



Local Unitarity

Computing cross-sections beyond NLO accuracy
with Local Unitarity

Zeno Capatti

In collaboration with:

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Orsay, November 28th, 2022

Photo: posing in front of Monte Carlo's casino, 2021

“It was at that time that I suggested an obvious name for the statistical method—a suggestion not unrelated to the fact that Stan had an uncle who would borrow money from relatives because he “just had to go to Monte Carlo.” The name seems to have endured.”

Nicholas Metropolis, “The beginning of the Monte Carlo method”,
Los Alamos Science Special Issue 1987

Local Unitarity: framing the problem

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A cross-section admits a perturbative expansion when $\alpha < 1$

$$\sigma = \sum_{L=1}^{\infty} \alpha^L \sigma^{(L)}$$

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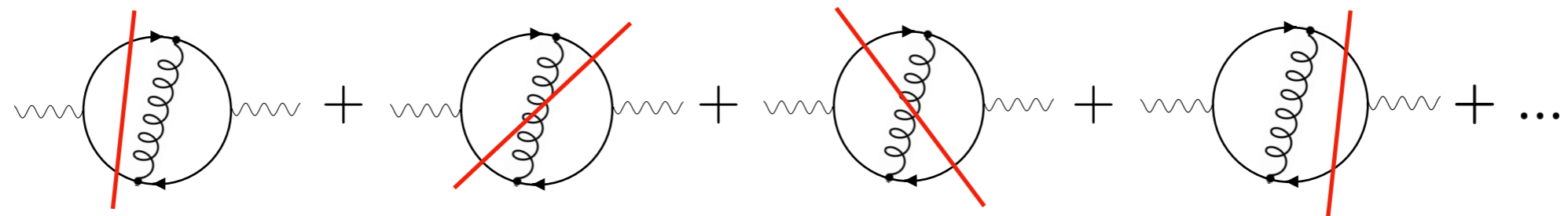
The coefficients can be represented as a sum of interference diagrams

Local Unitarity: framing the problem

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$$\sigma^{(2)} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \dots$$


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The diagrams are interference diagrams for $\sigma^{(2)}$, each consisting of a circular loop with a wavy line entering from the left and a wavy line exiting to the right. The loop contains a fermion line with arrows indicating a clockwise direction. A red vertical line is drawn through each diagram, representing a Cutkosky cut. The first diagram has the cut on the left side. The second and third diagrams have the cut on the right side. The fourth diagram has the cut on the left side. A red arrow points from the text "Cutkosky cut" to the red line in the fourth diagram.

Local Unitarity: framing the problem

A cross-section admits a perturbative expansion when $\alpha < 1$

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The coefficients can be represented as a sum of interference diagrams

$$\sigma^{(2)} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \dots$$

The diagram shows the second-order coefficient $\sigma^{(2)}$ as a sum of four interference diagrams. Each diagram consists of a circular loop with a wavy line entering from the left and a wavy line exiting to the right. Inside the loop, a curly line represents a fermion loop. A vertical red line is drawn through each loop, representing a cut. The first diagram has the red line on the left side. The second and third diagrams have the red line on the right side. The fourth diagram has the red line on the left side, and a red arrow points to it with the label "Cutkosky cut". The series continues with an ellipsis.

Interference diagrams themselves can be represented as integrals of amplitudes

Local Unitarity: framing the problem

A cross-section admits a perturbative expansion when $\alpha < 1$

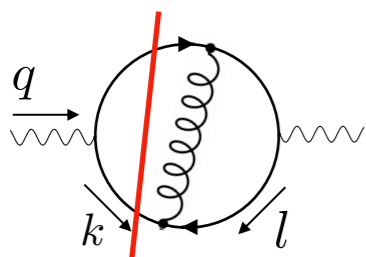
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The diagrams are interference diagrams for $\sigma^{(2)}$, each consisting of a circular loop with a wavy line entering from the left and a wavy line exiting to the right. A vertical red line is drawn through each loop. The first diagram has the red line on the left side. The second diagram has the red line on the right side. The third diagram has the red line on the left side. The fourth diagram has the red line on the right side. A red arrow points to the red line in the fourth diagram with the label "Cutkosky cut".

Interference diagrams themselves can be represented as integrals of amplitudes



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A cross-section admits a perturbative expansion when $\alpha < 1$

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The coefficients can be represented as a sum of interference diagrams

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The diagrams are interference diagrams for $\sigma^{(2)}$, each consisting of a circular loop with a wavy line entering from the left and a wavy line exiting to the right. A vertical red line is drawn through each loop. In the first diagram, the red line is on the left side. In the second, it is on the right side. In the third, it is on the left side. In the fourth, it is on the right side. A red arrow labeled "Cutkosky cut" points to the red line in the fourth diagram.

Interference diagrams themselves can be represented as integrals of amplitudes

$$\text{diagram} = \int d^4k \delta^+(k^2) \delta^+((q-k)^2)$$

The diagram is a circular loop with a wavy line entering from the left and a wavy line exiting to the right. A vertical red line is drawn through the loop. The momentum of the incoming wavy line is labeled q . The momentum of the internal fermion line is labeled k . The momentum of the outgoing wavy line is labeled l .

Local Unitarity: framing the problem

A cross-section admits a perturbative expansion when $\alpha < 1$

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The diagrams are interference diagrams for $\sigma^{(2)}$, each consisting of a circular loop with a wavy line entering from the left and a wavy line exiting to the right. The loop contains a fermion line with arrows. A vertical red line is drawn through each loop. In the first diagram, the red line is on the left side. In the second, it is on the right side. In the third, it is on the left side. In the fourth, it is on the right side. An arrow labeled "Cutkosky cut" points to the red line in the fourth diagram.

Interference diagrams themselves can be represented as integrals of amplitudes

$$\text{diagram} = \int d^4k \delta^+(k^2) \delta^+((q-k)^2)$$

Phase space integral

The diagram on the left shows a circular loop with a wavy line entering from the left with momentum q and a wavy line exiting to the right. The loop contains a fermion line with arrows. A vertical red line is drawn through the loop. The momentum k is labeled on the left side of the loop, and the momentum l is labeled on the right side of the loop.

Local Unitarity: framing the problem

A cross-section admits a perturbative expansion when $\alpha < 1$

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Cutkosky cut

Interference diagrams themselves can be represented as integrals of amplitudes

$$\text{diagram} = \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

Phase space integral

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Phase space integral

Loop integral

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Interference diagrams themselves can be represented as integrals of amplitudes

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Phase space integral

Loop integral

Problem: both types of integrals have **infrared** (collinear, soft) **divergences** and thresholds

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Phase space integral

Loop integral

Problem: both types of integrals have **infrared** (collinear, soft) **divergences** and thresholds

Many good methods around to deal with this that work **either** for loop or for phase-space integrals

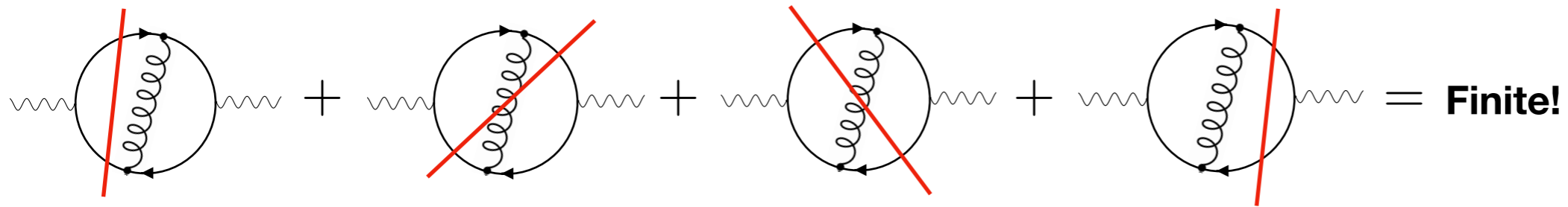
A local KLN cancellation mechanism

A local KLN cancellation mechanism

Our approach instead **combines** singularities of loop and phase-space integrals at the local level through KLN

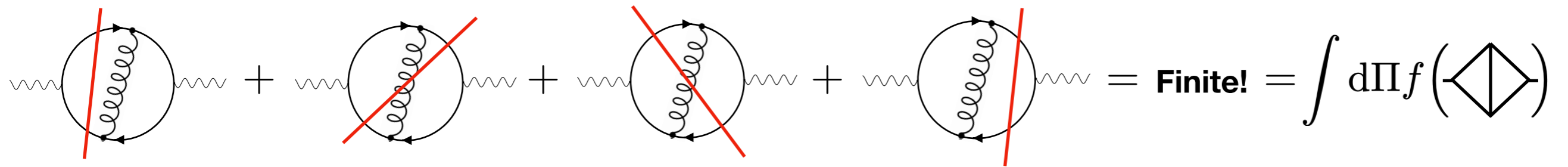
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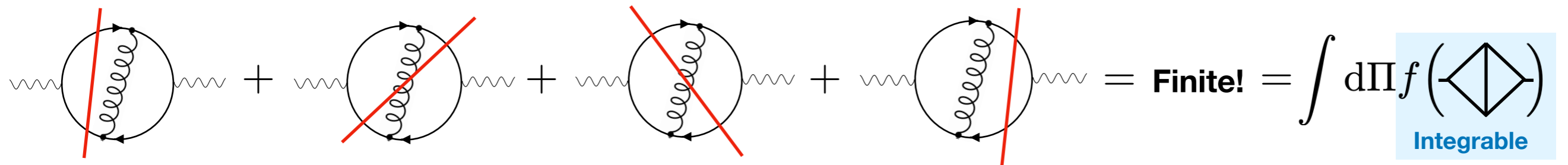


The diagram shows a sequence of four Feynman diagrams representing a sum. Each diagram consists of a circular loop with a wavy line entering from the left and another wavy line exiting to the right. Inside the loop, there is a vertical wavy line. A red diagonal line is drawn across each loop, representing a cut. The red line is vertical in the first and fourth diagrams, and diagonal in the second and third. The diagrams are separated by plus signs. To the right of the fourth diagram is an equals sign, followed by the word "Finite!". This is followed by another equals sign and an integral expression: $\int d\Pi f$ multiplied by a diamond-shaped diagram with a vertical line through its center.

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f \left(\text{Diamond Diagram} \right)$$

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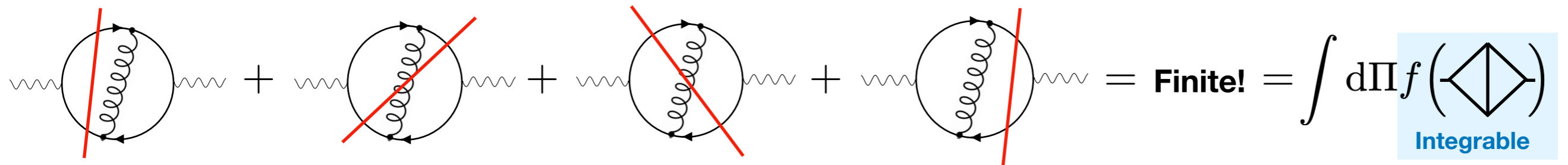
The diagram illustrates the KLN cancellation mechanism. It shows four Feynman diagrams in a row, each representing a different phase of a loop process. Each diagram consists of a circular loop with a wavy line (representing a photon) entering from the left and exiting to the right. The loop contains a fermion line with arrows indicating direction. A red diagonal line is drawn across each loop, representing a cut in the propagator. The four diagrams correspond to different phase regions: the first has a vertical cut, the second has a cut with a negative slope, the third has a cut with a positive slope, and the fourth has a vertical cut. These diagrams are summed together, as indicated by plus signs between them. The result is labeled as "Finite!" and is equated to an integral over phase space, $\int d\Pi f(\text{diamond})$. The diamond-shaped diagram inside the integral represents a four-point function with two incoming and two outgoing lines. The word "Integrable" is written in blue text below the diamond diagram, indicating that the combined integral is free of singularities.

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{diamond})$$

Integrable

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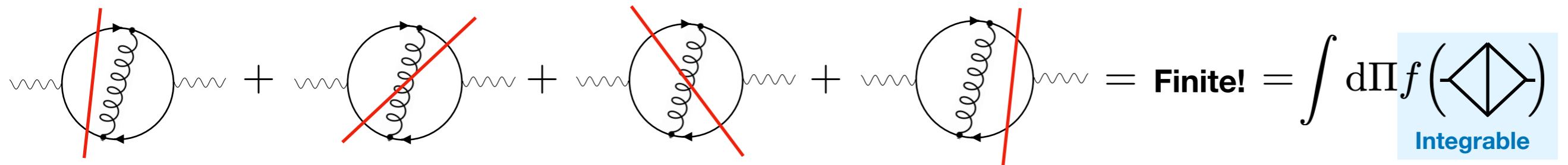
The diagram illustrates the KLN cancellation mechanism. It shows four Feynman diagrams in a sum, each representing a different phase of a loop process. Each diagram consists of a circular loop with a wavy line entering from the left and another wavy line exiting to the right. Inside the loop, there is a wavy line with an arrow pointing clockwise. A red diagonal line is drawn across each loop, representing a cut in the propagator. The red line is vertical in the first and fourth diagrams, and diagonal in the second and third. The sum of these four diagrams is equated to the word "Finite!". This is further equated to an integral over phase space, $\int d\Pi$, multiplied by a function f of a diamond-shaped diagram. The diamond diagram has a vertical line through its center and two diagonal lines forming a diamond shape. The word "Integrable" is written in blue text below the diamond diagram.

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Integrable})$$

Rough idea:

A local KLN cancellation mechanism

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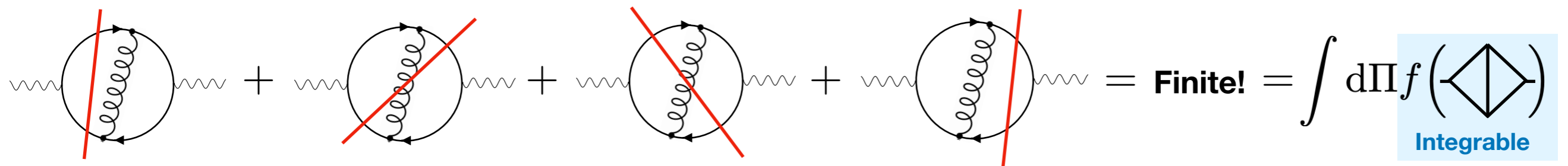

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Integrable})$$

Rough idea:

$$\sum_{i=1}^4 \int d\Pi_i f_i$$

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The diagram shows four Feynman diagrams, each consisting of a circular loop with a wavy line entering from the left and exiting to the right. Inside the loop is a vertical wavy line. A red diagonal slash is drawn across each loop. The diagrams are summed together with plus signs. This sum is equal to the word "Finite!" followed by an equals sign and an integral over phase space $\int d\Pi$ of a function f applied to a diamond-shaped diagram. The diamond diagram is enclosed in a light blue box with the word "Integrable" written below it.

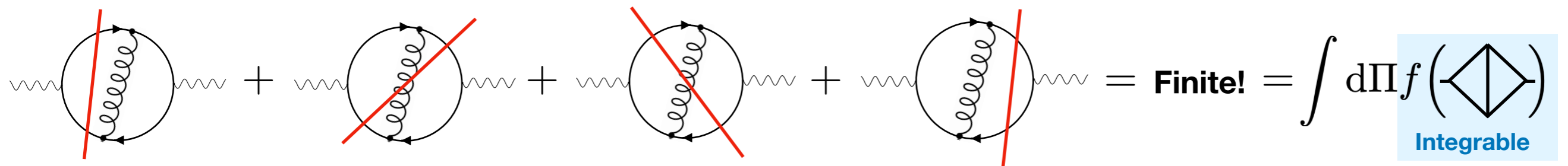
Rough idea:

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Different phase space measure

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The diagram illustrates the KLN cancellation mechanism. It shows four Feynman diagrams, each consisting of a circular loop with a wavy line entering from the left and exiting to the right. A red diagonal line is drawn through the loop in each diagram, representing a singularity. The diagrams are summed together, and the result is shown to be finite and equal to an integral over phase space of a diamond-shaped diagram. The diamond-shaped diagram is labeled as "Integrable".

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Integrable})$$

Rough idea:

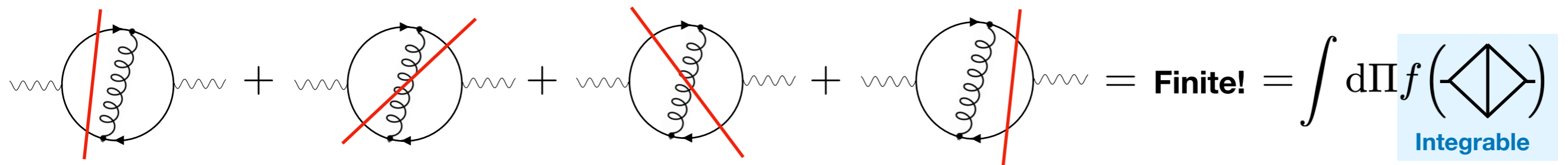
$$\sum_{i=1}^4 \underbrace{\int d\Pi_i f_i}_{\text{Non-integrable singularities}}$$

**Non-integrable
singularities**

Different phase space measure

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The diagram illustrates the KLN cancellation mechanism. It shows four Feynman diagrams, each consisting of a circular loop with a wavy line entering from the left and exiting to the right. A red diagonal line is drawn through the loop in each diagram, representing a singularity. The diagrams are summed together, and the result is shown to be finite and equal to an integral over phase space $d\Pi$ of a function f applied to a diamond-shaped diagram. The diamond diagram is labeled as "Integrable".

