Speeding up SM Amplitude Calculations with Chirality Flow

QCD@LHC 2022 28 NOVEMBER 2022 - ANDREW LIFSON
IN COLLABORATION WITH JOAKIM ALNEFJORD, CHRISTIAN REUSCHLE, MALIN SJÖDAHL, AND ZENNY WETTERSTEN
1 Introduction
   - Spinor-helicity recap
   - Colour flow reminder

2 Chirality Flow
   - Massless QED
   - Massless QCD

3 Automation
   - Aim and method
   - Results

4 Conclusions
Our Main Numerical Result (so far) (hep-ph:2203.13618)

Evaluation time for 100 000 matrix elements for $e^+e^- \to n$ photons

- MadGraph5, stand-alone version
- Chirality flow, no diagram removal
- Chirality flow, diagram removal
Spinor-Helicity: its Building Blocks

Lorentz algebra $so(3, 1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Spinors (use chiral basis):
\[ u^+(p) = v^-(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} \quad u^-(p) = v^+(p) = \begin{pmatrix} |p\rangle \\ 0 \end{pmatrix} \]
\[ \bar{u}^+(p) = \bar{v}^-(p) = \begin{pmatrix} |p\rangle \\ 0 \end{pmatrix} \quad \bar{u}^-(p) = \bar{v}^+(p) = \begin{pmatrix} 0 \\ \langle p| \end{pmatrix} \]

Amplitude written in terms of Lorentz-invariant spinor inner products
\[ \langle ij \rangle = -\langle ji \rangle \equiv \langle i||j \rangle \quad \text{and} \quad [ij] = -[ji] \equiv [i||j] \]

These are well known complex numbers, $\langle ij \rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$
Spinor-Helicity: Vectors and Removing $\mu$ Indices

Lorentz algebra $\text{so}(3, 1) \cong \text{su}(2) \oplus \text{su}(2)$

Dirac matrices in chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2} \tau^\mu \\ \sqrt{2} \bar{\tau}^\mu & 0 \end{pmatrix} \quad \sqrt{2} \tau^\mu = (1, \vec{\sigma}), \quad \sqrt{2} \bar{\tau}^\mu = (1, -\vec{\sigma}),$$

Remove vector indices with e.g.

$$\langle i | \bar{\tau}^\mu | j \rangle [k | \tau_\mu | l \rangle = \langle il | [kj \rangle, \quad \sqrt{2} \mathcal{P}^\mu \tau_\mu \equiv \mathcal{P} = |p\rangle \langle p|$$

Fierz identity

Contraction with Pauli

Polarisation vectors ($r \equiv$ gauge choice, $r^2 = 0, r \cdot p \neq 0$):

$$\mathcal{\epsilon}_+(p, r) = \frac{|p\rangle \langle r|}{\langle rp|}, \quad \mathcal{\epsilon}_-(p, r) = \frac{|r\rangle \langle p|}{[pr]}$$

Every object can be written as left or right spinor(s)
An Illuminating Example: $e^+e^- \rightarrow \gamma\gamma$

Helas/MadGraph: explicit matrix multiplication

\[ \sim [\bar{u}^-(p_1)\gamma^\mu u^+_\mu(p_4) (p_1^\nu + p_4^\nu) \gamma_\nu \gamma^\rho \epsilon^-_\rho(p_3) \nu^+(p_2)] \]

- Also cache and recycle various components
- No analytic simplification
- Most common numerical method
An Illuminating Example: $e^+e^- \rightarrow \gamma\gamma$

Helas/MadGraph: explicit matrix multiplication

$$\sim \left[ \bar{u}^- (p_1) \gamma^\mu \epsilon_\mu^+ (p_4) (p_1^\nu + p_4^\nu) \gamma_\nu \gamma^\rho \epsilon^- (p_3) \nu^+ (p_2) \right]$$

- Also cache and recycle various components
- No analytic simplification
- Most common numerical method

Can we systematically remove need for algebra or matrix multiplication?
Standard method in $SU(N)$-colour calculations:

Write all objects in terms of $\delta_{ij} \equiv \text{flows of colour}$ (for simplicity $T_R = 1$)

Calculations done pictorially, not via indices
**Chirality Flow Building Blocks**

