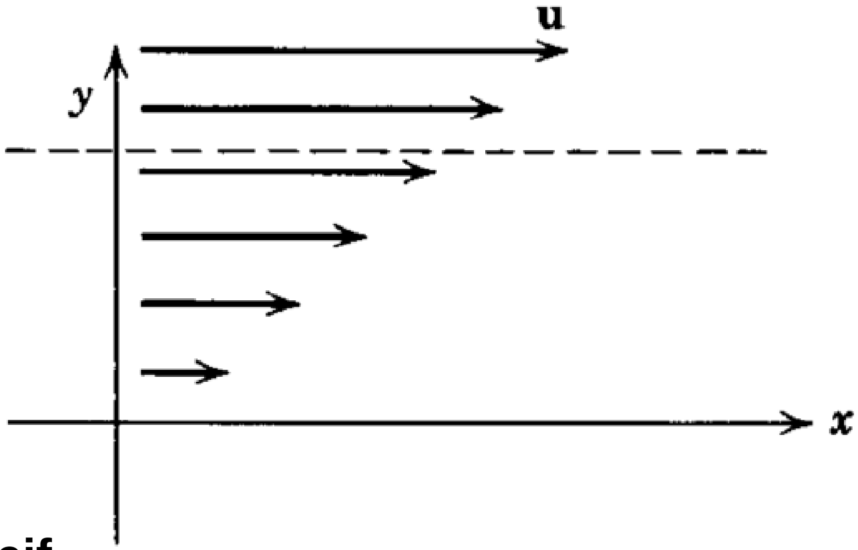
(i)Hard sphere gas, (ii) Maxwell gas,

(iii) Chiral pions & (iv) Massive pions.

These cases span the non-relativistic to the ultra-relativistic regimes.

Shear & Bulk Viscosity

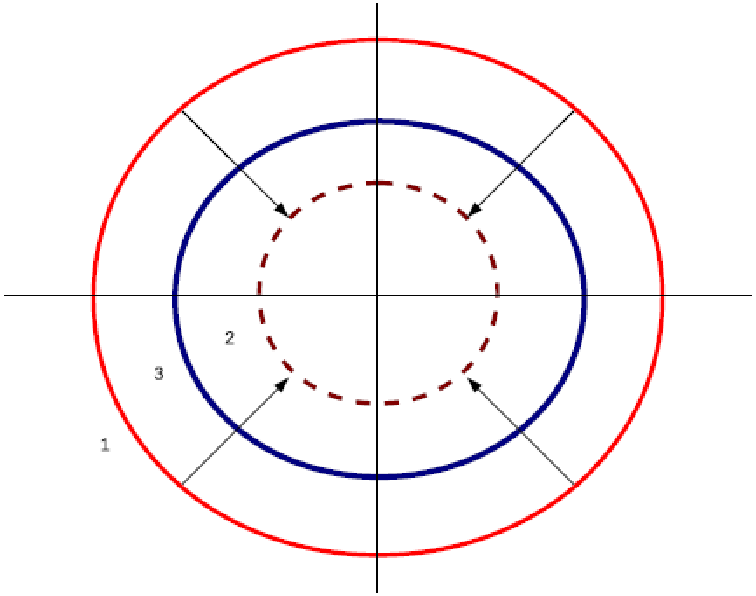
Shear Viscosity



F. Reif

$$F = -\eta \frac{dv_x}{dy}$$

Bulk Viscosity



$$F = -\eta_v \nabla \cdot v \,,$$

Temperature dependence of η

Estimates from kinetic theory:

$$\eta \approx \frac{1}{3}\frac{\bar{p}}{\bar{\sigma}_t(T)}\,; \quad \bar{\sigma}_t(T) = \frac{1}{3}\int_0^\infty dg\; g^7 e^{-g^2} \int d\theta\; \sin^2\theta\; \sigma\left(\frac{2g}{\sqrt{m\beta}}, \theta\right)$$

$\bar{p}$: Mean momentum, g : Scaled (w.r.t. $p_{m.p.}$) relative momentum.
$\bar{\sigma}_t(T) := T(=\beta^{-1})-$dependent transport cross section.