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Integrable})$$

Rough idea:

$$\sum_{i=1}^4 \underbrace{\int d\Pi_i f_i}_{\text{Non-integrable singularities}} = \int d\Pi \sum_{i=1}^4 g_i$$

**Non-integrable
singularities**

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The diagram illustrates the KLN cancellation mechanism. It shows four Feynman diagrams, each consisting of a circular loop with a wavy line (representing a photon) entering from the left and exiting to the right. A red diagonal line is drawn through the loop in each diagram, representing a singularity. The diagrams are summed together, and the result is labeled as "Finite!". This is equated to an integral over phase space, $\int d\Pi f(\text{diamond})$, where the diamond-shaped diagram is labeled as "Integrable".

Rough idea:

$$\sum_{i=1}^4 \underbrace{\int d\Pi_i f_i}_{\text{Non-integrable singularities}} = \underbrace{\int d\Pi \sum_{i=1}^4 g_i}_{\text{No non-integrable singularities}}$$

Different phase space measure

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The diagram illustrates the KLN cancellation mechanism. It shows four Feynman diagrams, each consisting of a loop with a wavy line and a red diagonal line through it. These diagrams are summed together, resulting in a finite result, which is represented as an integral over phase space $d\Pi$ of a function f applied to a diamond-shaped diagram. The diamond-shaped diagram is labeled as "Integrable".

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Integrable})$$

Rough idea:

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Different phase space measure

Problem: $d\Pi_i$ has to be aligned

A local KLN cancellation mechanism

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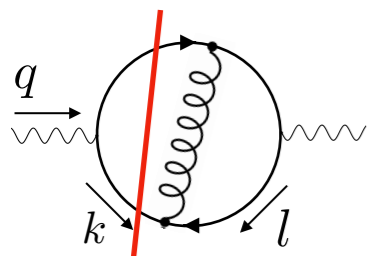
The diagram shows four Feynman diagrams, each consisting of a circle with a wavy line on the left and right, and a vertical red line passing through a loop inside. The diagrams are summed together, followed by an equals sign and the word "Finite!". This is followed by an integral over phase space $\int d\Pi$ of a function f applied to a diamond-shaped diagram. The diamond diagram is highlighted with a blue box and labeled "Integrable".

Rough idea:

$$\sum_{i=1}^4 \underbrace{\int d\Pi_i f_i}_{\text{Non-integrable singularities}} = \underbrace{\int d\Pi \sum_{i=1}^4 g_i}_{\text{No non-integrable singularities}}$$

Different phase space measure

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The diagram shows four Feynman diagrams, each consisting of a circle with a wavy line on the left and a wavy line on the right. Inside the circle is a loop with a wavy line. A red vertical line passes through the loop in each diagram. The diagrams are summed together, and the result is labeled "Finite!". This is equated to an integral over phase space $\int d\Pi$ of a function f applied to a diamond-shaped diagram. The diamond diagram is labeled "Integrable".

Rough idea:

$$\sum_{i=1}^4 \underbrace{\int d\Pi_i f_i}_{\text{Non-integrable singularities}} = \underbrace{\int d\Pi \sum_{i=1}^4 g_i}_{\text{No non-integrable singularities}}$$

Different phase space measure

Problem: $d\Pi_i$ has to be aligned

The diagram shows a Feynman diagram with a red vertical line through a loop. The incoming wavy line on the left is labeled q . The loop has two internal wavy lines, with momenta k and l labeled at the bottom. The diagram is equated to an integral over $d^4k d^4l$ with delta functions $\delta^{(+)}(k^2)$ and $\delta^{(+)}((k-q)^2)$, and a denominator $l^2(l+q)^2(k+l)^2$ and a numerator N .

$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2}$$

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$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Integrable})$

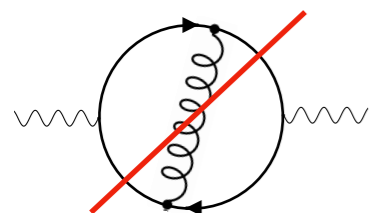
Rough idea:

$$\sum_{i=1}^4 \underbrace{\int d\Pi_i f_i}_{\text{Non-integrable singularities}} = \underbrace{\int d\Pi \sum_{i=1}^4 g_i}_{\text{No non-integrable singularities}}$$

Different phase space measure

Problem: $d\Pi_i$ has to be aligned

$$= \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2}$$



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$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Finite!} = \int d\Pi f(\text{Diamond})$

Integrable

Rough idea:

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Different phase space measure

Problem: $d\Pi_i$ has to be aligned

$$\text{Diagram} = \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k-q)^2) \frac{N}{l^2(l+q)^2(k+l)^2}$$

Problem 1:
Different number of deltas

$$\text{Diagram} = \int d^4k d^4l \delta^{(+)}(k^2) \delta^{(+)}((k+l)^2) \delta^{(+)}((l+q)^2) \frac{N}{l^2(k-q)^2}$$

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Problem 2:
Too few energy variables to solve the deltas

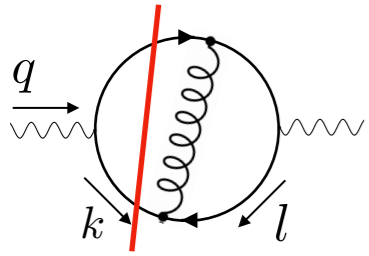
Problem 1: Different number of deltas

Problem 1: Different number of deltas

Observation:

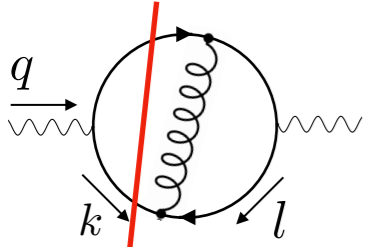
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Observation:

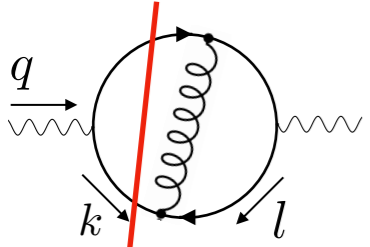


The diagram on the left shows a triangle loop with a wavy external line on the left labeled q . The bottom-left vertex is labeled k and the bottom-right vertex is labeled l . A red vertical line is drawn through the loop, representing a cut. The loop contains a wavy line on the right side.

$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

Problem 1: Different number of deltas

Observation:



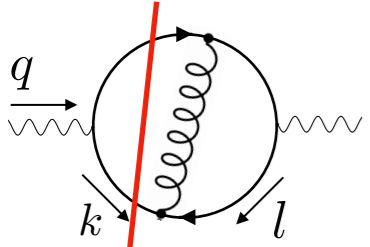
The diagram shows a triangle loop with a vertical red line through it. The left vertex has an incoming wavy line labeled q . The bottom-left vertex has an outgoing wavy line labeled k . The bottom-right vertex has an outgoing wavy line labeled l . The top vertex is connected to the bottom-left vertex by a straight line with an arrow pointing up. The right side of the loop is a wavy line. The bottom side is a straight line with an arrow pointing right.

$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

unconstrained integration over l^0

Problem 1: Different number of deltas

Observation:



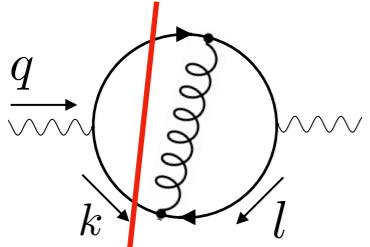
The diagram shows a triangle loop with a vertical red line through it. The left vertex has an incoming wavy line labeled q . The bottom-left vertex has an outgoing wavy line labeled k . The bottom-right vertex has an outgoing wavy line labeled l . The top vertex is connected to the bottom-left vertex by a wavy line, and to the bottom-right vertex by a wavy line. The right side of the loop is a wavy line. The vertical red line is a straight line passing through the loop.

$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2} \quad \text{unconstrained integration over } l^0$$

Solution 1: Perform integration over l^0

Problem 1: Different number of deltas

Observation:



The diagram shows a triangle loop with a vertical red line through it. The left vertex has an incoming wavy line with momentum q . The bottom-left vertex has an outgoing wavy line with momentum k . The bottom-right vertex has an outgoing wavy line with momentum l . The top vertex is connected to the bottom-left vertex by a wavy line, and to the bottom-right vertex by a wavy line. The right side of the triangle is a straight line with a wavy line on it. The red line is vertical and passes through the triangle.

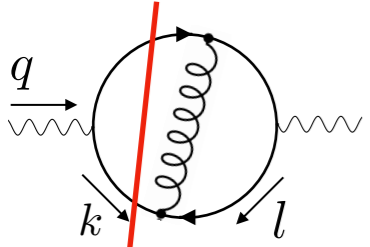
$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

unconstrained integration over l^0

Solution 1: Perform integration over l^0 (LTD, cLTD, TOPT...)

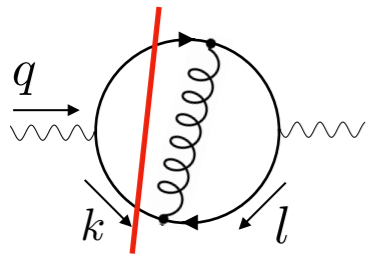
Problem 1: Different number of deltas

Observation:


$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

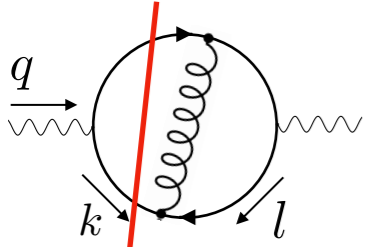
unconstrained integration over l^0

Solution 1: Perform integration over l^0 (LTD, cLTD, TOPT...)


$$=$$

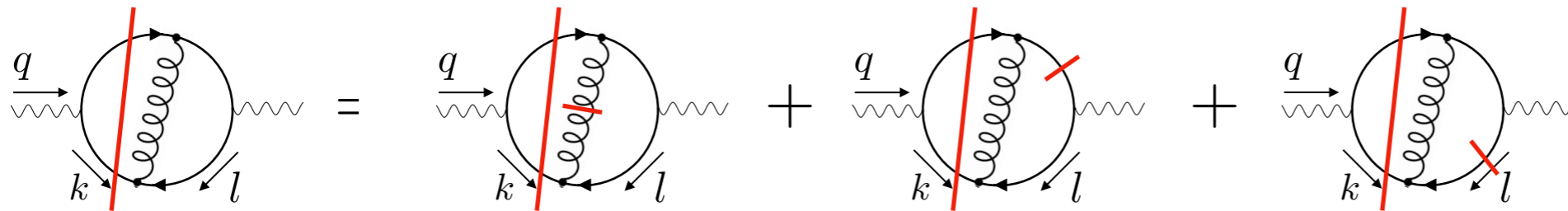
Problem 1: Different number of deltas

Observation:


$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

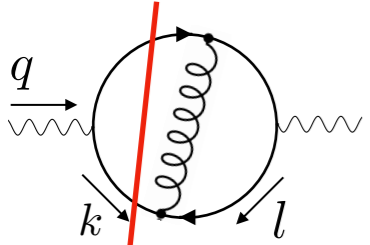
unconstrained integration over l^0

Solution 1: Perform integration over l^0 (LTD, cLTD, TOPT...)


$$= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3}$$

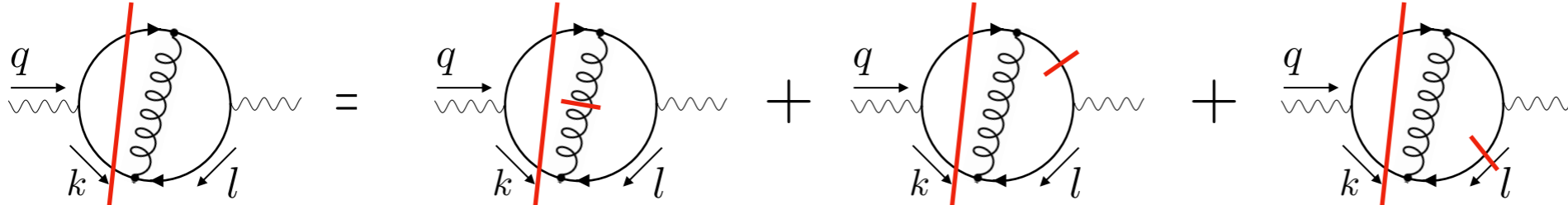
Problem 1: Different number of deltas

Observation:



$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2} \quad \text{unconstrained integration over } l^0$$

Solution 1: Perform integration over l^0 (LTD, cLTD, TOPT...)



Catani, Gleisberg, Krauss, Rodrigo, Winter
arXiv: [0804.3170](https://arxiv.org/abs/0804.3170) (2008)

Bierenbaum, Catani, Draggiotis, Rodrigo
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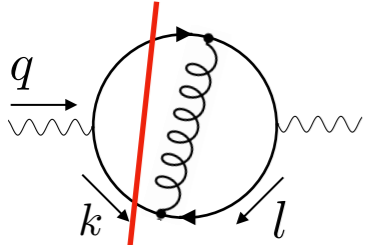
ZC, Hirschi, Kermanschah, Ruijl
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Verdugo, Driencout-Mangin, et al.
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Problem 1: Different number of deltas

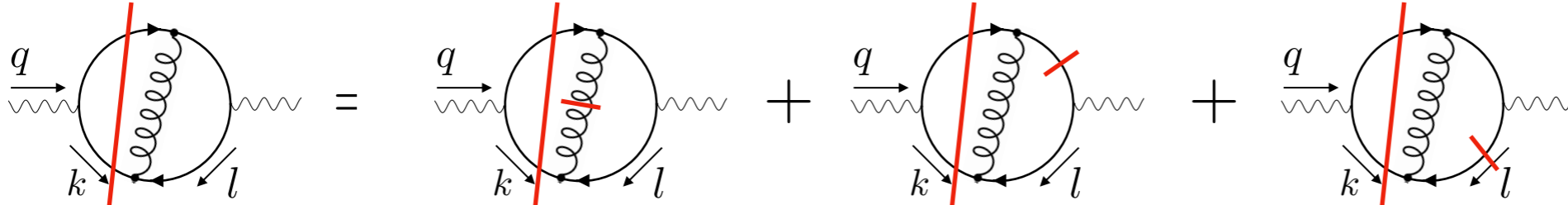
Observation:



$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

unconstrained integration over l^0

Solution 1: Perform integration over l^0 (LTD, cLTD, TOPT...)



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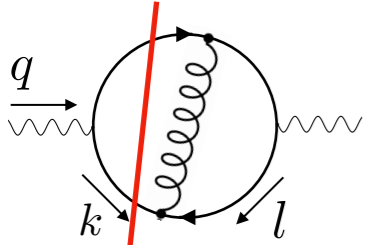
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Observation: Now real and virtual contributions have the same amount of deltas

Problem 1: Different number of deltas

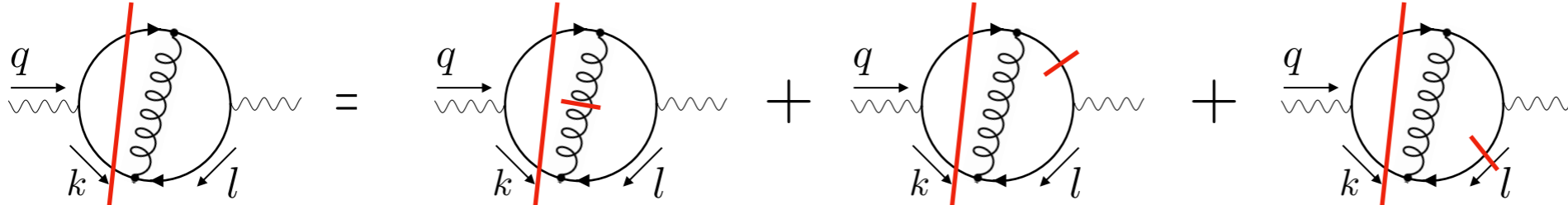
Observation:



$$= \int d^4k \delta^+(k^2) \delta^+((q-k)^2) \int d^4l \frac{N(k, l, q)}{l^2(l+q)^2(k+l)^2}$$

unconstrained integration over l^0

Solution 1: Perform integration over l^0 (LTD, cLTD, TOPT...)



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Observation: Now real and virtual contributions have the same amount of deltas



both have three cut lines!

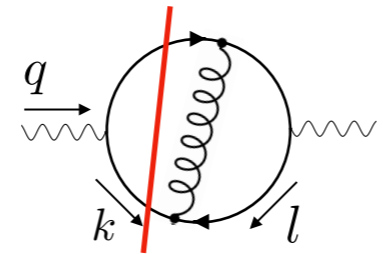
Problem 2: Delta enforcing on-shell energy conservation

Problem 2: Delta enforcing on-shell energy conservation

After killing all
energy integrations...