**Key idea:** $\text{su}(2) = \text{su}(N)$ (hep-ph:2003.05877)

Draw & connect lines to directly obtain inner products $\langle ij \rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$

Removes need to do algebra or matrix multiplication

- **Spinor inner products defined as connected lines**
  
  $\langle i|\alpha|j\rangle_\alpha \equiv \langle ij \rangle = -\langle ji \rangle = \begin{array}{c} i \rightarrow j \end{array}$
  
  $[i|\beta|j]\dot{\beta} \equiv [ij] = -[ji] = \begin{array}{c} i \ldots \ldots j \end{array}$

- **Spinors definitions follow**
  
  $\bar{u}_i^- = \bar{v}_i^+ = \langle i|\alpha = \begin{array}{c} \bigcirc \leftarrow i \end{array}$, $\quad u_j^+ = v_j^- = |j\rangle_\alpha = \begin{array}{c} \bigcirc \rightarrow j \end{array}$
  
  $\bar{u}_i^+ = \bar{v}_i^- = [i|\beta] = \begin{array}{c} \bigcirc \ldots \ldots i \end{array}$, $\quad u_j^- = v_j^+ = |j\rangle\dot{\beta} = \begin{array}{c} \bigcirc \ldots \ldots j \end{array}$

- **Define slashed momentum as dot**
  
  $p \equiv \sqrt{2p^\mu\tau^\alpha_\mu} = \begin{array}{c} - \bigcirc \rightarrow \end{array}$, $\quad \bar{p} \equiv \sqrt{2p^\mu\bar{\tau}^\mu_\alpha\beta} = \begin{array}{c} \bigcirc \rightarrow - \end{array}$
### The Massless QED Flow Rules: External Particles

<table>
<thead>
<tr>
<th>Species</th>
<th>Feynman</th>
<th>Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{u}^-(p_i)$</td>
<td><img src="image" alt="Feynman Diagram" /></td>
<td><img src="image" alt="Flow Diagram" /></td>
</tr>
<tr>
<td>$\nu^-(p_j)$</td>
<td><img src="image" alt="Feynman Diagram" /></td>
<td><img src="image" alt="Flow Diagram" /></td>
</tr>
<tr>
<td>$\nu^+(p_j)$</td>
<td><img src="image" alt="Feynman Diagram" /></td>
<td><img src="image" alt="Flow Diagram" /></td>
</tr>
<tr>
<td>$\bar{u}^+(p_i)$</td>
<td><img src="image" alt="Feynman Diagram" /></td>
<td><img src="image" alt="Flow Diagram" /></td>
</tr>
<tr>
<td>$\epsilon_-^\mu(p_i, r)$</td>
<td><img src="image" alt="Feynman Diagram" /></td>
<td><img src="image" alt="Flow Diagram" /></td>
</tr>
<tr>
<td>$\epsilon_+^\mu(p_i, r)$</td>
<td><img src="image" alt="Feynman Diagram" /></td>
<td><img src="image" alt="Flow Diagram" /></td>
</tr>
</tbody>
</table>

**Left-chiral** ≡ dotted lines  
**right-chiral** ≡ solid lines
The QED Flow Rules: Vertices and Propagators

Feynman | Flow
--- | ---
\[ i e \sqrt{2} \]
\[ i e \sqrt{2} \]
\[ \frac{i}{p^2} \quad \sum_i p_i \quad \text{or} \quad \frac{i}{p^2} \quad \sum_i p_i \]

Left-chiral $\equiv$ dotted lines  
right-chiral $\equiv$ solid lines
An Illuminating Example: $e^+e^- \rightarrow \gamma\gamma$

Spinor helicity analytically:

$$\sim \langle p_1 | \tilde{\tau}^\mu | [p_1] \langle p_1 | + [p_4] \langle p_4 | \rangle \tilde{\tau}^\nu | p_2 \rangle \frac{\langle r_3 | \tilde{\tau}^\nu | p_3 \rangle [r_4 | \tau_\mu | p_4 \rangle}{\langle r_3^3 | 4r_4 \rangle}$$