Relativistic expressions are more complicated, but have similar content.

$$\eta \approx \#\,\frac{\bar{p}}{\bar{\sigma}_t(T)} = \#\,\frac{\hbar}{\lambda\,\bar{\sigma}_t(T)} := \frac{\text{action}}{\text{physical volume}}$$

Above, λ is the thermal de-Broglie wavelength.

$$\lambda\;\&\;\eta \;\propto\; \begin{cases} \frac{1}{\sqrt{mT}} \;\;\&\;\; \frac{\sqrt{mT}}{\bar{\sigma}_t(T)} & \text{for NR} \\ \frac{1}{T} \;\;\&\;\; \frac{T}{\bar{\sigma}_t(T)} & \text{for UR} \end{cases}$$

Expectations for relativistic particles

$$\eta \approx \# \, \hbar/(\lambda \, \bar{\sigma}_t(T))$$

Unitary limit : For infinitely strong coupling, $\bar{\sigma}_t(T) \to \infty$, but η remains finite as λ^2 replaces $\bar{\sigma}_t(T)$. Consequently,

$$\eta = \# \, \frac{\hbar}{\lambda^3} = \# \, T^3$$

S = # T^3 $\longrightarrow$ **η/s = # 1**

(see also, Danielewicz & Gyulassy (1985) & earlier works).

Pion gas: For both chiral pions ($m_\pi = 0$) and for massive pions treated using current algebra, $\sigma \propto E^2_{c.m}$; thus, $\sigma_t(T) \propto T^2$. Hence,

$$\eta = \# \, \frac{\hbar T}{T^2} = \# \, \frac{\hbar}{T}$$

With experimental cross sections featuring the $\rho-$resonance prominently,

$$\eta = \# \, \frac{(\sqrt{m_\pi T} \text{ to } T)}{\sigma_t(T)}$$

Chapman-Enskog Approximation

"For small deviations from equilibrium, the distribution function can be expressed in terms of hydrodynamic variables (f(x,p) μ(x), u(x), T(x)) and their gradients. Transport coefficients (e.g., bulk & shear viscosities) are then calculable from relativistic kinetic theory."

$$f(x,p) = f^0[1 + \boxed{\varphi^1(x,p)]}$$

Deviation function

$$f^{(0)}(x,p) = \frac{1}{(2\pi\hbar)^3}\exp\left[\frac{m\mu(x)+p_\alpha u^\alpha(x)}{kT}\right]$$

Equilibrium distribution function

μ(x) : Chemical potential
u(x) : Flow velocity
T(x) : Temperature

The Boltzmann Equation

$$p_\alpha \partial^\alpha f^{(0)} = -f^{(0)} \int f_1^{(0)} \left(\varphi^{(1)} + \varphi_1^{(1)} - \varphi'^{(1)} - \varphi_1'^{(1)}\right) W' \frac{d^3p_1}{p_1^0}\frac{d^3p'}{p'^0}\frac{d^3p_1'}{p_1'^0}$$

The Collision Integral

$$Q\partial_\alpha U^\alpha - \left(p_\gamma U^\gamma + mh\right) p_\alpha \Delta^{\alpha\beta}\left(T^{-1}\partial_\beta T + c^{-2} DU_\beta\right) + \langle p_\alpha p_\beta\rangle \langle \partial^\alpha U^\beta\rangle = -kT\mathcal{L}[\varphi^{(1)}]$$

Bulk Viscosity

Heat Conductivity

Shear Viscosity

$$Q = -\frac{1}{3}(mc)^2 + c^{-2} p_\alpha U^\alpha[(1-\gamma)mh + \gamma kT] + c^{-2}\left(\frac{4}{3}-\gamma\right)(p_\alpha U^\alpha)^2$$

the solution (deviation function) has the general structure

$$\varphi^{(1)} = A\partial_\alpha U^\alpha - B\Delta_{\alpha\beta} p^\beta \Delta^{\alpha\beta}\left(T^{-1}\partial_\beta T + c^{-2} DU_\beta\right) + C\langle p_\alpha p_\beta\rangle\langle \partial^\alpha U^\beta\rangle$$