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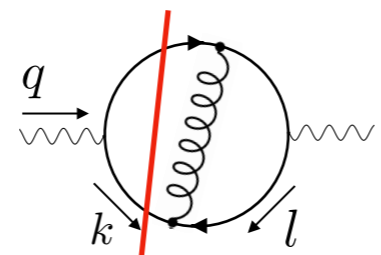


The diagram shows a circular loop with a wavy line entering from the left labeled q and another wavy line exiting to the right. A vertical red line is drawn through the loop, representing a cut. Two arrows, labeled k and l , indicate the direction of momentum flow around the loop.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

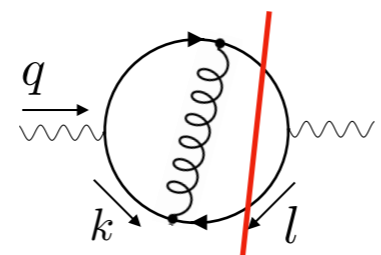
Problem 2: Delta enforcing on-shell energy conservation

After killing all energy integrations...



A Feynman diagram for the function $f_{v,1}$. It consists of a circle with a wavy line on the left labeled q and a wavy line on the right. Inside the circle, there is a vertical red line. A wavy line labeled k goes from the bottom of the red line to the top of the circle. Another wavy line labeled l goes from the top of the red line to the bottom of the circle. A vertical wavy line connects the top and bottom of the circle.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

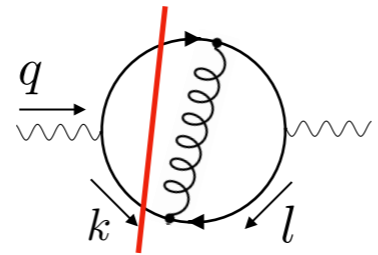


A Feynman diagram for the function $f_{v,2}$. It is identical to the diagram for $f_{v,1}$, but the vertical red line is on the right side of the circle. The wavy line k goes from the bottom of the red line to the top of the circle, and the wavy line l goes from the top of the red line to the bottom of the circle.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Problem 2: Delta enforcing on-shell energy conservation

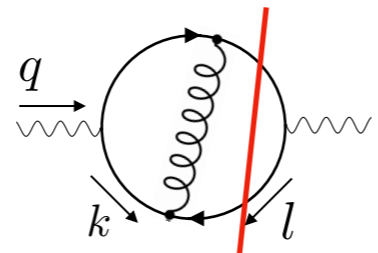
After killing all energy integrations...



A Feynman diagram showing a circular loop with a wavy line entering from the left labeled q and exiting on the right. Inside the loop, a wavy line is connected to a vertical red line. Two arrows labeled k and l indicate the direction of the loop. The red line is positioned on the left side of the loop.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

Solving in the norm misaligns measure

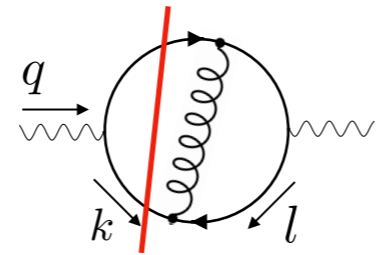


A Feynman diagram similar to the first one, but the vertical red line is positioned on the right side of the loop.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Problem 2: Delta enforcing on-shell energy conservation

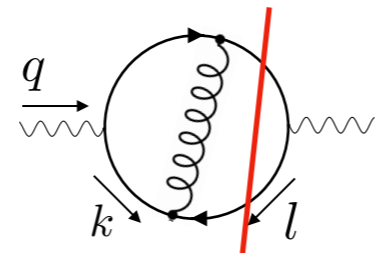
After killing all energy integrations...



A Feynman diagram showing a circle with a wavy line on the left labeled q , a wavy line on the right, and two external lines at the bottom labeled k and l . A vertical red line is drawn through the circle, intersecting the wavy lines and the bottom lines. A wavy line is drawn inside the circle, parallel to the red line.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

Solving in the norm misaligns measure



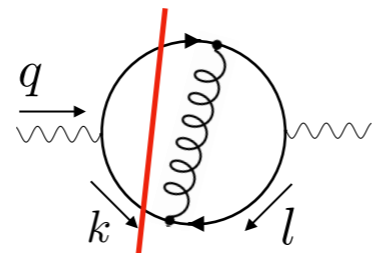
A Feynman diagram identical to the one above, but the vertical red line is shifted to the right, intersecting the wavy line on the right and the bottom line l .

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

Problem 2: Delta enforcing on-shell energy conservation

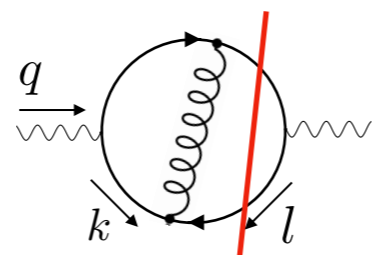
After killing all energy integrations...



A Feynman diagram showing a circular loop with a wavy line on the left and a wavy line on the right. The top part of the loop is a straight line with an arrow pointing right, labeled q . The bottom part of the loop is a straight line with an arrow pointing left, labeled l . A vertical red line is drawn through the loop, intersecting the top and bottom straight lines. The left side of the loop is labeled k .

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

Solving in the norm misaligns measure



A Feynman diagram similar to the one above, but the vertical red line is shifted to the right, intersecting the top and bottom straight lines at a different position. The left side of the loop is labeled k .

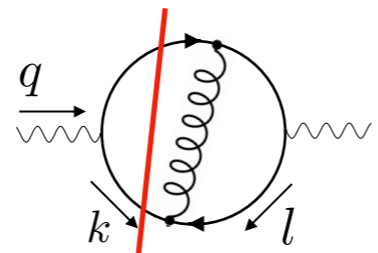
$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k})$$

Problem 2: Delta enforcing on-shell energy conservation

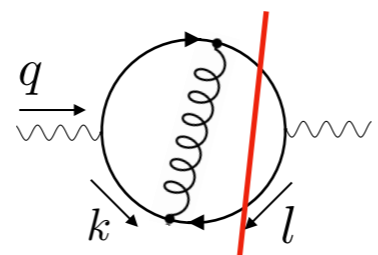
After killing all energy integrations...



A Feynman diagram showing a circle with a wavy line on the left labeled q , a wavy line on the right, and two external lines at the bottom labeled k and l . A vertical red line is drawn through the circle. A wavy line is drawn inside the circle, parallel to the red line.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

Solving in the norm misaligns measure



A Feynman diagram similar to the one above, but the vertical red line is shifted to the right, closer to the l line.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k}) = \int d^3\vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k})$$

$$1 = \int dt h(t)$$

Problem 2: Delta enforcing on-shell energy conservation

After killing all energy integrations...

A Feynman diagram showing a circle with a wavy line on the left labeled q , a wavy line on the right, and two external lines at the bottom labeled k and l . A vertical red line is drawn through the circle, intersecting the wavy lines. A curly line is drawn inside the circle, parallel to the red line.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

Solving in the norm misaligns measure

A Feynman diagram identical to the one above, but the vertical red line is shifted to the right, intersecting the wavy line labeled q and the wavy line on the right.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k}) = \int d^3\vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k})$$

$$= \int d^3\vec{k} dt t^3 h(t) \delta(t|\vec{k}| - Q_0) f(t\vec{k})$$

$$1 = \int dt h(t)$$

$$\vec{k} \rightarrow t\vec{k}$$

Problem 2: Delta enforcing on-shell energy conservation

After killing all energy integrations...

A Feynman diagram showing a circular loop with a wavy line labeled 'q' entering from the left and another wavy line exiting to the right. Inside the loop, there are two vertices labeled 'k' and 'l'. A vertical red line is drawn through the loop, intersecting the wavy line 'q' and the vertex 'k'.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

Solving in the norm misaligns measure

A Feynman diagram identical to the one above, but the vertical red line is shifted to the right, intersecting the wavy line 'q' and the vertex 'l'.

$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k}) = \int d^3\vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k})$$

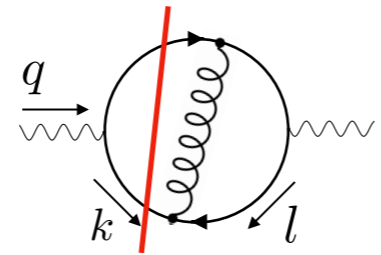
$$1 = \int dt h(t)$$

$$= \int d^3\vec{k} dt t^3 h(t) \delta(t|\vec{k}| - Q_0) f(t\vec{k}) = \int d^3\vec{k} h\left(\frac{Q_0}{|\vec{k}|}\right) \frac{Q_0^3}{|\vec{k}|^4} f\left(\frac{Q_0}{|\vec{k}|}\vec{k}\right)$$

$$\vec{k} \rightarrow t\vec{k}$$

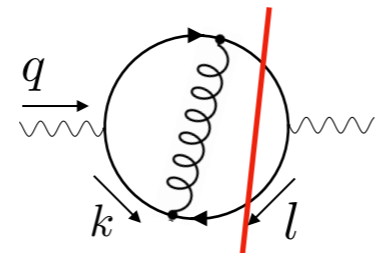
Problem 2: Delta enforcing on-shell energy conservation

After killing all energy integrations...



$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

Solving in the norm misaligns measure



$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k}) = \int d^3\vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k})$$

$$1 = \int dt h(t)$$

$$= \int d^3\vec{k} dt t^3 h(t) \delta(t|\vec{k}| - Q_0) f(t\vec{k}) = \int d^3\vec{k} h\left(\frac{Q_0}{|\vec{k}|}\right) \frac{Q_0^3}{|\vec{k}|^4} f\left(\frac{Q_0}{|\vec{k}|}\vec{k}\right)$$

$$\vec{k} \rightarrow t\vec{k}$$

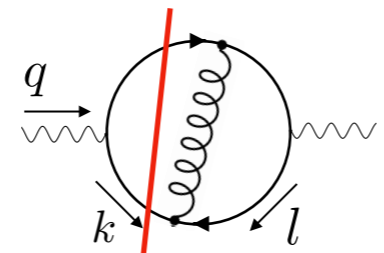
Soper,
arXiv: [9804454](https://arxiv.org/abs/9804454) (1998)

Soper,
arXiv: [0102031](https://arxiv.org/abs/0102031) (2001 @ RADCOR)

ZC, Hirschi, Pelloni, Ruijl
arXiv: [2010.01068](https://arxiv.org/abs/2010.01068) (2020)

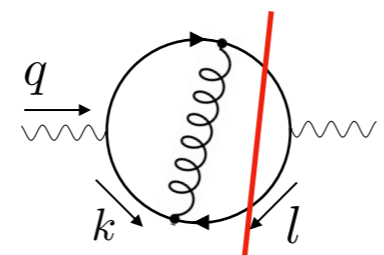
Problem 2: Delta enforcing on-shell energy conservation

After killing all energy integrations...



$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

Solving in the norm misaligns measure



$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k}) = \int d^3\vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k}) \quad 1 = \int dt h(t)$$

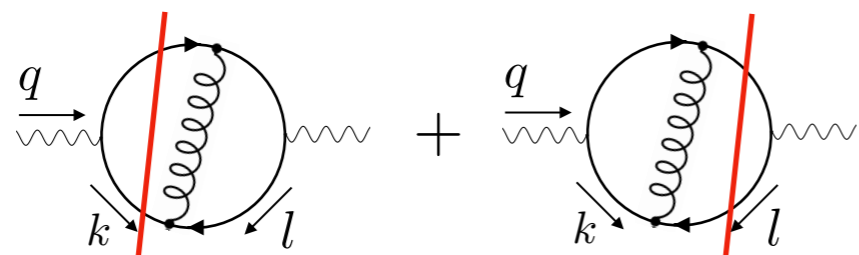
$$= \int d^3\vec{k} dt t^3 h(t) \delta(t|\vec{k}| - Q_0) f(t\vec{k}) = \int d^3\vec{k} h\left(\frac{Q_0}{|\vec{k}|}\right) \frac{Q_0^3}{|\vec{k}|^4} f\left(\frac{Q_0}{|\vec{k}|}\vec{k}\right) \quad \vec{k} \rightarrow t\vec{k}$$

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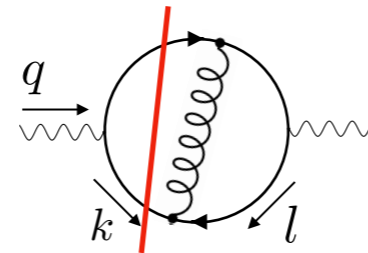
In the end:



$$= \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) \right]$$

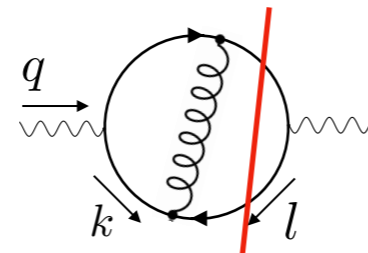
Problem 2: Delta enforcing on-shell energy conservation

After killing all energy integrations...



$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{k}| - Q_0) f_{v,1}$$

Solving in the norm misaligns measure



$$= \int d^3\vec{k} d^3\vec{l} \delta(2|\vec{l}| - Q_0) f_{v,2}$$

Solution 2: Introduce fictitious variable to solve deltas

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k}) = \int d^3\vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k}) \quad 1 = \int dt h(t)$$

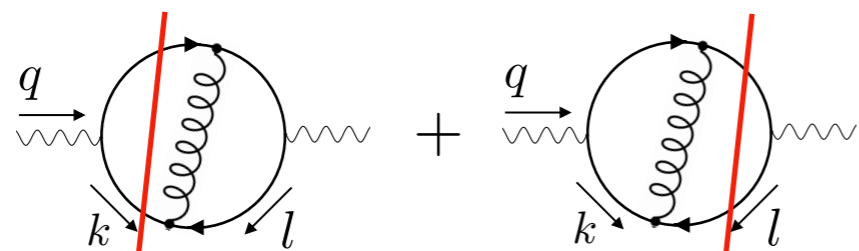
$$= \int d^3\vec{k} dt t^3 h(t) \delta(t|\vec{k}| - Q_0) f(t\vec{k}) = \int d^3\vec{k} h\left(\frac{Q_0}{|\vec{k}|}\right) \frac{Q_0^3}{|\vec{k}|^4} f\left(\frac{Q_0}{|\vec{k}|}\vec{k}\right) \quad \vec{k} \rightarrow t\vec{k}$$

Soper,
arXiv: [9804454](https://arxiv.org/abs/9804454) (1998)

Soper,
arXiv: [0102031](https://arxiv.org/abs/0102031) (2001 @ RADCOR)

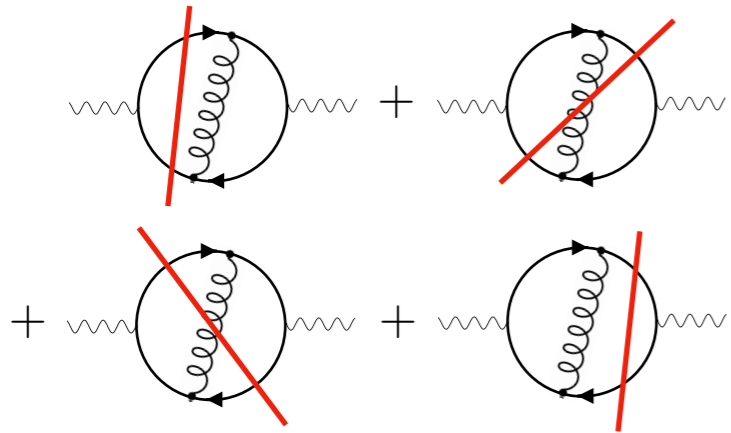
ZC, Hirschi, Pelloni, Ruijl
arXiv: [2010.01068](https://arxiv.org/abs/2010.01068) (2020)

In the end:

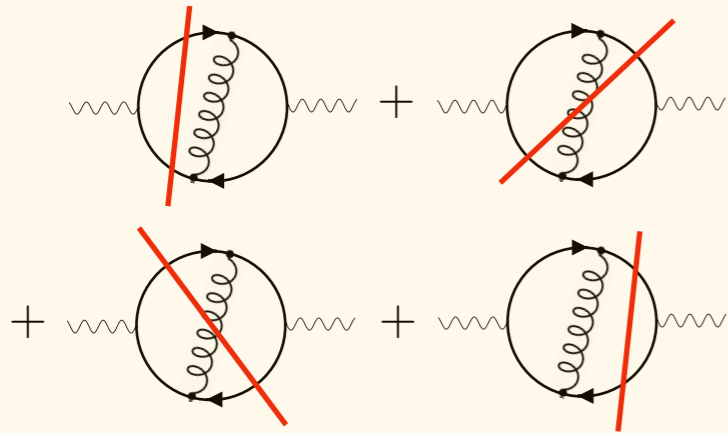


$$= \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) \right]$$

Observation: solved deltas, phase-space has same dimensionality (redundant dimension)

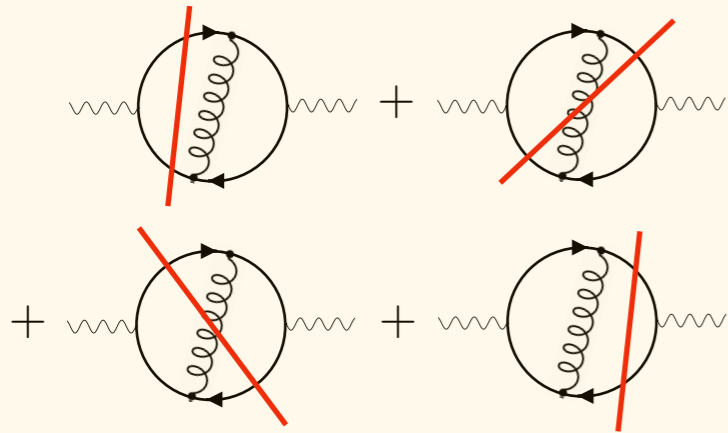


$$= \int d^3\vec{k} d^3\vec{l} \left[g_{v,1}(\vec{k}, \vec{l}) + g_{v,2}(\vec{k}, \vec{l}) + g_{r,1}(\vec{k}, \vec{l}) + g_{r,2}(\vec{k}, \vec{l}) \right]$$



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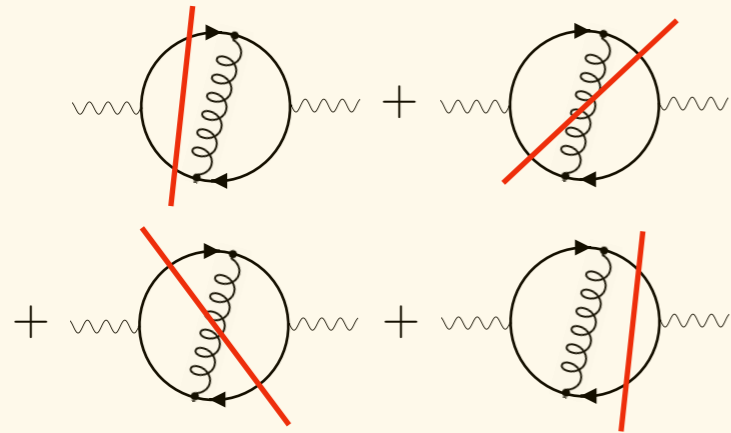
Measure is aligned now!