$$= \left( \langle p_1 | \tilde{\tau}^\mu | p_1 \rangle + \langle p_1 | \tilde{\tau}^\mu | p_4 \rangle \right) \frac{\langle r_4 | \tau_\mu | p_4 \rangle}{\langle r_3^3 | 4r_4 \rangle} \left( \langle p_1 | \tilde{\tau}^\nu | p_2 \rangle + \langle p_4 | \tilde{\tau}^\nu | p_2 \rangle \right) \left[ p_3 | \tau_\nu | r_3 \right]$$

$$= \frac{\langle 1r_4 \rangle ([41] \langle 13 \rangle + [44] \langle 43 \rangle) \langle r_3^2 | 4r_4 \rangle}{\langle r_3^3 | 4r_4 \rangle} = \frac{\langle 1r_4 \rangle [41] \langle 13 \rangle [r_3^2]}{\langle r_3^3 | 4r_4 \rangle}$$

Fierz identities like $\langle i | \tilde{\tau}^\mu | j \rangle [k | \tau_\mu | l \rangle = \langle i l | [k j]$
An Illuminating Example: \( e^+ e^- \rightarrow \gamma \gamma \)

Chirality flow analytically:

\[
\sim \frac{1}{\langle r_3 3 \rangle [4r_4]}
\]
An Illuminating Example: $e^+e^- \rightarrow \gamma\gamma$

Chirality flow analytically:

Recall:

$\langle ij \rangle = - \langle ji \rangle = i \xrightarrow{} j$

$[ij] = -[ji] = i \xrightarrow{} j$
A complicated QED Example

Spinor-helicity analytic:
- 5 charge conjugation/Fierz + rearranging
- Not possible to fit on single slide!

\[
\begin{align*}
&= (\sqrt{2}ei)^8 (\frac{-i}{2})^3 (i)^4 \left[\sum_{\text{vertices}} \sum_{\text{photon propagators}} \sum_{\text{fermion propagators}} \sum_{\text{polarization vectors}} \right] \times \\
&\times \left( \langle r_99 \rangle [r_89] + \langle r_910 \rangle [10r_8] \right) \left( [33] \langle 37 \rangle + [34] \langle 47 \rangle + [36] \langle 67 \rangle \right) \times \\
&\times \left( - \langle 89 \rangle [91] \langle 12 \rangle - \langle 89 \rangle [95] \langle 52 \rangle - \langle 80 \rangle [10 1] \langle 12 \rangle - \langle 80 \rangle [10 5] \langle 52 \rangle \right)
\end{align*}
\]
The Non-abelian Massless QCD Flow Vertices

<table>
<thead>
<tr>
<th>Feynman</th>
<th>Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ - \frac{g_s f^{abc}}{2} ]</td>
<td>[ \begin{align*} \sum_{Z(2,3,4)} f a_1 a_2 b f b a_4 a_3 \end{align*} ]</td>
</tr>
<tr>
<td>[ g_{12}(p_1 - p_2)_3 ]</td>
<td>[ g_{12}(p_1 - p_2)_3 ]</td>
</tr>
<tr>
<td>[ g_{23}(p_2 - p_3)_1 ]</td>
<td>[ g_{23}(p_2 - p_3)_1 ]</td>
</tr>
<tr>
<td>[ g_{13}(p_3 - p_1)_2 ]</td>
<td>[ g_{13}(p_3 - p_1)_2 ]</td>
</tr>
</tbody>
</table>

Arrow directions only consistently set within full diagram
QCD Example: $q_1 \bar{q}_1 \rightarrow q_2 \bar{q}_2 g$

\[
\begin{align*}
\left[ \cdots \right] & \equiv \left\{ 2[q_1 \bar{q}_2] \langle q_2 \bar{q}_1 \rangle \left( [1 q_1] \langle q_1 r \rangle + [1 \bar{q}_1] \langle 1 r \rangle \right) \\
& \quad - 2[q_1 1] \langle 1 \bar{q}_1 \rangle \langle q_2 r \rangle [1 \bar{q}_2] + 2[q_1 1] \langle r \bar{q}_1 \rangle \langle q_2 1 \rangle [1 q_2] \right\}
\end{align*}
\]
Summary

- So far: Numerical calculations use explicit multiplication rather than spin algebra analytically because quicker
- We have made the analytical spin algebra trivial
- Can we use this to make even faster numerics?
MadGraph and the Automation of Chirality Flow

Summary

- So far: Numerical calculations use explicit multiplication rather than spin algebra analytically because quicker
- We have made the analytical spin algebra trivial
- Can we use this to make even faster numerics?