Shear Viscosity (working expression)

1st approximation

The shear viscosity

$$[\eta]_1 = \frac{1}{10} kT \frac{\gamma_0^2}{c_{00}} \,,$$

$$\gamma_0 \quad = \quad -10\hat{h}$$

Thermal variable

$$\hat{h} \;=\; \frac{K_3(z)}{K_2(z)}$$

Reduced enthalpy

$$c_{00} \quad = \quad 16\left(w_2^{(2)} - \frac{1}{z} w_1^{(2)} + \frac{1}{3z} w_0^{(2)} \right)$$

$w_i^{(s)}$: Relativistic omega integrals ; Contain collision cross sections

The Omega Integrals

Relativistic Omega Integrals

$$w_i^{(s)} = \frac{2\pi z^3 c}{[K_2(z)]^2}\int_0^\infty d\psi \boxed{\sinh^7\psi\, \cosh^i\psi}\boxed{K_j(2z\, \cosh\,\psi)}\int_0^\pi d\Theta \boxed{\sin\Theta\, \sigma(\psi,\Theta)(1-\cos^s\Theta)}$$

Relative momentum dependence

Thermal weight

Transport cross section

$z = \frac{mc^2}{kT}$ = **Relativity parameter ,**

g = mc sinh ψ = Relative momentum

P = 2mc cosh ψ = Total momentum

$$j = \frac{5}{2} + \frac{(-1)^i}{2},$$

$$i = 0, \pm 1, \pm 2, \ldots$$

$$s = 2,4,6, \ldots$$

Reduction to the non-relativistic case

Introducing the dimensionless quantity

$$\begin{aligned}
\phi &= \frac{g}{\sqrt{mT}} = \frac{mc\sinh\,\psi}{\sqrt{mT}} \\
\sinh\,\psi &= \frac{\sqrt{mT}}{mc}\,\phi \;\rightarrow\; \cosh\,\psi\;d\psi = \frac{\sqrt{mT}}{mc}\,d\phi \\
\cosh\,\psi &= \sqrt{1+\sinh^2\psi} = \sqrt{1+\frac{mT}{m^2c^2}\phi^2} = \sqrt{1+z^{-1}\phi^2}\;.
\end{aligned}$$

The omega integral in the non-relativistic limit :

$$\Omega_2^{(s)} = 2\sqrt{\frac{\pi\,T}{m}}\;\int d\phi\;\phi^7\,e^{-\phi^2}\int_0^\pi d\theta\;\sin\theta\,\sigma(\phi,\theta)\,(1-\cos^s\theta)$$

Note the g^7 – dependence in the kernel, which favors high relative velocity particles in the heat bath (energy density & pressure carry a g^4 – dependence).

Note also the importance of the relative velocity and angle dependence in the transport cross section which favors transport of momentum perpendicular to the flow direction.

The Relaxation Time Approximation

Chakraborty & Kapusta , arXiv:1006.0257

Here, the collision term in the Boltzmann equation is approximated as

$$C[f] \cong \frac{f - f_0^{eq}}{\tau(p)} = D_t f$$

Shear viscosity (for a+b --> c+d) is given by

$$\eta_s \quad = \quad \frac{1}{15T} \int_0^\infty \frac{d^3 p_a}{(2\pi)^3} \frac{|p_a|^4}{E_a^2} \frac{1}{w_a(E_a)} f_a^{eq} \; ,$$

$w_a(E_a)$ **is the collision frequency**

$$w_a(E_a) = \sum_{bcd} \frac{1}{2} \int \frac{d^3 p_b}{(2\pi)^3} \frac{d^3 p_c}{(2\pi)^3} \frac{d^3 p_d}{(2\pi)^3} W(a, b|c, d) \, f_b^{eq}$$

$$W(a, b|c, d) = \frac{(2\pi)^4 \delta^4 (p_a + p_b - p_c - p_d)}{2E_a 2E_b 2E_c 2E_d} |\mathcal{M}|^2$$

The collision frequency can be simplified to

$$w_a(E_a) \quad = \int \frac{d^3 p_b}{(2\pi)^3} \frac{\sqrt{s(s - 4m^2)}}{2E_a \, 2E_b} f_b^{eq} \boxed{\sigma_T}$$