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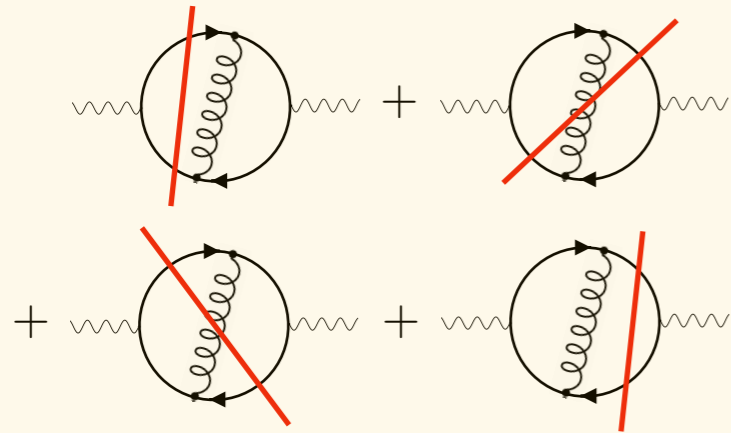


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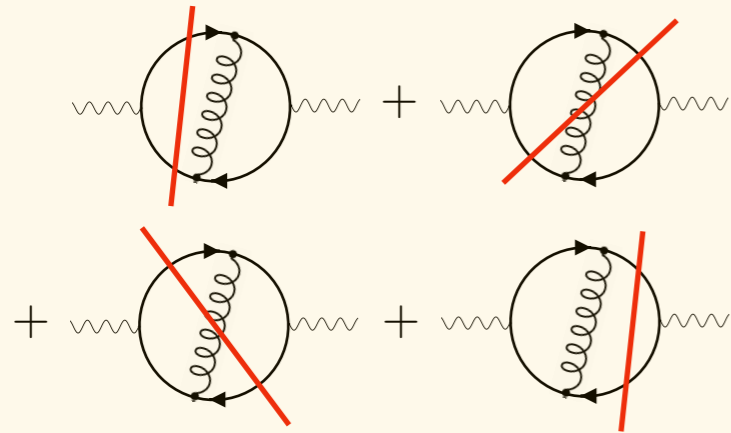


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- ◆ The causal flow is generalised to generic kinematics



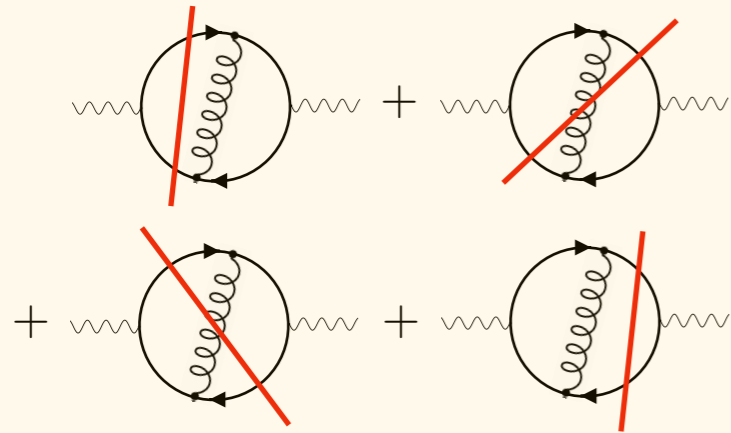
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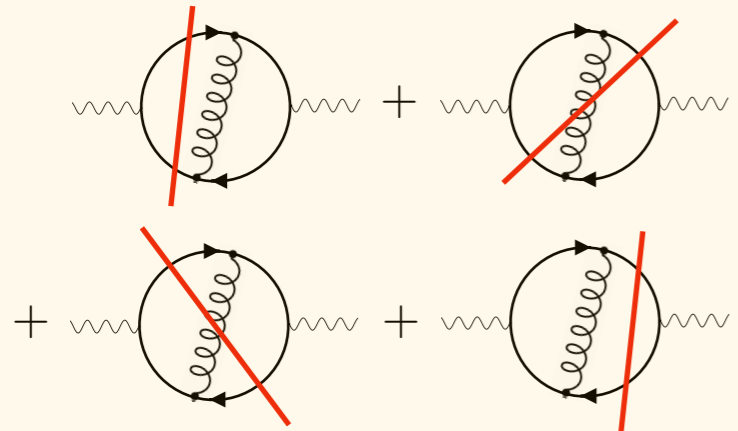
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Proof \Rightarrow [ZC, Hirschi, Pelloni, Ruijl](#)
[arXiv: 2010.01068 \(2020\)](#)

Local IR cancellations, what next?



Local IR cancellations, what next?

- **Automated UV renormalisation:**

▶

▶

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 - Local subtraction of UV divergences

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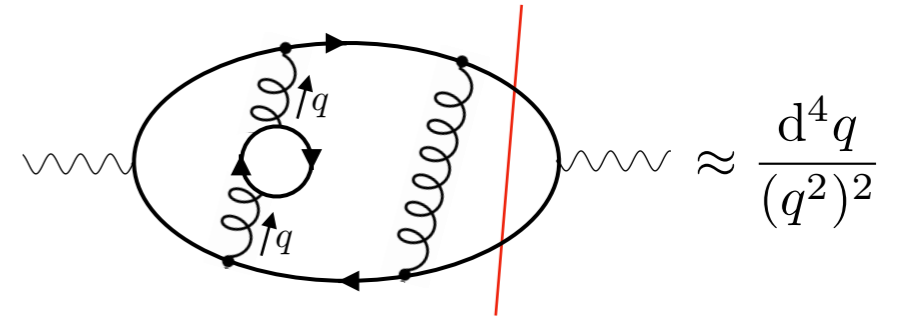
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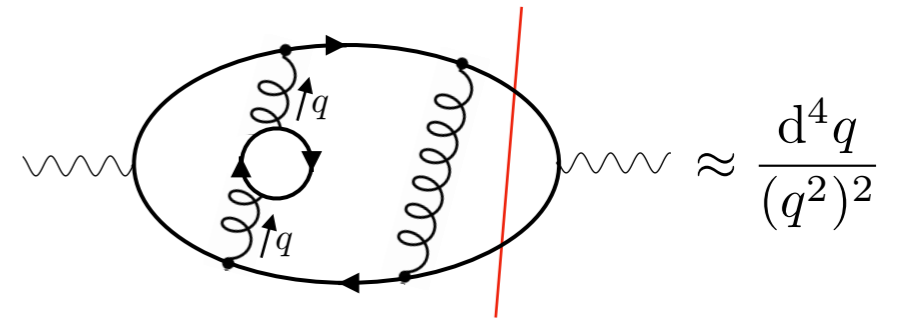
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- **Automated UV renormalisation:**

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- Local subtraction of spurious soft divergences
- Retain local IR cancellations



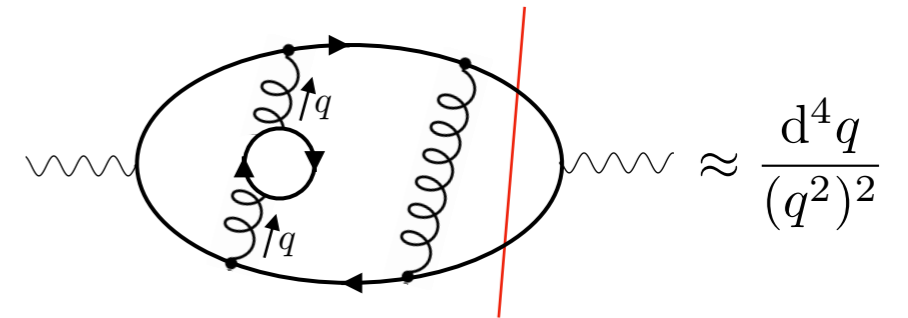
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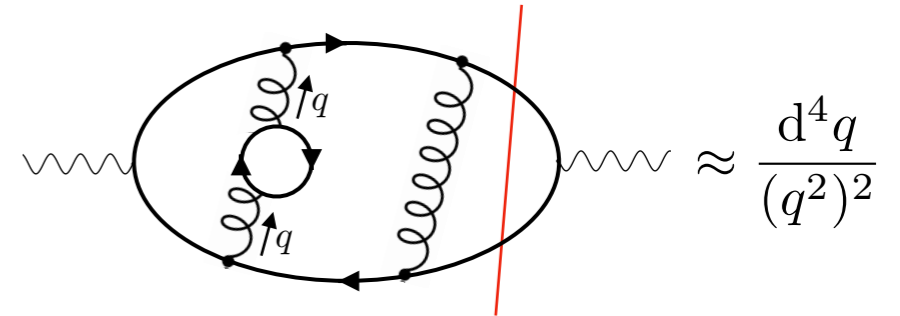
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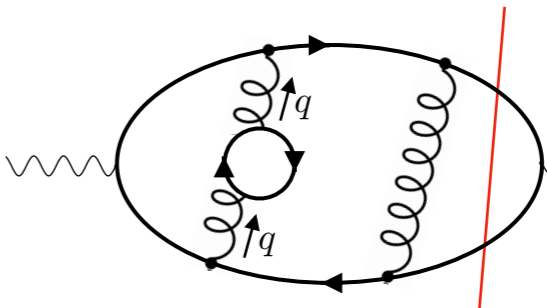


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$\approx \frac{d^4 q}{(q^2)^2}$

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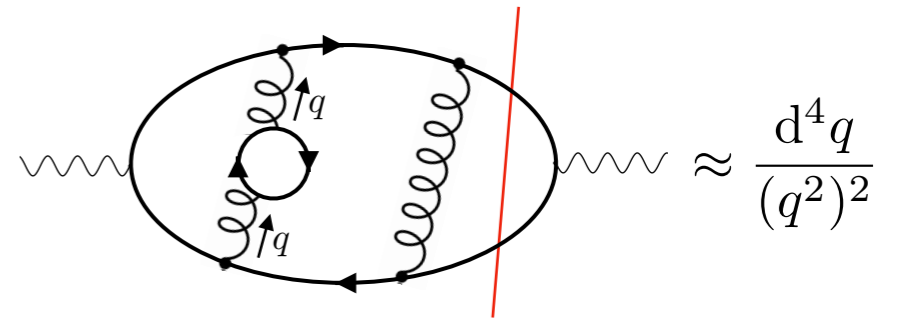
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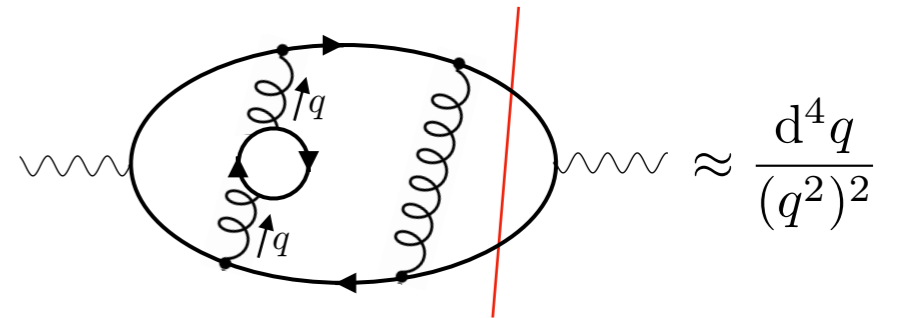
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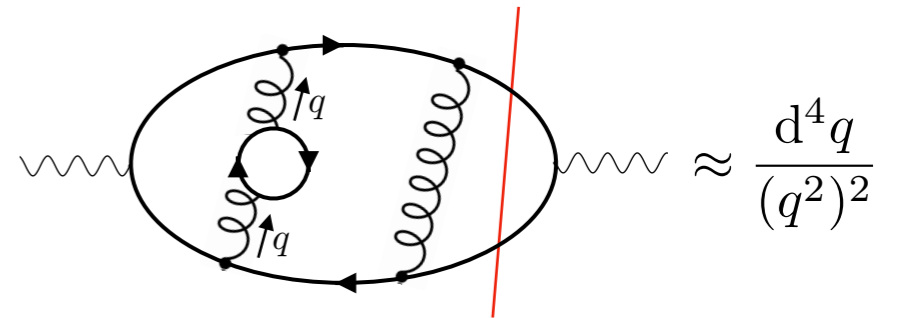
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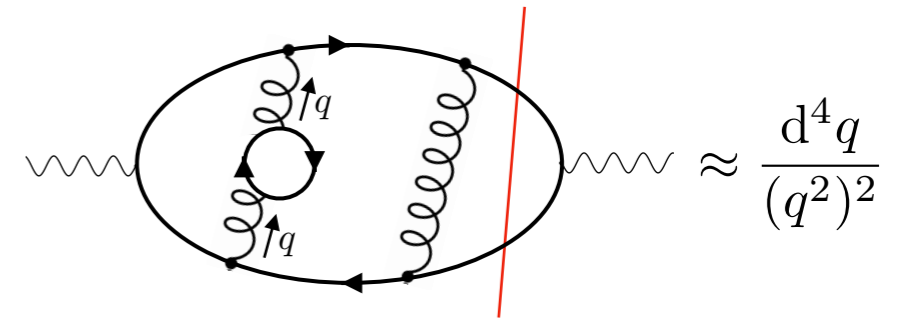
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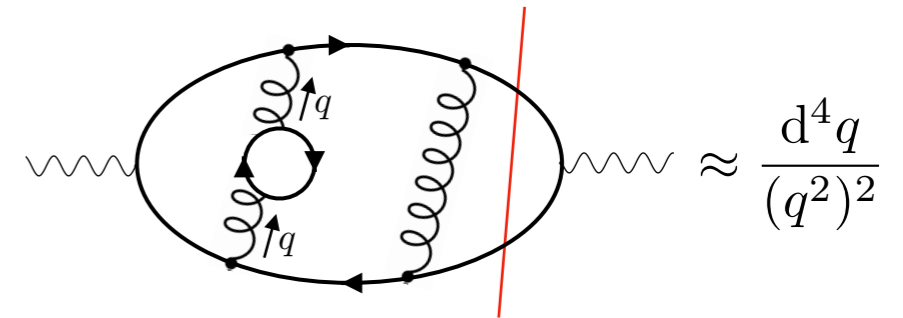
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$$= \frac{1}{(p^2)^2}$$