Use MadGraph5_aMC@NLO (MG5aMC) for proof of concept automation

- Make minimal changes to massless QED in MG5aMC
- Pro: any difference in speed from our changes ⇒ sound conclusions
Sources of Expect Speed Gains

1. Simplified vertices and propagators
   - We minimise matrix multiplication
   - Each component of a calculation is simpler
Sources of Expect Speed Gains

1. Simplified vertices and propagators
   - We minimise matrix multiplication
   - Each component of a calculation is simpler

2. Gauge-based diagram removal
   - Polarisation vectors contain arbitrary gauge-reference spinor of momentum $r$
   - Spinor inner products antisymmetric $\Rightarrow \langle ii \rangle = [jj] = 0$
   - Chirality-flow makes optimal choice of $r$ obvious $\Rightarrow$ remove diagrams!

\[ \sim \frac{1}{\langle r_3 3 \rangle [4 r_4]} \]

\[ \langle 1 r_4 \rangle [4 1] (13) [r_3 2] \]

\[ r_4 \equiv 1 \quad 0 \]
Our Main Result (hep-ph:2203.13618)

Evaluation time for 100 000 matrix elements for $e^+ e^- \rightarrow n$ photons

- MadGraph5, stand-alone version
- Chirality flow, no diagram removal
- Chirality flow, diagram removal

Number of external photons

Time [s]

Evaluation time for 100 000 matrix elements for $e^+ e^- \rightarrow n$ photons

MadGraph5, stand-alone version
Chirality flow, no diagram removal
Chirality flow, diagram removal
Conclusions and Outlook

**Conclusions:**
- Chirality flow is the shortest route from Feynman diagram to complex number
- We have flow rules for full SM including masses at tree level
- We automised it for massless QED, found significant gains in MadGraph

**Outlook and other work in this area:**
- Use method analytically to calculate loop amplitudes
  - Ongoing work by AL, Simon Plätzer, and Malin Sjödahl,
- Automate for rest of (tree-level) Standard Model
  - Two current master students working to achieve this
  - First implementation of QCD working
- Simon Plätzer and Malin Sjödahl used chirality flow as basis for resummation (hep-ph:2204.03258)
Spinor-Helicity: Gauge Bosons in Terms of Spinors

Lorentz algebra $so(3, 1) \cong su(2) \oplus su(2)$

Consider massless particles: chirality $\sim$ helicity

Outgoing polarisation vectors:

$$\epsilon^\mu_+(p, r) = \frac{\langle r | \bar{r}^\mu | p \rangle}{\langle rp \rangle}, \quad \epsilon^\mu_-(p, r) = \frac{[r | \tau^\mu | p \rangle}{[pr]}$$

- $r$ is a (massless) arbitrary reference momentum ($p \cdot r \neq 0$)
- Different $r$ choices correspond to different gauges
  $$\epsilon^\mu_+(p, r') - \epsilon^\mu_+(p, r) = -p^\mu \frac{\langle r' r \rangle}{\langle r' p \rangle \langle rp \rangle}$$
- Gauge invariant quantities must be $r$-invariant
  - Choose $r$ as conveniently as possible (remember $\langle ij \rangle = -\langle ji \rangle$ s.t. $\langle ii \rangle = 0$)
    (4-gluon amplitude: can make 20/21 terms vanish)
  - Variance under $r \to r'$ good check of gauge invariance of (partial) amplitude
Spinor-Helicity: Vectors and Removing $\mu$ Indices

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$

Consider massless particles: chirality $\sim$ helicity

Dirac matrices in chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix} \quad \sqrt{2}\tau^\mu = (1, \bar{\sigma}), \quad \sqrt{2}\bar{\tau}^\mu = (1, -\bar{\sigma}),$$

Remove $\tau/\bar{\tau}$ matrices in amplitude with

$$\langle i | \bar{\tau}^\mu | j \rangle [k | \tau_\mu | l \rangle = \langle il | [kj] \rangle, \quad \langle i | \bar{\tau}^\mu | j \rangle = [j | \tau_\mu | i \rangle$$