Total CM cross section

Non-relativistic collision frequency

For non-relativistic particles,

1. $E_a \approx m$ and $E_b \approx m$
2. $\sqrt{s(s-4m^2)} \;=\; \sqrt{4(m^2+q_{cm}^2)\,4q_{cm}^2} \approx 4\,m\,|\vec{q}_{cm}|$

The collision frequency

$$w_a(v_a) \;=\; \int_0^\infty d^3v_b\,\sigma_T\,f_b^{eq}\,|\vec{v}_a-\vec{v}_b|$$

Symbols :

q_{cm} :Rel. vel. in the CM frame

v_i :Vel. of the ith particle.

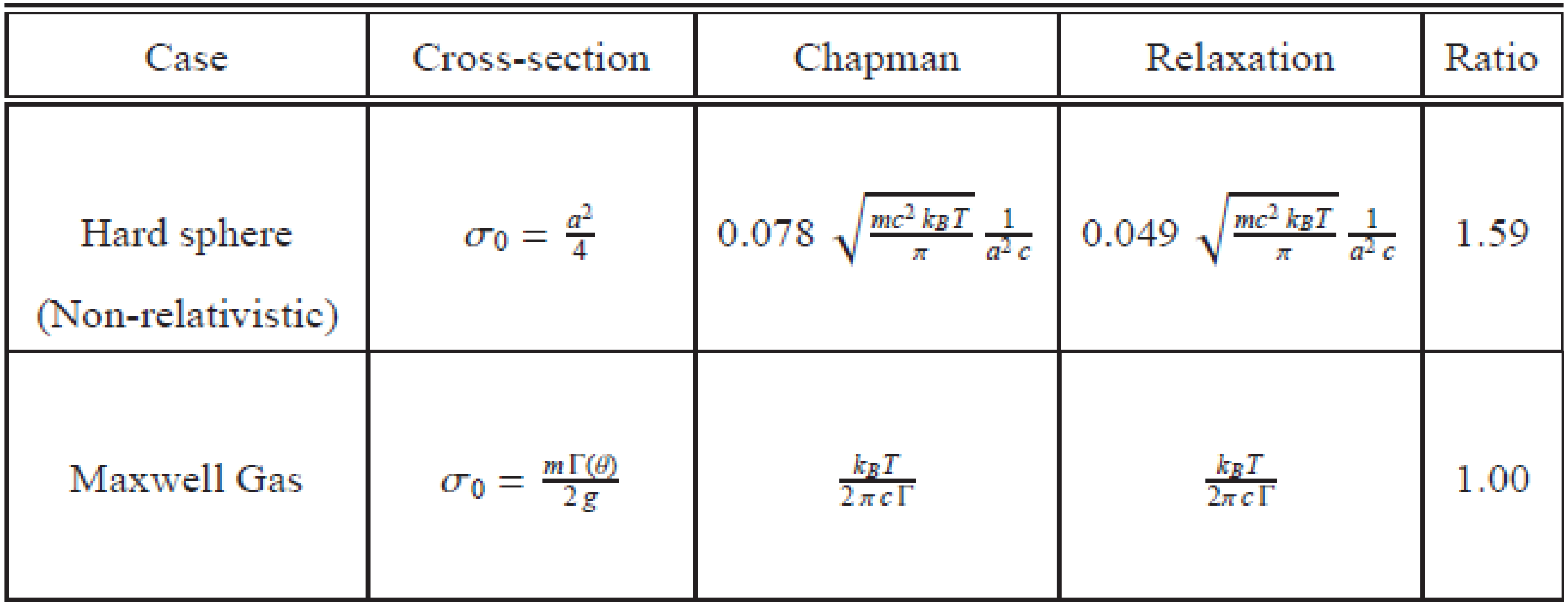
The shear viscosity

$$\eta_s \;=\; \frac{1}{30\pi^2}\frac{m^5}{T}\int dv_a\,v_a^6\,\frac{f_a}{w_a(v_a)}$$