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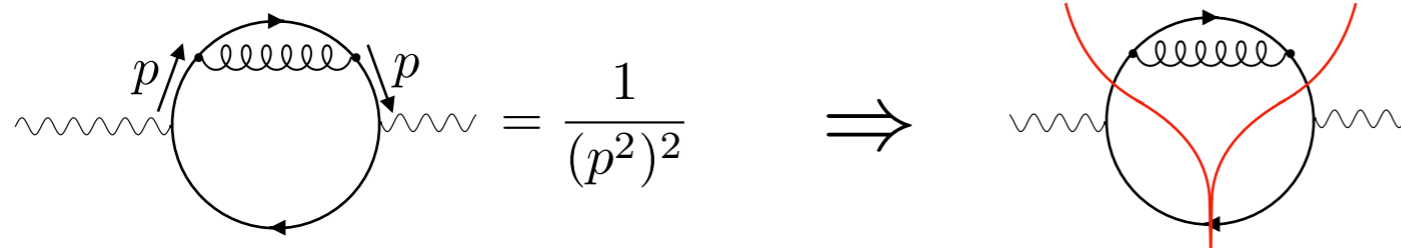
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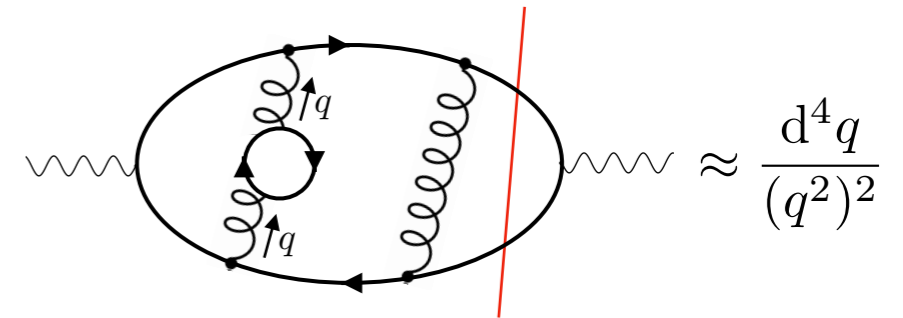
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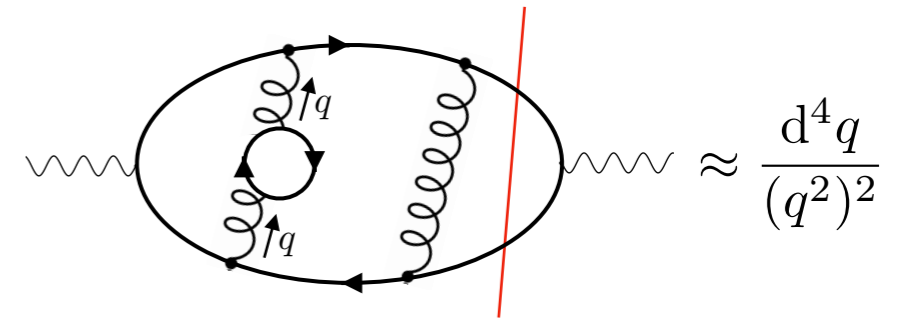
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 & \Rightarrow \text{Diagram: } \text{wavy line} \rightarrow \text{circle with wavy line and red raised propagators} \rightarrow \text{wavy line} = \frac{-2\pi i}{(2-1)!} \frac{1}{(2E_{\vec{p}})^2} \frac{d}{dp^0} \delta(p^0 - E_{\vec{p}})
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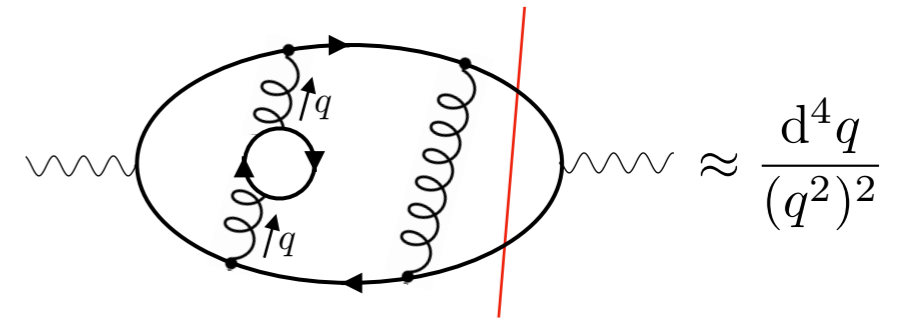
$$\text{Diagram} = \frac{1}{(p^2)^2} \Rightarrow \text{Diagram} = \frac{-2\pi i}{(2-1)!} \frac{1}{(2E_{\vec{p}})^2} \frac{d}{dp^0} \delta(p^0 - E_{\vec{p}})$$

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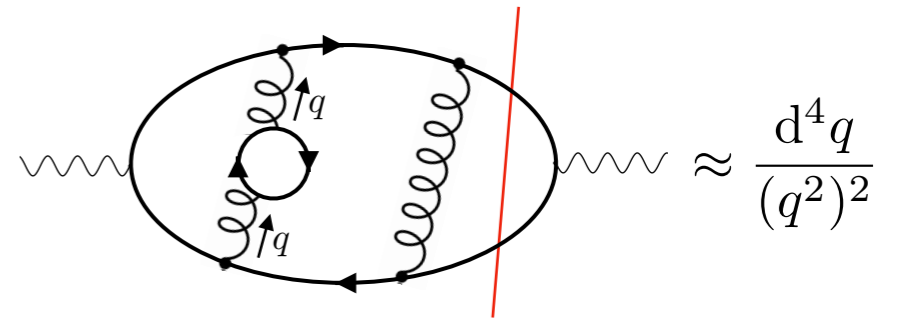
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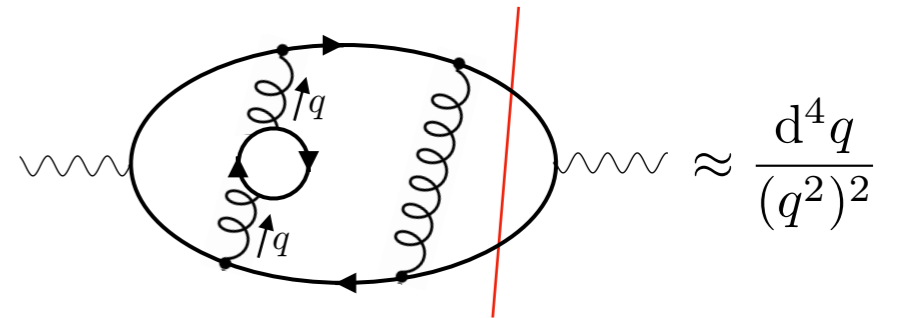
$$\text{Bubble with raised propagators} = \frac{1}{(p^2)^2} \Rightarrow \text{Bubble with raised propagators} = \frac{-2\pi i}{(2-1)!} \frac{1}{(2E_{\vec{p}})^2} \frac{d}{dp^0} \delta(p^0 - E_{\vec{p}})$$

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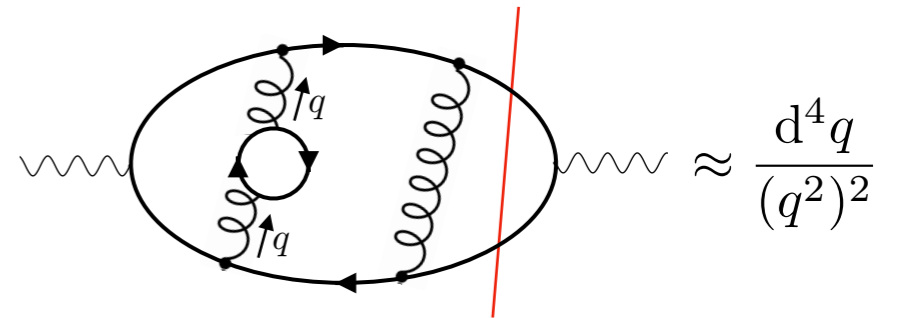
$$\begin{aligned}
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- Generalised cutting rules and LU representation*
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Everything solved generically in [arXiv: 2203.11038](https://arxiv.org/abs/2203.11038)

Tests and results

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NNLO $e^+e^- \rightarrow jj$

and $e^+e^- \rightarrow t\bar{t}$

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NNLO $e^+e^- \rightarrow jj$ Herzog, Ruijl, Ueda, Vermaseren, Vogt
arXiv:1707.01044

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- ✓ NNLO IR cancellations
- ✓ 2-loop UV renormalisation
- ✓ 1,2 loop self-energies

Tests and results

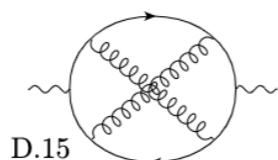
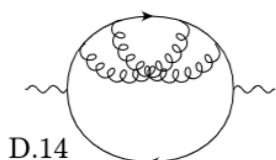
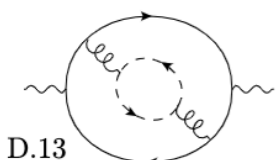
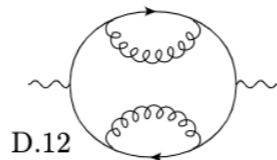
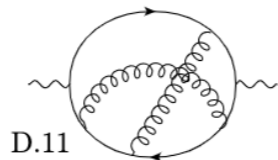
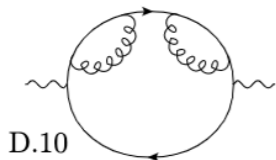
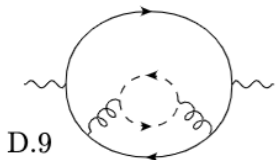
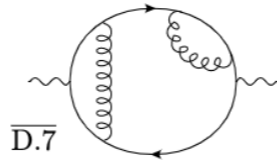
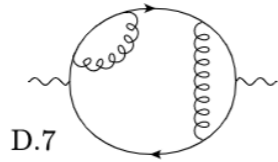
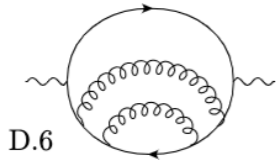
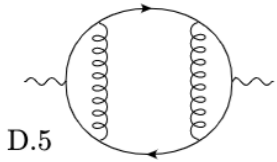
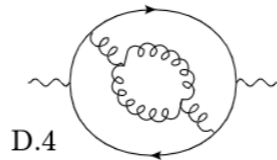
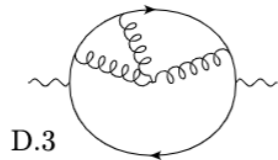
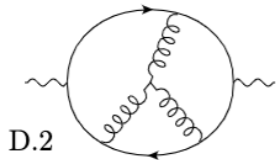
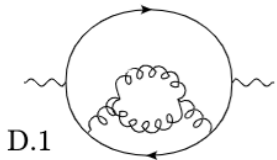
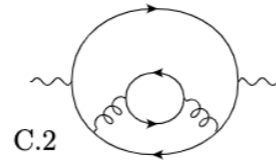
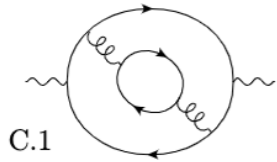
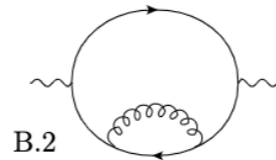
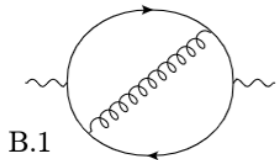
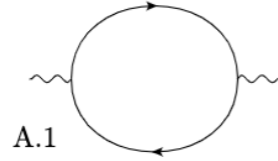
NNLO $e^+e^- \rightarrow jj$

Herzog, Ruijl, Ueda, Vermaseren, Vogt
arXiv:1707.01044

and $e^+e^- \rightarrow t\bar{t}$

Chetykrin, Kuehn, Steinhauser,
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- ✓ NNLO IR cancellations
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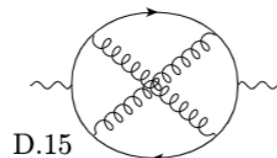
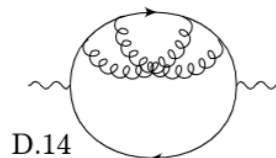
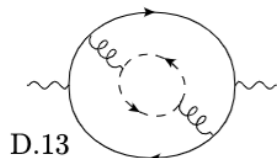
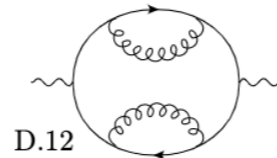
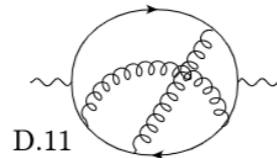
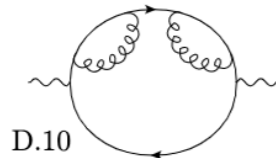
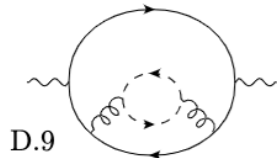
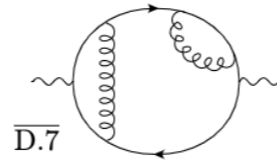
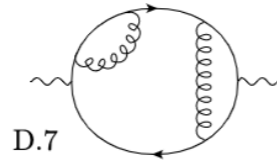
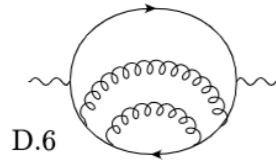
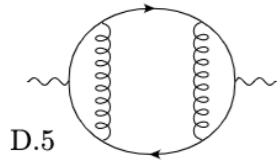
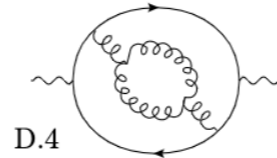
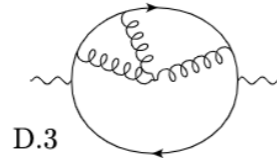
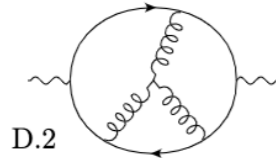
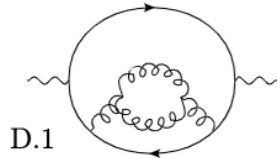
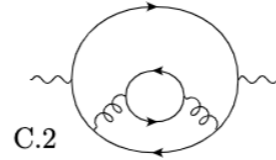
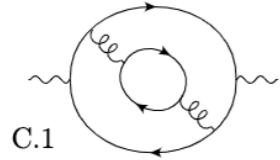
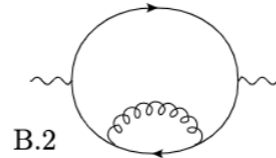
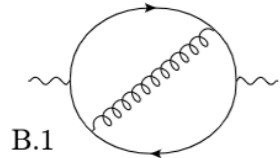
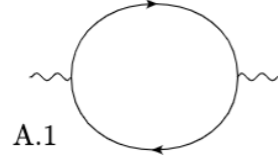
Tests and results

NNLO $e^+e^- \rightarrow jj$
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Herzog, Ruijl, Ueda, Vermaseren, Vogt
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SG id	Ξ	$\sigma_{\gamma^* \rightarrow t\bar{t}}^{[\alpha_s^{(\overline{\text{MS}})}, m_t^{(\text{OS})}]} [\text{GeV}^{-2}]$ $\mu_r^2 = m_t^2, p_{\gamma^*}^2 = (400 \text{ GeV})^2$	Δ [%]	$\sigma_{\gamma^* \rightarrow t\bar{t}}^{[\alpha_s^{(\overline{\text{MS}})}, m_t^{(\text{OS})}]} [\text{GeV}^{-2}]$ $\mu_r^2 = m_t^2, p_{\gamma^*}^2 = (3000 \text{ GeV})^2$	Δ [%]
LO		$\mathcal{O}(\alpha_s^0)$			
A.1	1	$1.387586 \cdot 10^{+00}$	0.0011	$1.509262 \cdot 10^{+01}$	0.000064
Total		$1.387586 \cdot 10^{+00}$	0.0011	$1.509262 \cdot 10^{+01}$	0.000064
NLO		$\mathcal{O}(\alpha_s)$			
B.1	1	$2.52705 \cdot 10^{-01}$	0.034	$-6.3725 \cdot 10^{-01}$	0.071
B.2	2	$1.80050 \cdot 10^{-01}$	0.049	$1.22702 \cdot 10^{+00}$	0.039
Total		$4.3276 \cdot 10^{-01}$	0.028	$5.8977 \cdot 10^{-01}$	0.11
Benchmark		$4.32831 \cdot 10^{-01}$	-0.018	$5.9047 \cdot 10^{-01}$	-0.12
NNLO		$\mathcal{O}(\alpha_s^2)$ ($n_f = 1$ contribution)			
C.1	1	$-1.0022 \cdot 10^{-03}$	0.17	$2.6658 \cdot 10^{-02}$	0.059
C.2	2	$-4.6982 \cdot 10^{-03}$	0.081	$-8.388 \cdot 10^{-03}$	0.30
Total		$-5.7004 \cdot 10^{-03}$	0.073	$1.8270 \cdot 10^{-02}$	0.16
Benchmark		$-5.6982 \cdot 10^{-03}$	0.038	$1.8296 \cdot 10^{-02}$	-0.15
NNLO		$\mathcal{O}(\alpha_s^2)$ (all other contributions)			
D.1	2	$3.8886 \cdot 10^{-02}$	0.031	$6.3163 \cdot 10^{-02}$	0.11
D.2	2	$5.6351 \cdot 10^{-03}$	0.14	$-3.52337 \cdot 10^{-01}$	0.027
D.3	2	$1.76075 \cdot 10^{-02}$	0.055	$5.6646 \cdot 10^{-02}$	0.14
D.4	1	$8.8163 \cdot 10^{-03}$	0.078	$-1.83770 \cdot 10^{-01}$	0.023
D.5	1	$9.200 \cdot 10^{-04}$	0.79	$-7.9531 \cdot 10^{-02}$	0.054
D.6	2	$5.1058 \cdot 10^{-03}$	0.15	$1.1244 \cdot 10^{-02}$	0.51
D.7	2	$6.7284 \cdot 10^{-03}$	0.10	$5.2105 \cdot 10^{-02}$	0.094
$\overline{D.7}$	2	$6.7300 \cdot 10^{-03}$	0.10	$5.2171 \cdot 10^{-02}$	0.094
D.9	2	$2.3361 \cdot 10^{-03}$	0.12	$2.520 \cdot 10^{-03}$	0.73
D.10	2	$3.7418 \cdot 10^{-03}$	0.14	$3.4996 \cdot 10^{-02}$	0.11
D.11	2	$2.0845 \cdot 10^{-03}$	0.083	$2.5486 \cdot 10^{-02}$	0.060
D.12	1	$3.5114 \cdot 10^{-03}$	0.12	$2.8263 \cdot 10^{-02}$	0.10
D.13	1	$8.222 \cdot 10^{-04}$	0.19	$-7.994 \cdot 10^{-03}$	0.13
D.14	2	$1.76075 \cdot 10^{-02}$	0.055	$9.106 \cdot 10^{-03}$	0.19
D.15	1	$-7.242 \cdot 10^{-04}$	0.14	$-1.96633 \cdot 10^{-02}$	0.044
Total		$1.04214 \cdot 10^{-01}$	0.024	$-3.0760 \cdot 10^{-01}$	0.061
Benchmark		$1.0386 \cdot 10^{-01}$	0.34	$-3.0818 \cdot 10^{-01}$	-0.19