Fierz identity

Charge Conjugation

Express (massless) $p^\mu$ in terms of spinors

$$p^\mu = \frac{[p | \tau^\mu | p \rangle}{\sqrt{2}} = \frac{[p | \bar{\tau}^\mu | p \rangle}{\sqrt{2}} \quad , \quad \sqrt{2}p^\mu \tau_\mu \equiv \phi = |p\rangle\langle p| \quad , \quad \sqrt{2}p^\mu \bar{\tau}_\mu \equiv \bar{\phi} = |p\rangle\langle p|$$
An Illuminating Example: $e^+e^- \rightarrow \gamma\gamma$

- $|p\rangle \equiv$ right-chiral spinor
- $[p] \equiv$ left-chiral spinor
- $\tau^\mu$, $\bar{\tau}^\mu \equiv$ Pauli matrices
- $\langle ij \rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$

**Spinor helicity: analytic**

\[
\sim \langle p_1 | \bar{\tau}^\mu (|p_1\rangle \langle p_1| + |p_4\rangle \langle p_4|) \bar{\tau}^\nu |p_2\rangle \frac{\langle r_3 | \bar{\tau}_\nu | p_3 \rangle [r_4 | \tau_\mu | p_4 \rangle}{\langle r_3 3 \rangle [4 r_4]} \frac{\epsilon^-_3}{\epsilon^+_4}
\]

\[
= \frac{\langle p_1 | \bar{\tau}^\mu | p_1 \rangle + \langle p_1 | \bar{\tau}^\mu | p_4 \rangle} {\langle r_3 | p_3 \rangle [4 r_4]} \langle r_3 3 \rangle [4 r_4]
\]

\[
= \frac{1}{r_4} \frac{[41] \langle 13 \rangle + [44] \langle 43 \rangle} {\langle r_3 | p_3 \rangle [4 r_4]} [r_3 2] = \frac{1}{r_4} [41] \langle 13 \rangle [r_3 2]
\]

Fierz identities like $\langle i | \bar{\tau}^\mu | j \rangle [k | \tau_\mu | l \rangle = \langle ii | [kj]$

$[ii] = 0$
An Illuminating Example: \( e^+e^- \rightarrow \gamma\gamma \)

- \( |p\rangle \equiv \) right-chiral spinor
- \( |\bar{p}\rangle \equiv \) left-chiral spinor
- \( \tau^\mu, \bar{\tau}^\mu \equiv \) Pauli matrices
- \( \langle ij \rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j} \)

**Spinor helicity: explicit matrix multiplication**

\[
\sim [\bar{u}^-(p_1)\gamma^\mu\epsilon^+_{\mu}(p_4)(p_1^\nu + p_4^\nu)\gamma^\nu\gamma^\rho\epsilon^-_{\rho}(p_3)v^+(p_2)]
\]

- Also cache and recycle various components
- Most common numerical method
An Illuminating Example: $e^+e^- \rightarrow \gamma\gamma$

- $|\rho\rangle \equiv$ right-chiral spinor
- $|\rho\rangle \equiv$ left-chiral spinor
- $\tau^\mu, \bar{\tau}^\mu \equiv$ Pauli matrices
- $\langle ij \rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$

**Spinor helicity: explicit matrix multiplication**

$$\sim \left[ \bar{u}^-(p_1) \gamma^\mu \epsilon^+_\mu(p_4) \left(p_1^\nu + p_4^\nu\right) \gamma^\nu \gamma^\rho \epsilon^-\rho(p_3) \nu^+(p_2) \right]$$

- Also cache and recycle various components
- Most common numerical method

---

Can we systematically remove need for algebra or matrix multiplication?
Spinor-Helicity: Gauge Bosons in Terms of Spinors