Test Cases

The non relativistic cases

Table 3.1: Summary of results for shear viscosity



Case	Cross-section	Chapman	Relaxation	Ratio
Hard sphere (Non-relativistic)	$\sigma_0 = \frac{a^2}{4}$	$0.078 \sqrt{\frac{mc^2 k_B T}{\pi}} \frac{1}{a^2 c}$	$0.049 \sqrt{\frac{mc^2 k_B T}{\pi}} \frac{1}{a^2 c}$	1.59
Maxwell Gas	$\sigma_0 = \frac{m \Gamma(\theta)}{2 g}$	$\frac{k_B T}{2 \pi c \Gamma}$	$\frac{k_B T}{2 \pi c \Gamma}$	1.00

These results highlight the role of the energy/momentum dependence in the the differential cross section.

Massless pions

The differential cross section

$$\sigma(s,\theta) = \frac{s}{64\,\pi^2 f_\pi^4}\left(1 + \frac{1}{3}\cos^2\theta\right)$$

$f_\pi = 93$ MeV is the pion decay consnstant.

Chapman-Enskog approximation

Shear viscosity :

$$\eta_s \quad = \quad -\frac{4\,n}{c^2}\,(kT)^3\,c_0 \ ,$$

$$c_0 = -\frac{40}{n}\,\frac{1}{c^2}\,\frac{(kT)^2}{c_{00}} \qquad\qquad c_{00} = (\frac{1}{3}w_{63}^2 + \frac{1}{2}\,w_{72}^2 + \frac{1}{4}\,w_{81}^2)/\beta^2 c^2$$

The omega integral for the massless case :

$$w_{ij}^k = \frac{1}{16}\,\frac{\pi}{\beta}\int_{-1}^{1} d\cos\Theta \int_0^\infty dp\,\sigma(p,\Theta)\,(1-\cos^k\Theta)\,(\beta\,c\,p)^i\,K_j(\beta\,c\,p)$$

where $\beta = 1/T$

Interacting Pions (Experimental Cross Sections)

$$\sigma^{\pi\pi}(s,\theta) = \frac{4}{q_{CM}^2}\left(\frac{1}{9}sin^2\delta_0^0 + \frac{5}{9}sin^2\delta_0^2 + \frac{1}{3}\,\frac{1}{9}\;sin^2\delta_1^1\right)$$

$$\delta_0^0(\varepsilon) = \frac{\pi}{2} + \arctan\left(\frac{\varepsilon - m_\sigma}{\Gamma_\sigma/_2}\right)$$

$$\delta_0^0(\varepsilon) = \frac{-0.12q}{m_\pi}$$

$$\delta_1^1(\varepsilon) = \frac{\pi}{2} + \arctan\left(\frac{\varepsilon - m_\rho}{\Gamma_\rho/_2}\right)$$