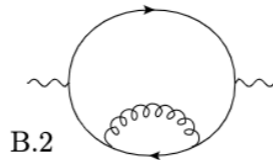
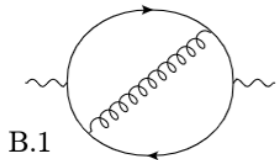
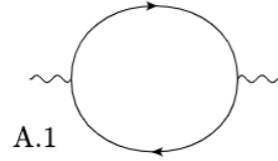
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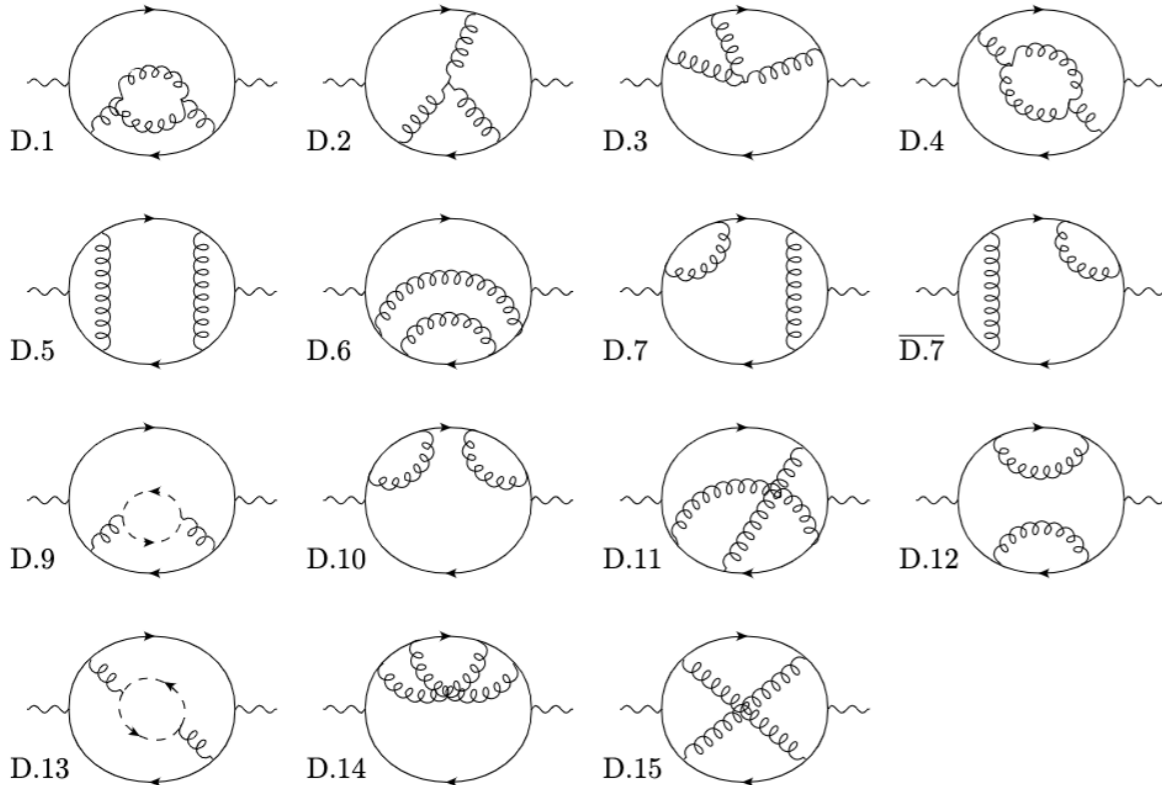
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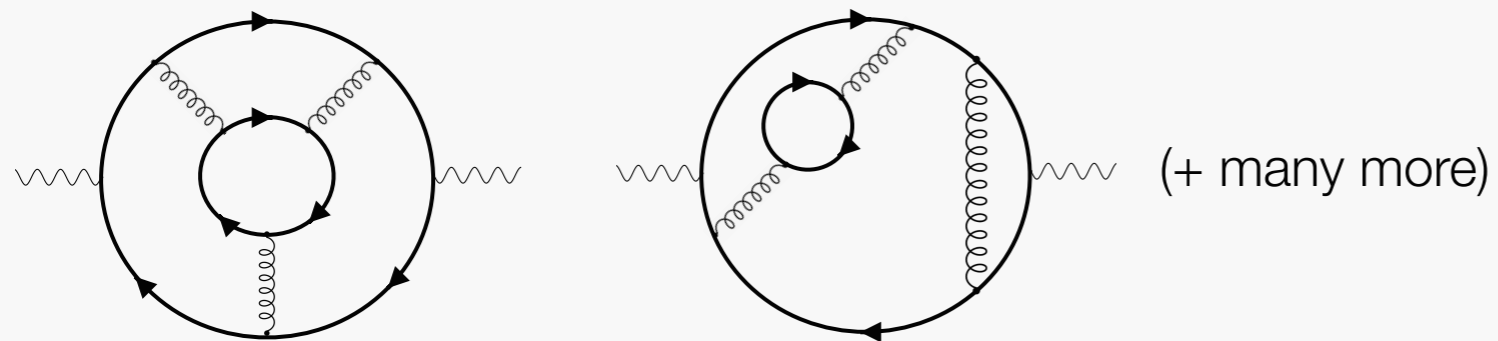
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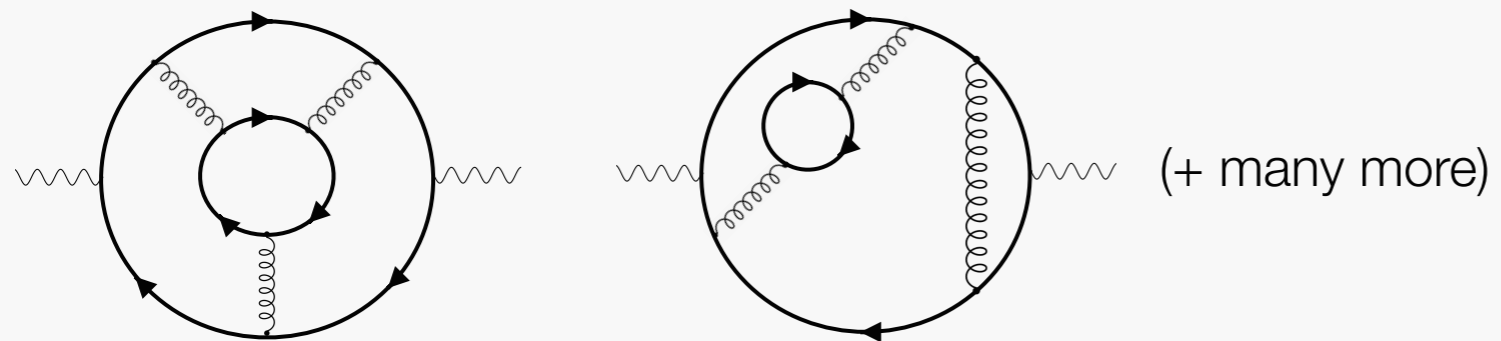
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N_f part @N³LO $e^+e^- \rightarrow jj$

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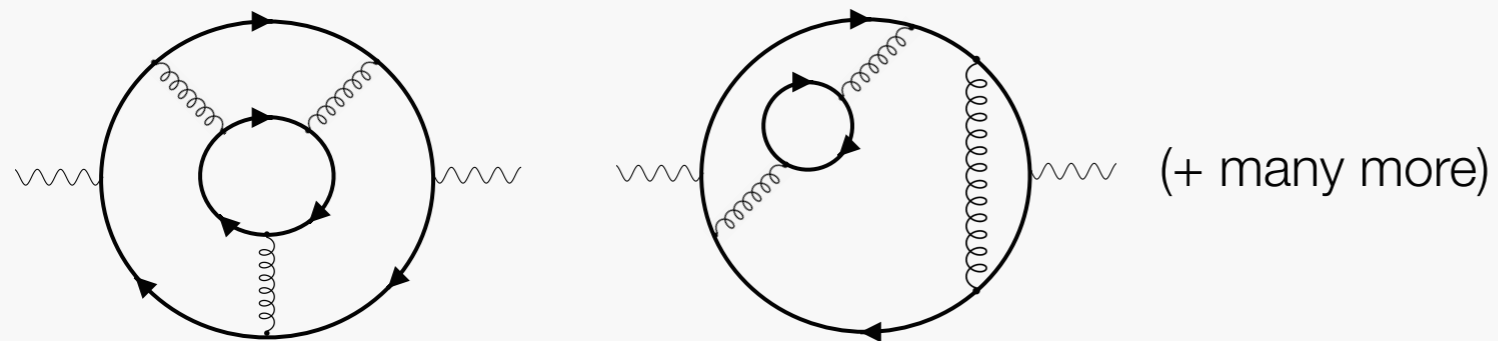


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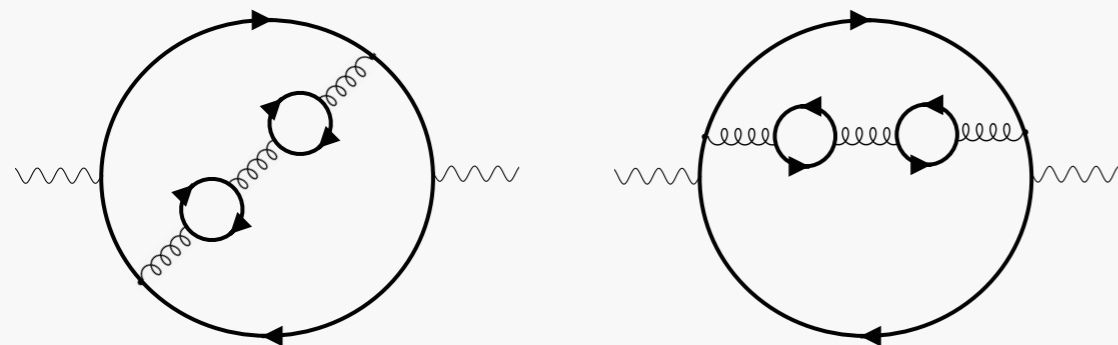


N_f^2 part @N³LO $e^+e^- \rightarrow jj$

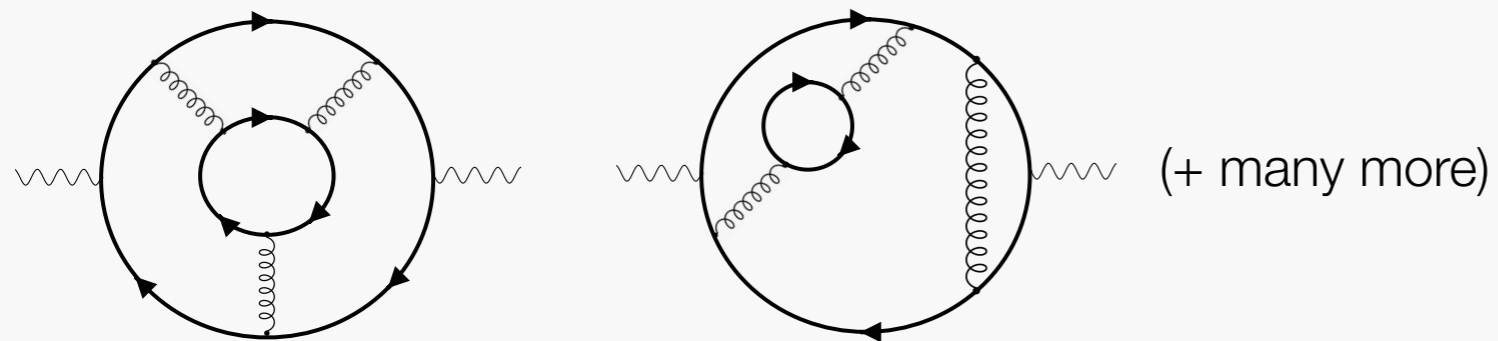
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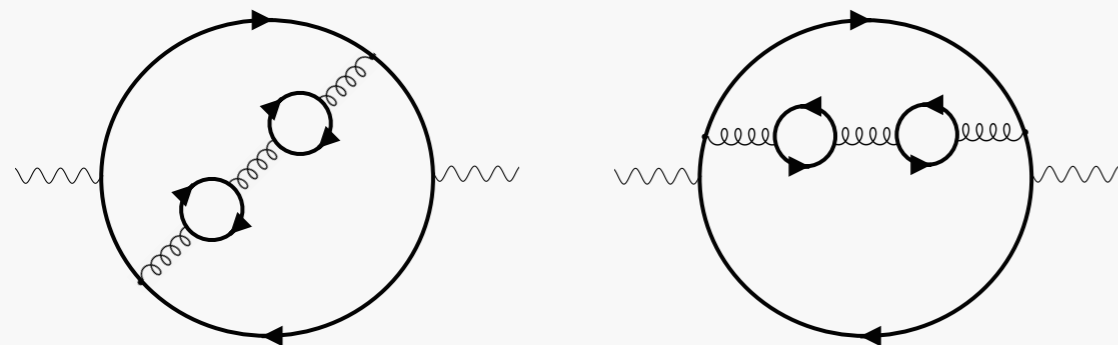
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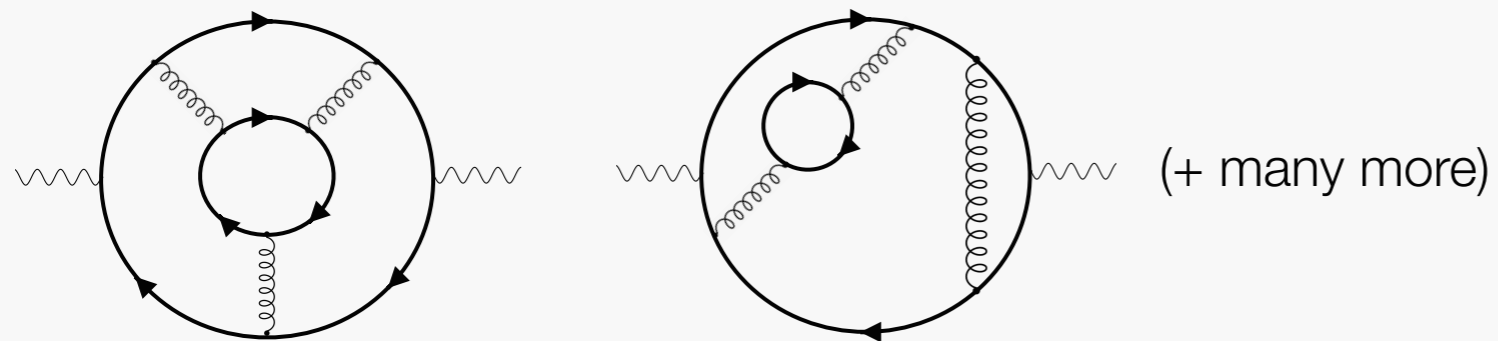