Lorentz algebra $so(3, 1) \cong su(2) \oplus su(2)$

Consider massless particles: chirality $\sim$ helicity

Outgoing polarisation vectors ($r \equiv$ gauge choice, $r^2 = 0$, $r \cdot p \neq 0$):

$$
\epsilon^\mu_+(p, r) = \frac{\langle r | \bar{\tau}^\mu | p \rangle}{\langle rp \rangle}, \quad \epsilon^\mu_-(p, r) = \frac{[r | \tau^\mu | p]}{[pr]}
$$

$$
p \cdot \epsilon_+(p, r) = \frac{\langle r | p^\mu \bar{\tau}_\mu | p \rangle}{\langle rp \rangle} = 0 \quad \text{Weyl eq. } p^\mu \bar{\tau}_\mu | p \rangle = 0
$$

$$
p \cdot \epsilon_-(p, r) = \frac{[r | p^\mu \tau_\mu | p]}{[pr]} = 0 \quad \text{Weyl eq. } p^\mu \tau_\mu | p \rangle = 0
$$

$$
\epsilon_+(p, r) \cdot (\epsilon_-)^*(p, r) = \frac{\langle r | \bar{\tau}^\mu | p \rangle [r | \tau^\mu | p]}{\langle rp \rangle [pr]} \quad = \frac{\langle rp \rangle [rp]}{\langle rp \rangle [pr]} = -1 \quad \text{[pr] } = -[rp]
$$
Colour Flow: a Quick Introduction

Standard method in $SU(N)$-colour calculations:

Write all objects in terms of $\delta_{ij} \equiv$ flows of colour (for simplicity $T_R = 1$)

Calculations done pictorially, not via indices

$$\delta_{ij} = \bar{j} \rightarrow i$$

$$\sum_i \delta_{ii} = N = \sum_i \delta_{ii} = N = \sum_i \delta_{ii} = N$$

$$t^a_{ij} =$$

$$\delta_{ij} \delta_{kl}$$

$$\frac{1}{N}$$
Colour Flow: a Quick Introduction

Standard method in SU(N)-colour calculations:

Calculations done pictorially, not via indices $\sum_i \delta_{ii} = N = \begin{array}{c}
\end{array}$

\[
\begin{align*}
\text{Tr}(t^a t^a) &= \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} - \frac{1}{N} \begin{array}{c}
\end{array} = N^2 - 1
\end{align*}
\]

Decompose massive momentum into massless ones

\[ p^\mu = p^{b,\mu} + \alpha q^\mu, \quad (p^b)^2 = q^2 = 0, \quad \alpha = \frac{p^2}{2p^b \cdot q} \]

- Spinors contain both chiralities, e.g.
  
  \[
  \bar{v}^-(p) = \begin{pmatrix} p^{b} & p \end{pmatrix} = \begin{pmatrix} \bar{\psi}^{\dot{\alpha}} \psi^{\dot{\beta}} \end{pmatrix} \begin{pmatrix} m \delta_{\dot{\alpha} \dot{\beta}} \end{pmatrix} \begin{pmatrix} \Sigma_i p_i \end{pmatrix} \begin{pmatrix} m^\alpha \beta \end{pmatrix}
  \]

- Add new polarisation vector \( \epsilon_0 = \frac{1}{m\sqrt{2}} \)
- Need matrix structure in fermion propagators and vertices, e.g.
Main conclusion

Matrix structure unavoidable with massive fermions
Proceed as before to calculate without algebra
Consider the same diagram of $f_1^+ \bar{f}_2^- \to \gamma_3^+ \gamma_4^-$ as before but include mass $m_f$

- Obtain 3 new terms
- Simplify with choices of $q_1, q_2, r_3, r_4$

$$e^{i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{\langle p_i^0 q_i \rangle}, \quad e^{-i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^0]}$$

$$e^{2i\varphi} = \frac{-2ie^2}{(s_{23} - m_f^2)} \langle r_3 3 \rangle \langle 4 r_4 \rangle$$

$$+ m_f \left( \sqrt{\alpha_2} e^{i\varphi_2} - \sqrt{\alpha_1} e^{-i\varphi_2} \right)$$

$$= \sqrt{\alpha_1 \alpha_2} e^{i(\varphi_2 - \varphi_1)}$$
### Incoming Massive Spinors in Chirality Flow

$$p^\mu = p^{b,\mu} + \alpha q^\mu, \quad (p^b)^2 = q^2 = 0, \quad e^{i\phi} \sqrt{\alpha} = \frac{m}{\langle p^b q \rangle}, \quad e^{-i\phi} \sqrt{\alpha} = \frac{m}{[q p^b]}$$

Spin operator
$$\Sigma^\mu s_\mu = \gamma^5 s^\mu \gamma^\mu, \quad s^\mu = \frac{1}{m} (p^{b,\mu} - \alpha q^\mu)$$