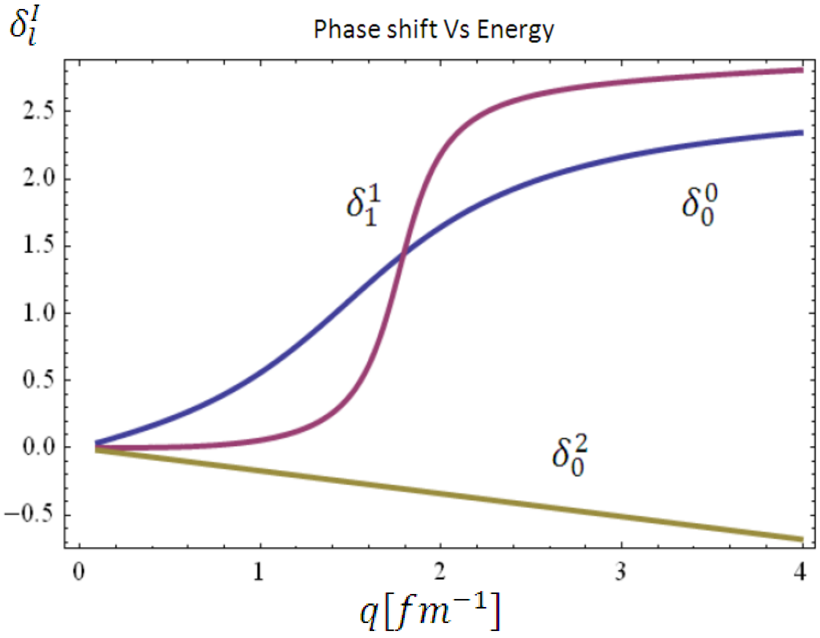
$$m_\sigma = 5.8m_\pi$$

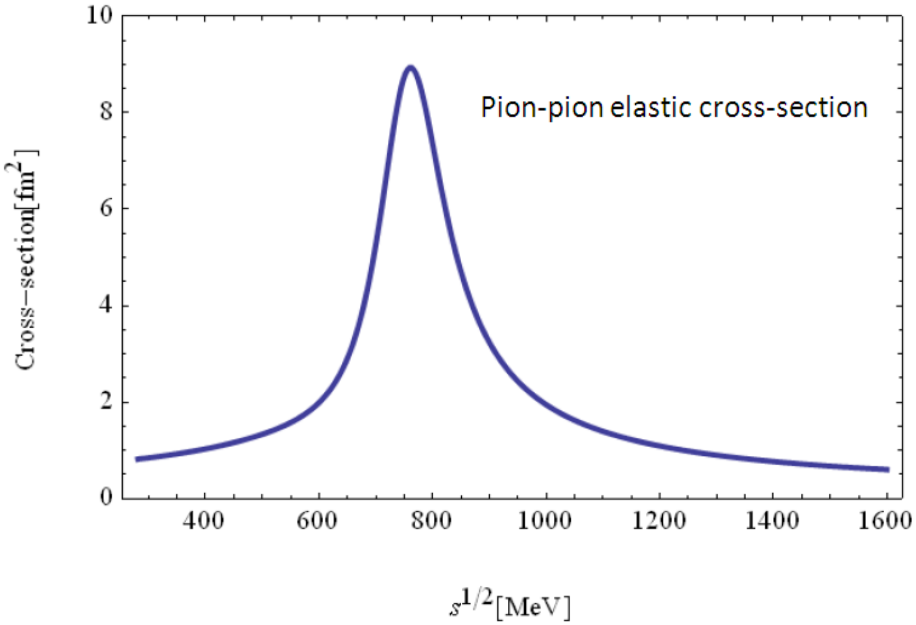
$$\Gamma_\sigma = 2.06\,q$$

$$m_\rho = 5.53m_\pi$$

$$\Gamma_\rho = 2.06\,q\left(\frac{q/m_\pi}{1+\left(q/m_\rho\right)^2}\right)^2$$

Parametrization from Bertsch et al. ,
PR D37 (1988) 1202.