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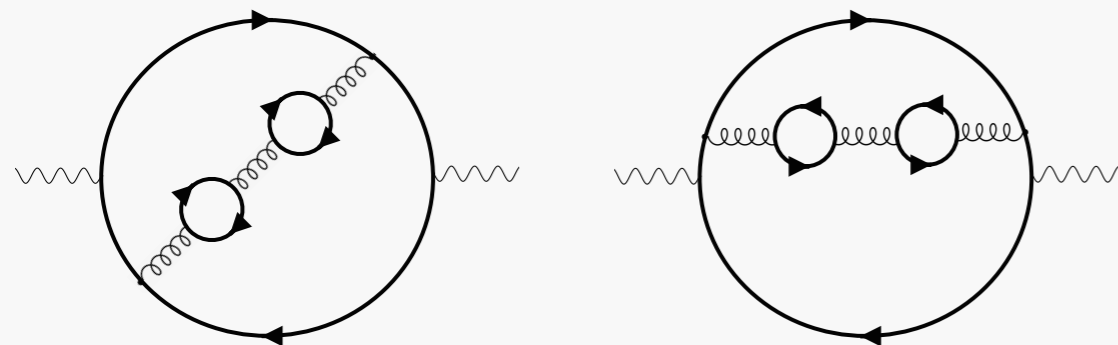


singlet part @N³LO $e^+e^- \rightarrow jj$

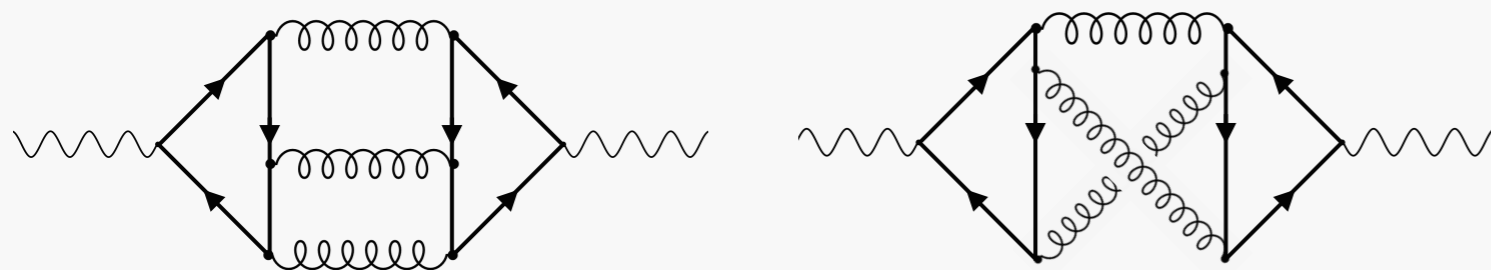
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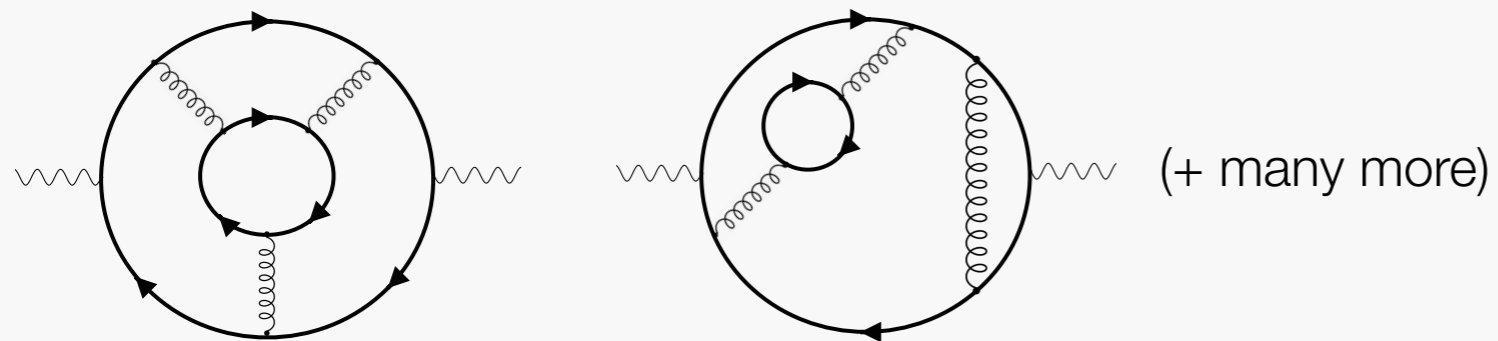
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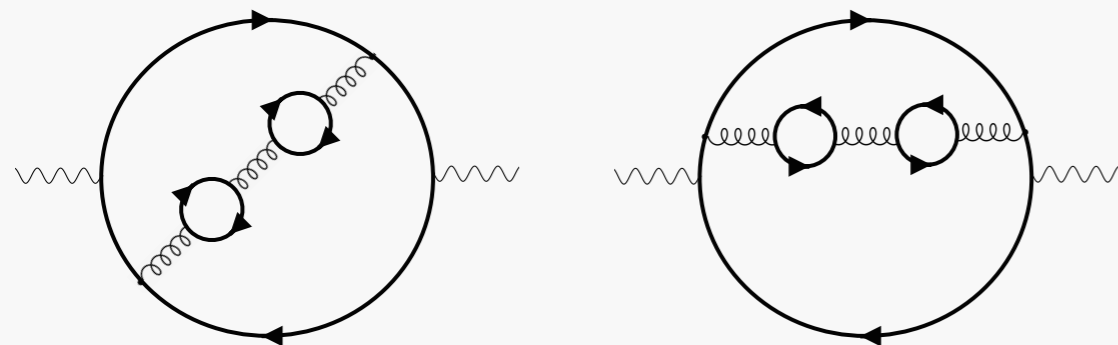


N_f part @N³LO $e^+e^- \rightarrow jj$



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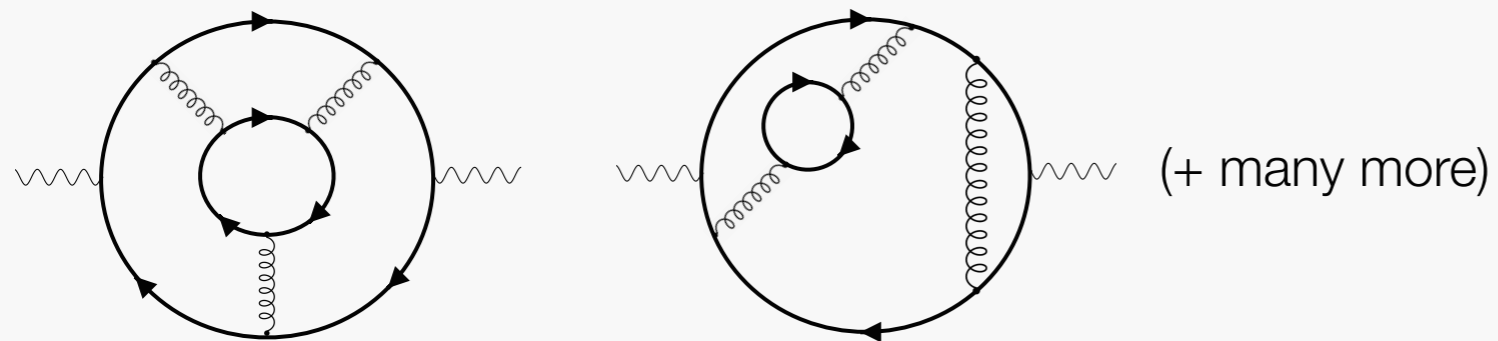
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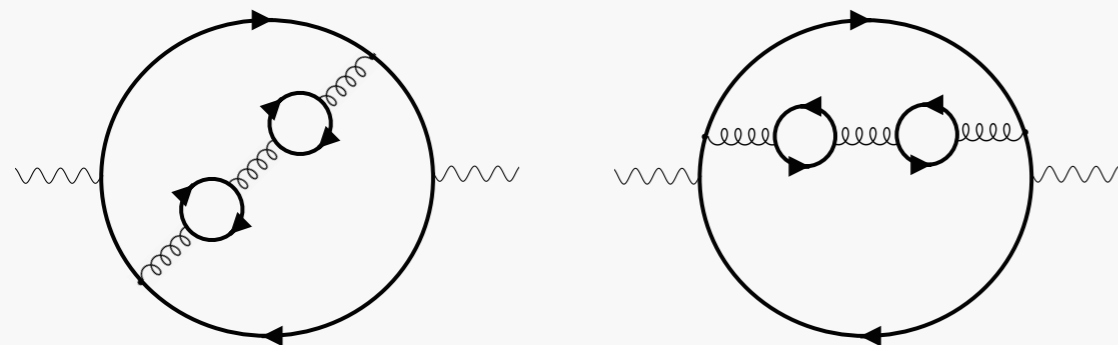
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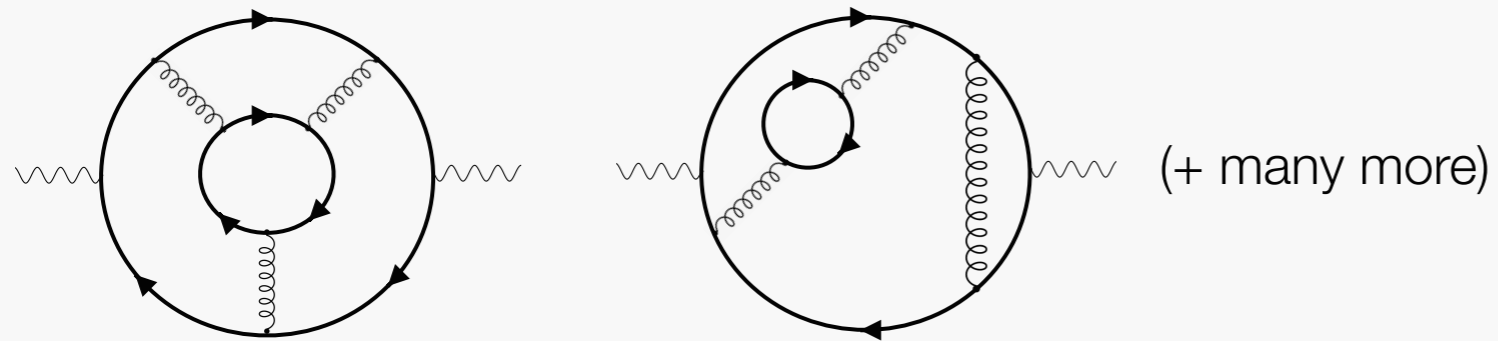
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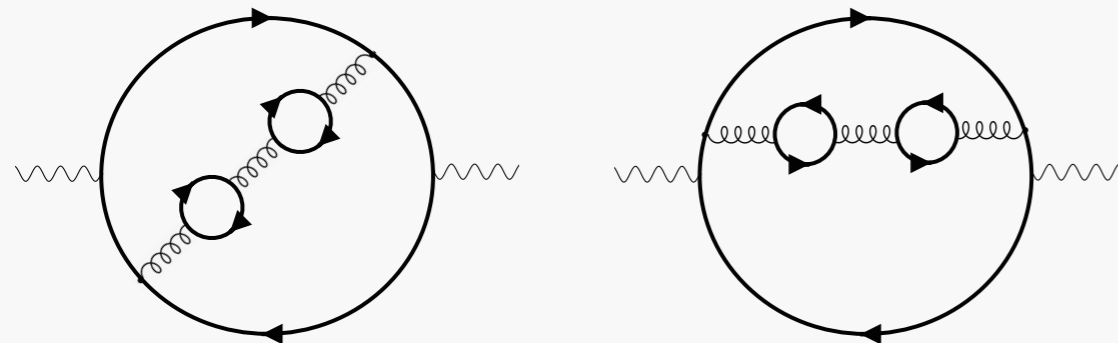
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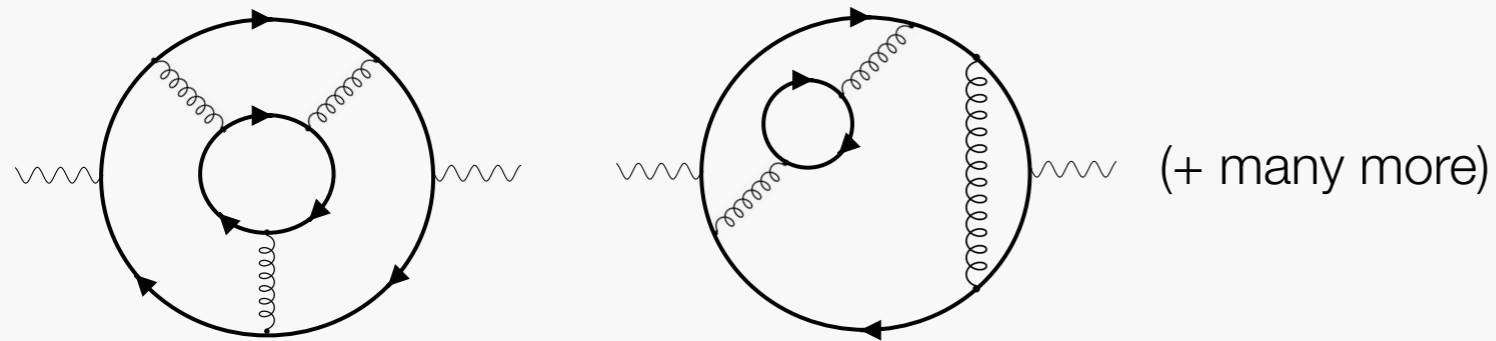
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Benchmarks:

Herzog, Ruijl, Ueda, Vermaseren, Vogt :
arXiv:1707.01044

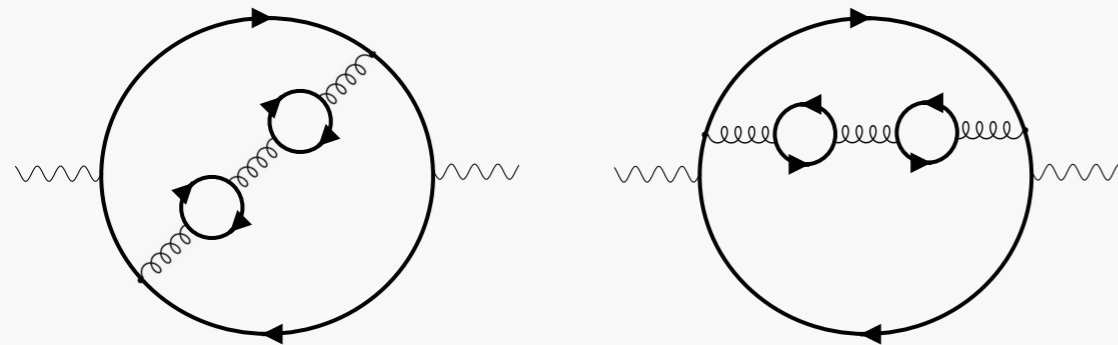
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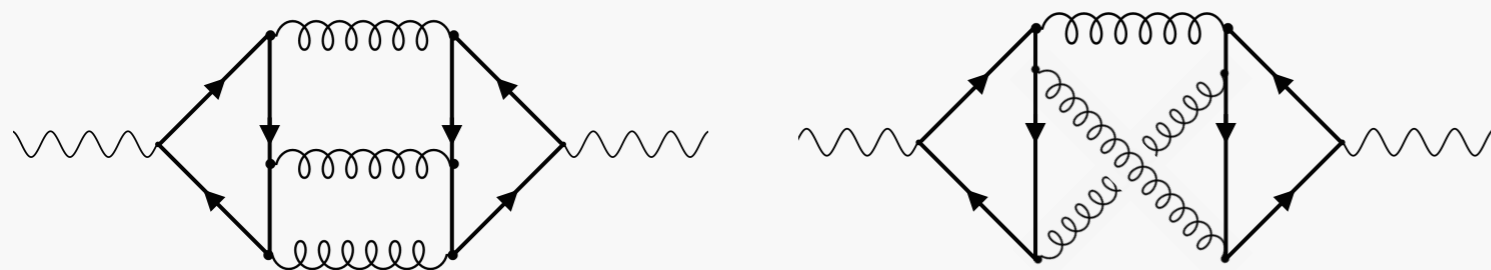
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Testing {

- ✓ N3LO IR cancellations
- ✓ 3-loop UV renormalisation
- ✓ 1,2,3-loop self-energies

Outlook

Outlook

Local Unitarity soon ready for automation of processes without initial-state singularities