<table>
<thead>
<tr>
<th>Spinor</th>
<th>Feynman</th>
<th>Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{v}^-(p)$</td>
<td>$\bar{v}^+(p)$</td>
<td>$u^-(p)$</td>
</tr>
<tr>
<td>$\bar{v}^-(p)$</td>
<td>$\bar{v}^+(p)$</td>
<td>$u^-(p)$</td>
</tr>
</tbody>
</table>

Andrew Lifson  
Automating Chirality Flow  
28th November 2022  
9/19
Some Fermion Flow Rules

\[ p^\mu = p^\flat,\mu + \alpha q^\mu, \quad (p^\flat)^2 = q^2 = 0, \quad \alpha = \frac{p^2}{2p\cdot q} \neq 0 \]

Fermion-vector vertex

\[
\begin{pmatrix}
\text{Fermion propagator}
\end{pmatrix}
\]

Left and right chiral couplings may differ
A Massive Illuminating Example

Consider the same diagram of $f_1^+ \bar{f}_2^- \rightarrow \gamma_3^+ \gamma_4^-$ as before but include mass $m_f$

- Obtain 3 new terms
- Simplify with choices of $q_1$, $q_2$, $r_3$, $r_4$

\[
e^{i \varphi_i} \sqrt{\alpha_i} = \frac{m_i}{\langle p_i^p q_i \rangle}, \quad e^{-i \varphi_i} \sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^p]}\]

\[
= -2ie^2 \left( s_{23} - m_f^2 \right) \left< r_3^3 \right> [4r_4] \left\{ p_2^b \right\}
\]

\[
+ m_f \left( \sqrt{\alpha_2} e^{i \varphi_2} - \sqrt{\alpha_1} e^{-i \varphi_2} \right)
\]
A Second Massive Example: \( f_1 \bar{f}_2 \rightarrow W \rightarrow f_3 \bar{f}_4 h_5 \)

- W bosons simplifies (\( C_R = 0 \))
- Simplify with choices of \( q_1, \ldots, q_5 \)
  \[ e^{i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{\langle p_i^\flat q_i \rangle}, \quad e^{-i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^\flat]} \]
- Scalar has no flow line

Step 1: Draw fermion lines: \( \sim C_{L,12} \sqrt{\alpha_2} e^{i\varphi_2} \times C_{L,34} \sqrt{\alpha_3} (\!-\! e^{i\varphi_3}) \sqrt{\alpha_4} (\!-\! e^{i\varphi_4}) \)
A Second Massive Example: $f_1 \bar{f}_2 \rightarrow W \rightarrow f_3 \bar{f}_4 h_5$

- W bosons simplifies ($C_R = 0$)
- Simplify with choices of $q_1, \cdots q_5$
  
  $e^{i\phi_i} \sqrt{\alpha_i} = \frac{m_i}{\langle p_i^b q_i \rangle}$, $e^{-i\phi_i} \sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^b]}$

- Scalar has no flow line

Step 2: Flip arrows and connect: $C_{L,12} C_{L,34} \sqrt{\alpha_2 \alpha_3} e^{i(\phi_2 + \phi_3)}$

$$\times \begin{bmatrix} q_2 & q_3 \\ \sqrt{\alpha_4} e^{i\phi_4} & q_4 \end{bmatrix}$$
Lorentz Group Representations

Lorentz group elements: $e^{i(\theta_i J_i + \eta_i K_i)}$, $J_i \equiv$ rotations, $K_i \equiv$ boosts

- Lorentz group generators $\simeq$ 2 copies of $su(2)$ generators
  - $so(3, 1)_C \simeq su(2) \oplus su(2)$

Group algebra defined by commutator relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k$$

$$N_i^\pm = \frac{1}{2}(J_i \pm iK_i), \quad [N_i^-, N_j^+] = 0,$$

$$[N_i^-, N_j^-] = i\epsilon_{ijk}N_k^- \quad \text{and} \quad [N_i^+, N_j^+] = i\epsilon_{ijk}N_k^+$$

- Representations (i.e. realisations of $N_i^\pm$)
  - $(0, 0)$ scalar particles
  - $(\frac{1}{2}, 0)$ left-chiral and $(0, \frac{1}{2})$ right-chiral Weyl (2-component) spinors.
  - $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, Dirac (4-component) spinors.
  - $(\frac{1}{2}, \frac{1}{2})$ vectors, e.g. gauge bosons
How to Calculate? Spinor-Helicity

Give each particle a defined helicity ⇒ amplitude now a number!