Results for pions

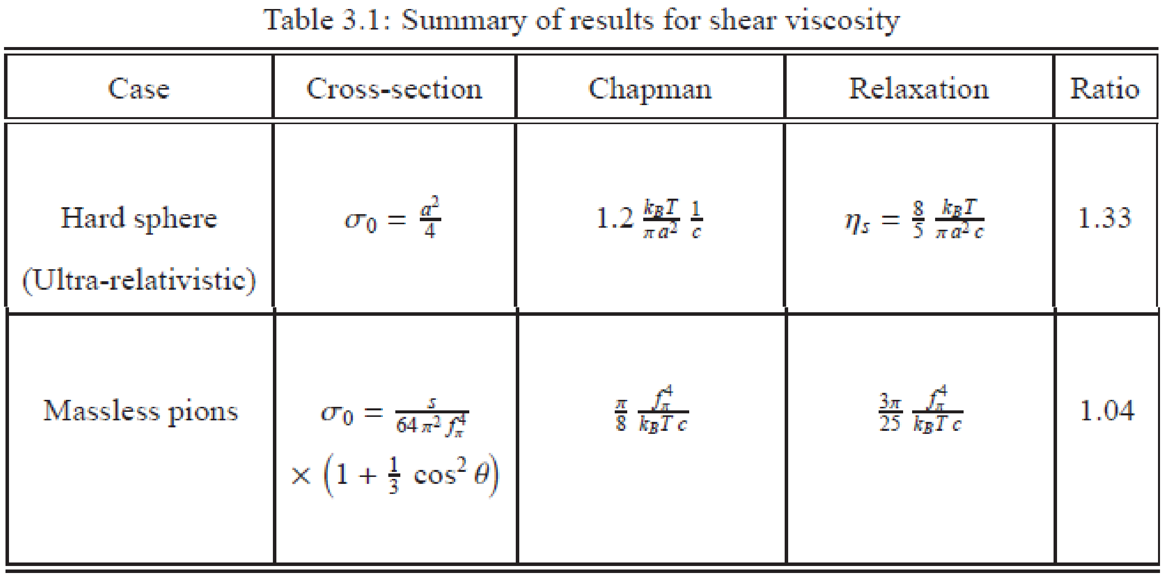
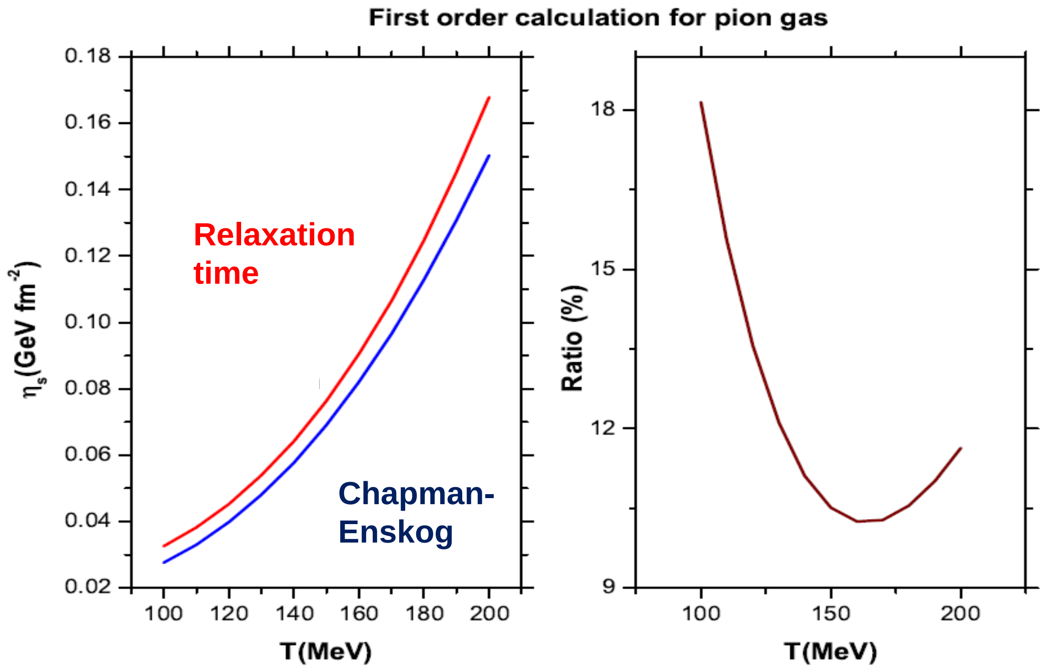


Table 3.1: Summary of results for shear viscosity

Case	Cross-section	Chapman	Relaxation	Ratio
Hard sphere (Ultra-relativistic)	$\sigma_0 = \frac{a^2}{4}$	$1.2\,\frac{k_B T}{\pi a^2}\,\frac{1}{c}$	$\eta_s = \frac{8}{5}\,\frac{k_B T}{\pi a^2 c}$	1.33
Massless pions	$\sigma_0 = \frac{s}{64\,\pi^2 f_\pi^4}$ $\times \left(1 + \frac{1}{3}\cos^2\theta\right)$	$\frac{\pi}{8}\,\frac{f_\pi^4}{k_B T\, c}$	$\frac{3\pi}{25}\,\frac{f_\pi^4}{k_B T\, c}$	1.04

Pions with experimental cross sections.