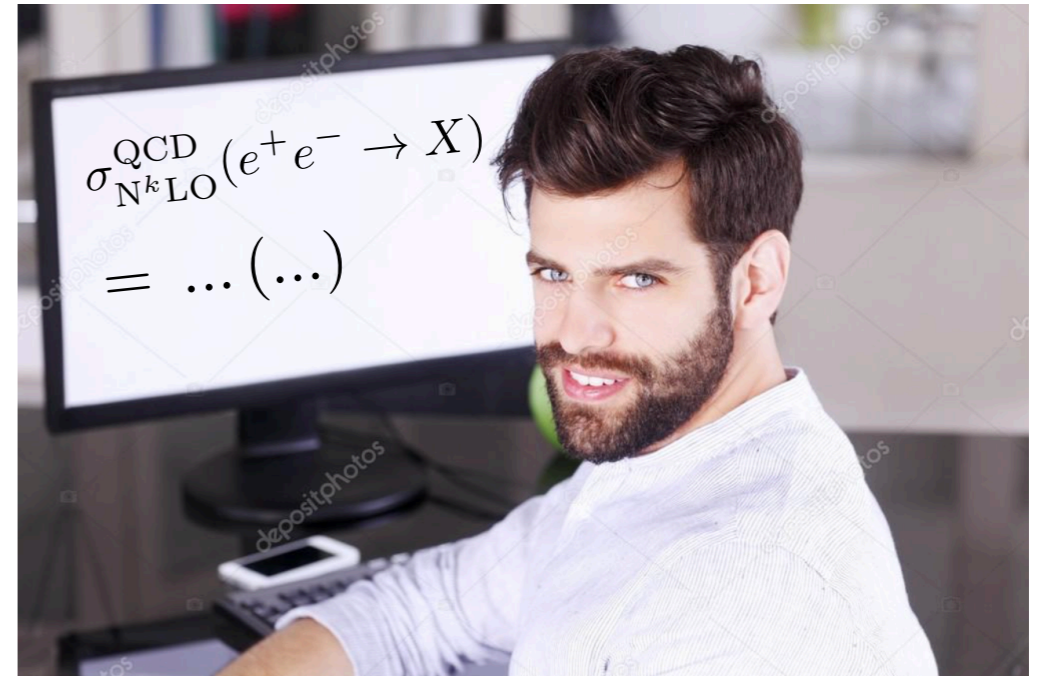
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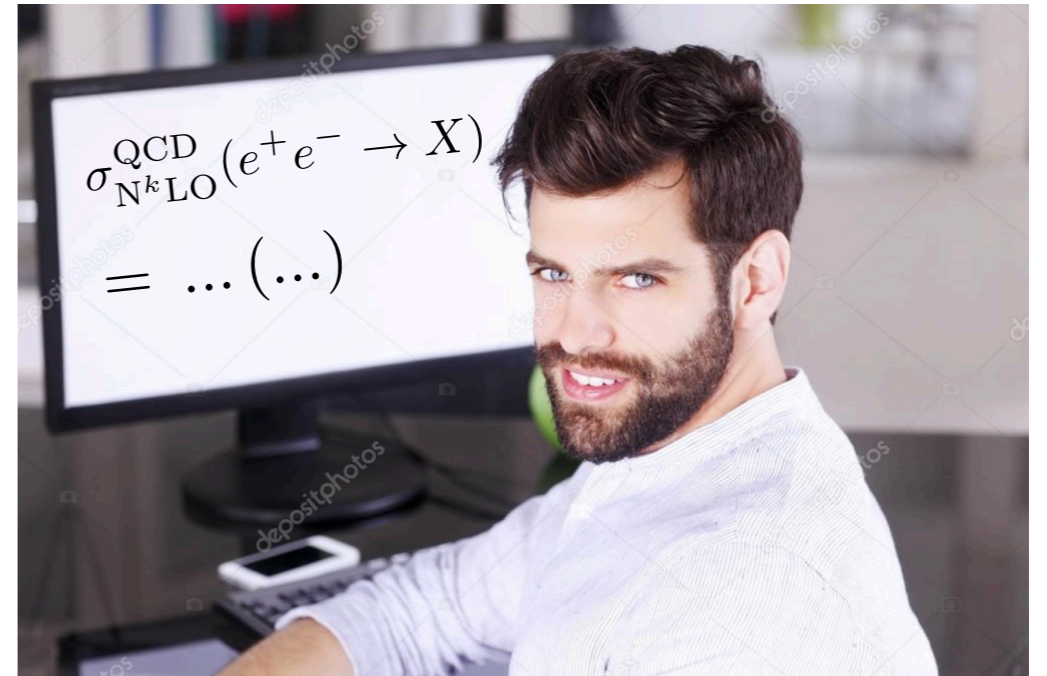
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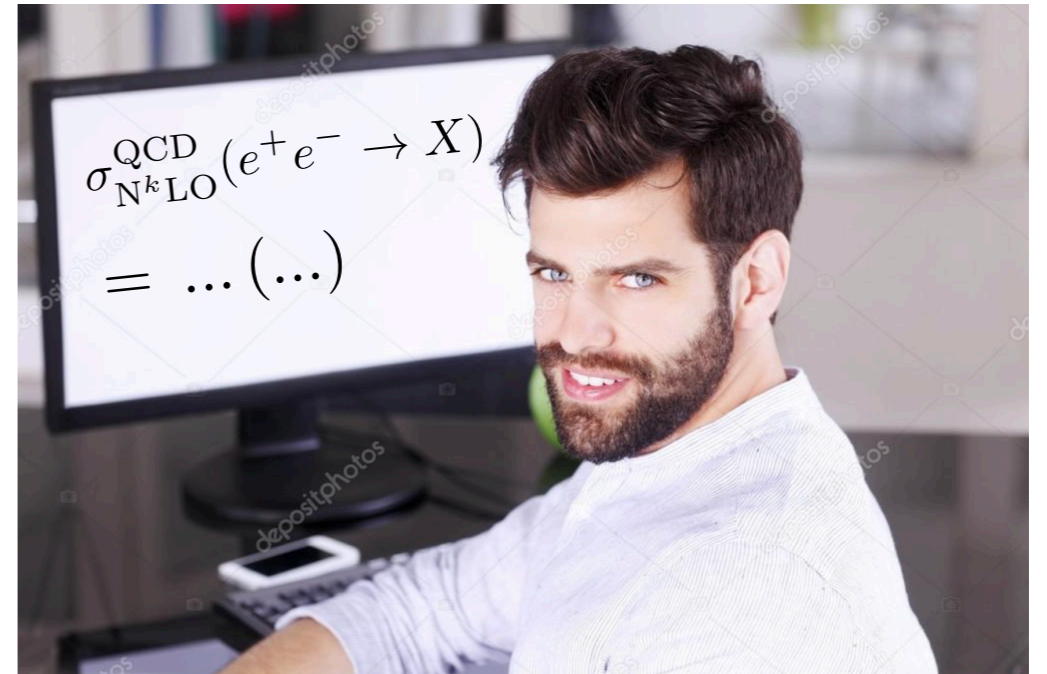
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Future theory work:

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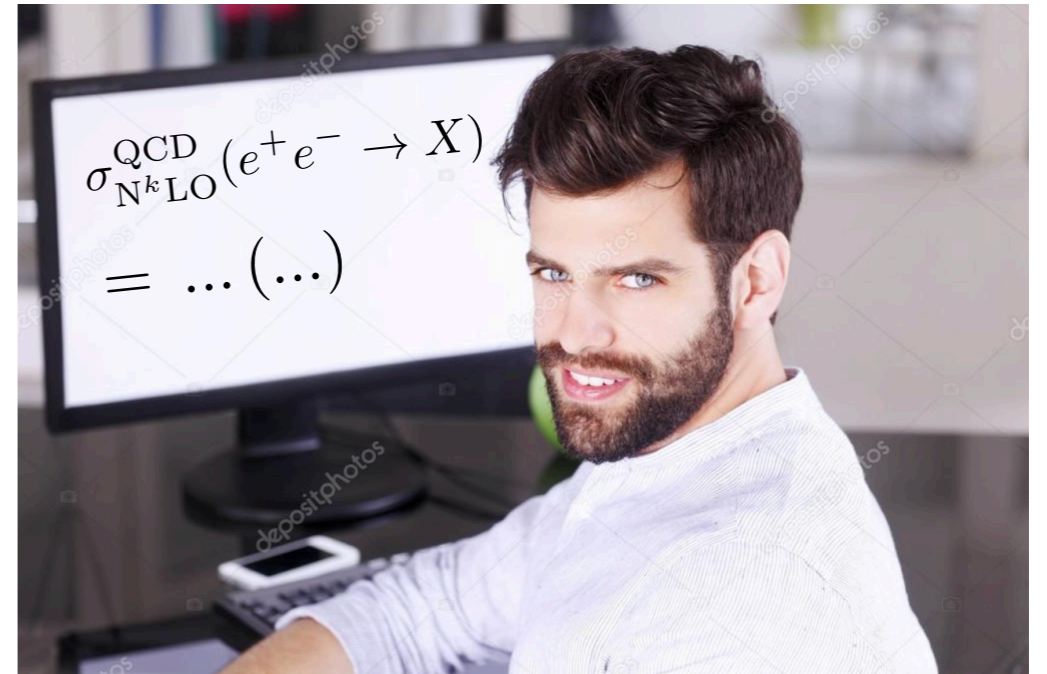


Future theory work:

Resummation: we would like an automatable solution, that only relies on the general factorisation properties of amplitudes

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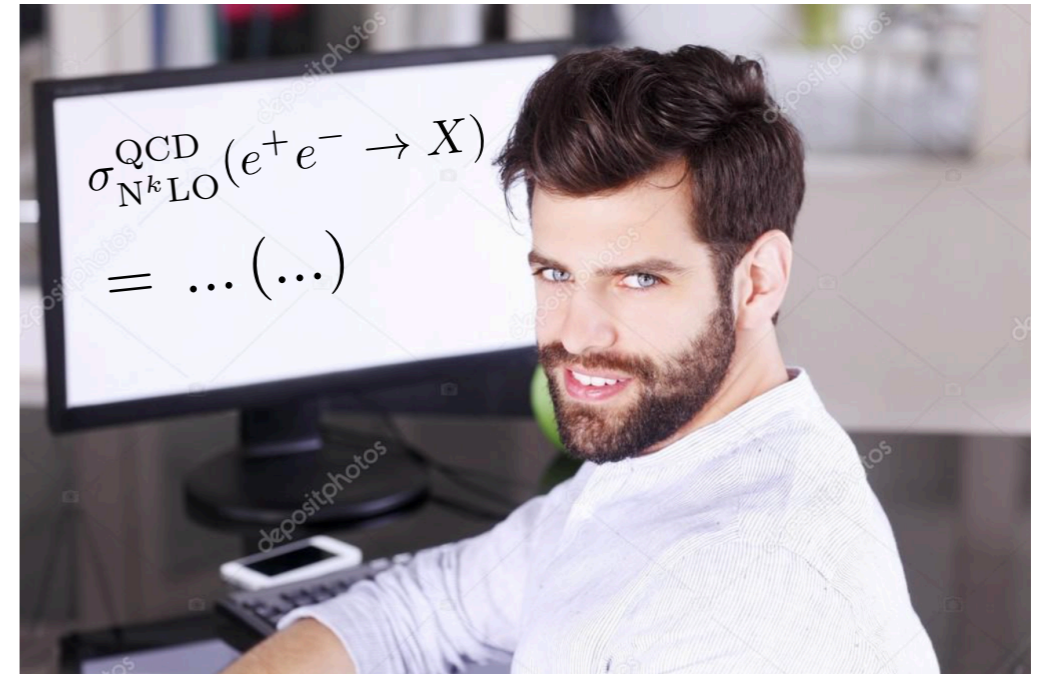
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Banfi, Salam, Zanderighi, [arXiv:0304148](https://arxiv.org/abs/0304148)

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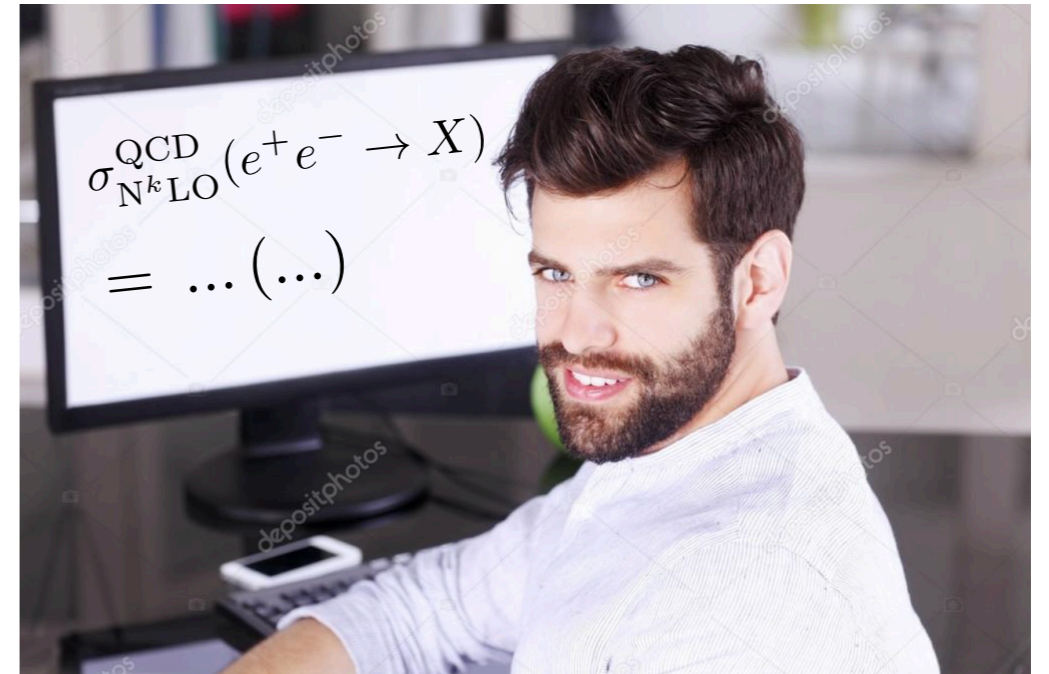
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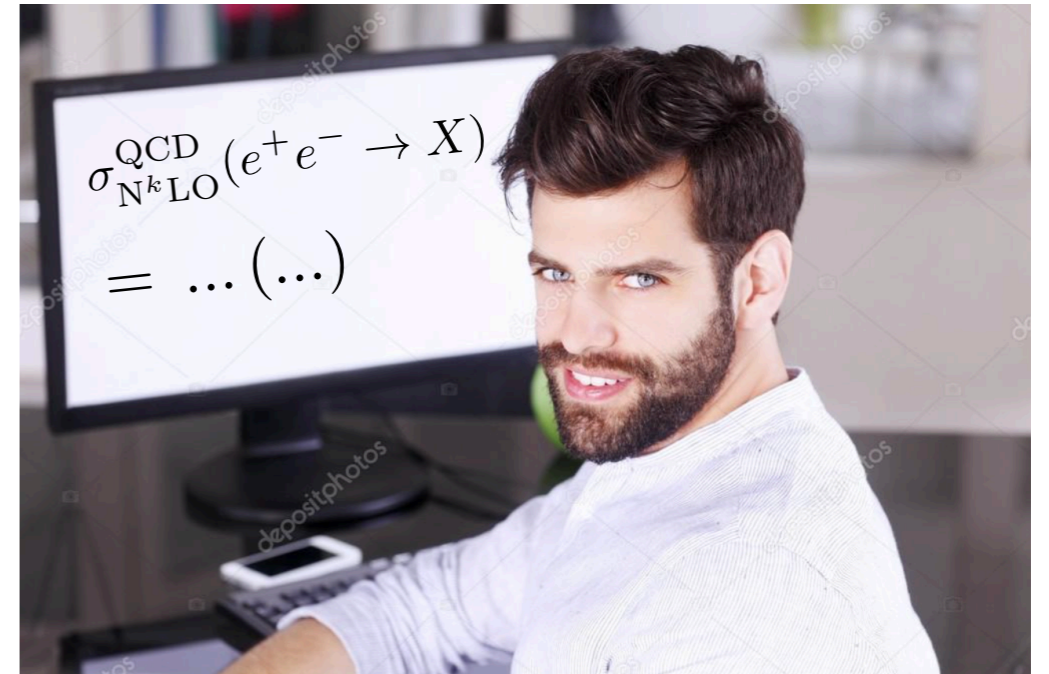
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- **KLN for initial states:** the KLN cancellation mechanism can be extended to include initial state singularities, but requires a significant departure from the Parton model

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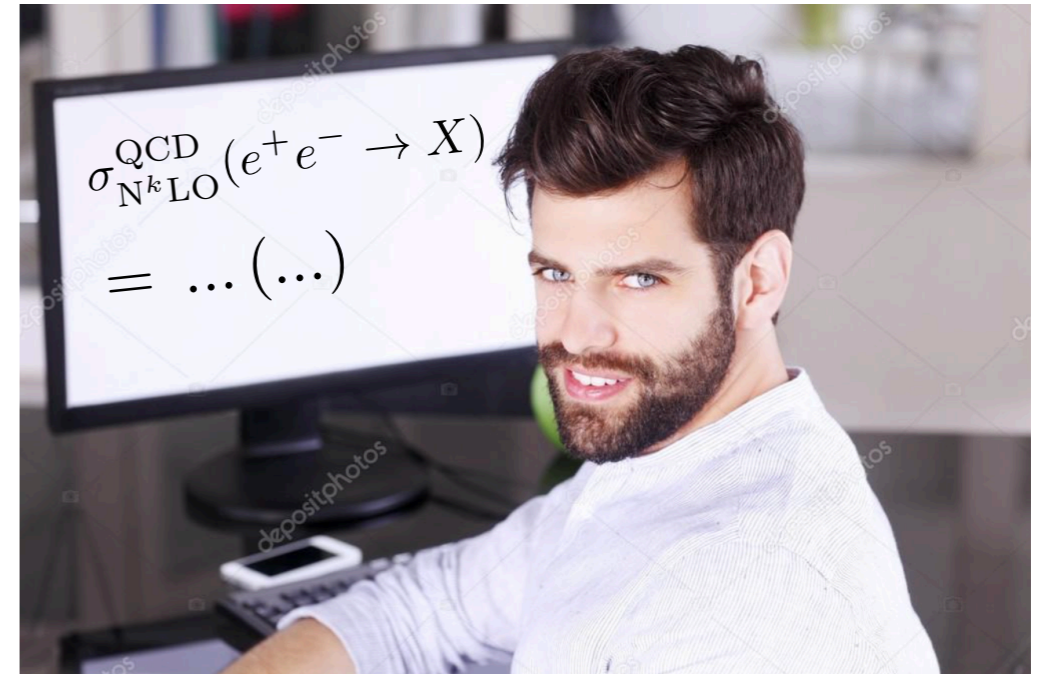
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a lot of physics and cool mathematics!

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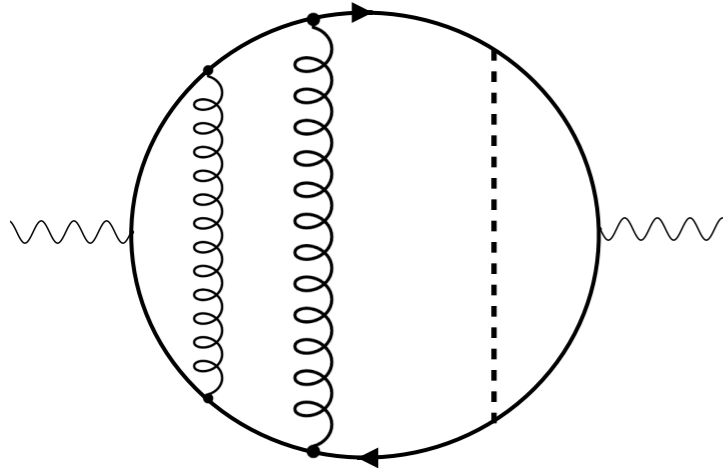
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- **Local PDF counterterms:** develop a competitive subtraction method or integrate existing ones

Thank you!

$$N^2LO \quad \gamma^* \rightarrow t\bar{t}H$$

Top energy distribution
for the supergraph



$$\sqrt{s} = 1\text{TeV}$$