Spinors (in chiral basis):

\[
\begin{align*}
    u^+(p) &= v^-(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} \\
    u^-(p) &= v^+(p) = \begin{pmatrix} |p\rangle \\ 0 \end{pmatrix} \\
    \bar{u}^+(p) &= \bar{v}^-(p) = \begin{pmatrix} |p| & 0 \end{pmatrix} \\
    \bar{u}^-(p) &= \bar{v}^+(p) = \begin{pmatrix} 0 & \langle p| \end{pmatrix}
\end{align*}
\]

\[\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix}, \quad \sqrt{2}\tau^\mu = (1, \vec{\sigma}), \quad \sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma})\]

- Amplitude written in terms of Lorentz-invariant spinor inner products

\[\langle ij \rangle = -\langle ji \rangle \equiv \langle i| j \rangle \quad \text{and} \quad [ij] = -\langle ji \rangle \equiv [i| j]\]

- These are well known complex numbers, \[\langle ij \rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}\]

- Remove \(\tau/\bar{\tau}\) matrices in amplitude with

\[\langle i| \bar{\tau}^\mu | j \rangle [k| \tau_\mu | l \rangle = \langle il | [kj] \rangle, \quad \langle i| \bar{\tau}^\mu | j \rangle = [j| \tau^\mu | i \rangle\]
How to Calculate a Process

Sum all Feynman diagrams, square, and integrate

Often spin structure is non-trivial

\[ \sum \text{all Feynman diagrams, square, and integrate} \]

e.g.

\[ \sim \left[ \bar{u}(p_1) \gamma^\mu (p_1^\nu + p_4^\nu) \gamma_\nu \gamma^\rho v(p_2) \right] \epsilon_\rho(p_3) \epsilon_\mu(p_4) \]

A mathematical expression we have simplify and square

Most common method: use helicity basis

Each diagram is a complex number, easy to square
Can use algebra to simplify first, or brute force matrix multiplication
Define Problem

Kinematic part of amplitude slowed by spin and vector structures

- Can we still improve on this?
  - Deriving spinor inner products $\langle ij \rangle, [kl]$ requires at least 2 steps
    - Re-write every object as spinors
    - Use Fierz identity $\bar{\tau}^{\mu}_{\alpha \beta} \tau^{\mu}_{\dot{\alpha} \dot{\beta}} = \delta^{\beta}_{\dot{\alpha}} \delta^{\dot{\alpha}}_{\beta}$
  - Not intuitive which inner products we obtain

- In SU(N) use graphical reps for calculations
  - E.g. using the colour-flow method
  - (Also birdtracks etc.)

- Spinor-helicity $\equiv su(2) \oplus su(2)$
  - Can we use graphical reps?
Creating Chirality Flow: Building Blocks

A flow is a directed line from one object to another. $su(2)$ objects have dotted indices and $su(2)$ objects undotted indices.

- First step: Ansatz for spinor inner products (only possible Lorentz invariant)
  \[ \langle i | \alpha | j \rangle \equiv \langle ij \rangle = -\langle ji \rangle = \overrightarrow{i} \quad \overleftarrow{j} \]
  \[ [i | \dot{\beta} | j] \dot{\beta} \equiv [ij] = -[ji] = \overrightarrow{i} \quad \overleftarrow{j} \]

- Spinors and Kronecker deltas follow
  \[ \langle i | \alpha \rangle = \overrightarrow{i} \quad \overleftarrow{j} \]
  \[ [i | \dot{\beta} \rangle = \overrightarrow{i} \quad \overleftarrow{j} \]
  \[ \delta_{\alpha}^{\beta} \equiv 1_{\alpha}^{\beta} = \overrightarrow{\alpha} \quad \overleftarrow{\beta} \]
  \[ \delta_{\dot{\alpha}}^{\dot{\beta}} \equiv 1_{\dot{\alpha}}^{\dot{\beta}} = \overrightarrow{\dot{\alpha}} \quad \overleftarrow{\dot{\beta}} \]