Summary & Outlook

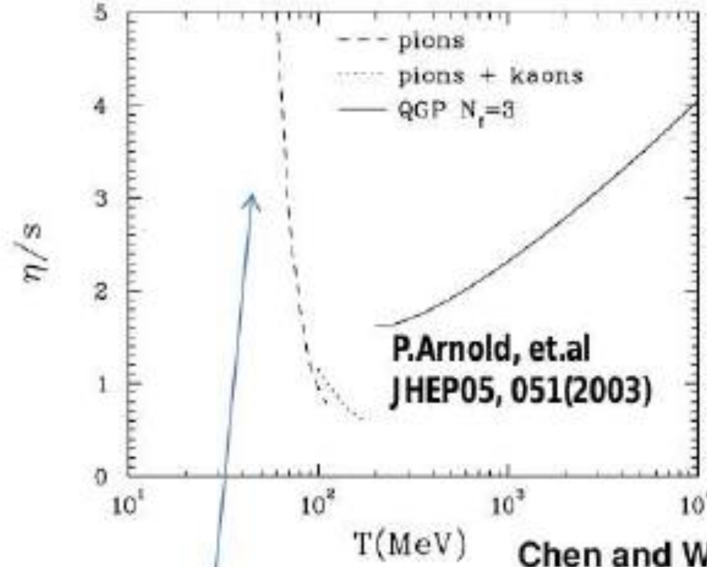
- ☐ **Depending on the relativistic nature of the system, different enegy/momentum dependence in the cross sections seemingly lead to different levels of agreement/disagreement.**

- ☐ **An in-depth analysis of the origin of these differences is underway.**

- ☐ **Comparison with Green-Kubo is on the way.**

Thank you

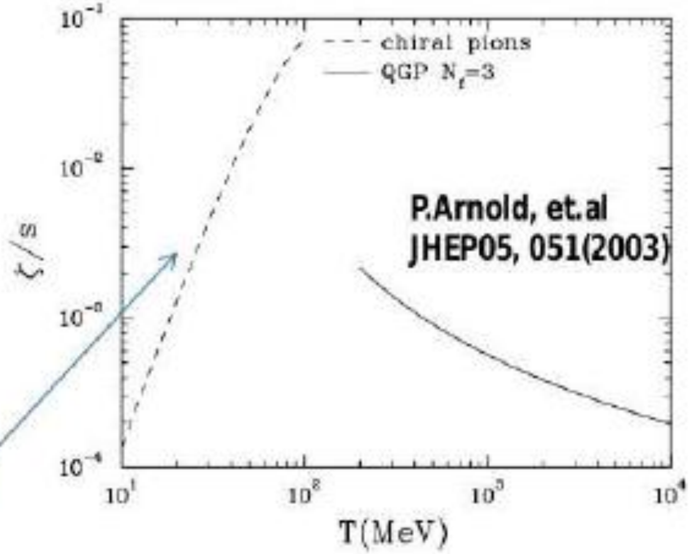
Examples of theoretical results



P.Arnold, et.al
JHEP05, 051(2003)

Prakash et al. ,
Phys. Rep. 227 (1993) 321

Chen and Wang,
Arxiv:0711.4824v1



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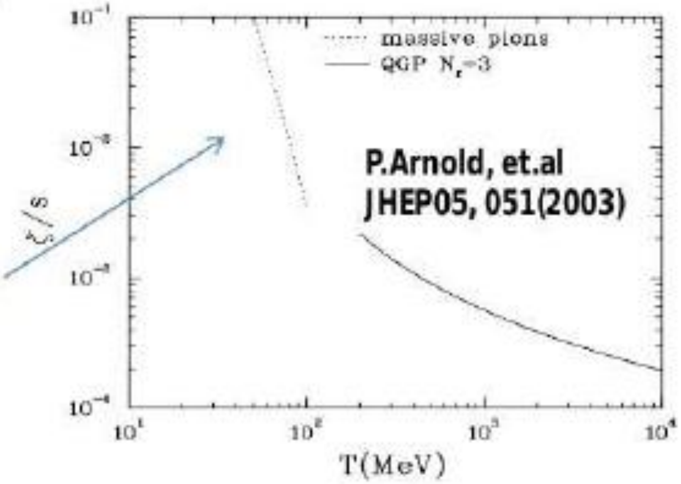

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Kapusta, arxiv:0809.3746v2