NLO computation of diffractive di-hadron production in the shockwave framework

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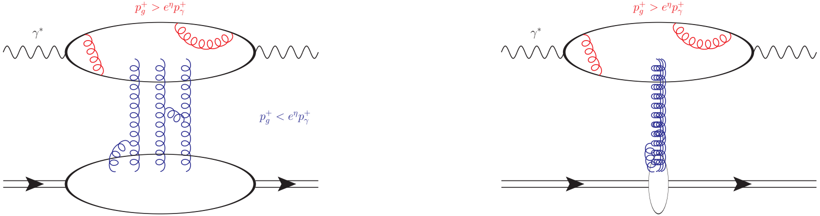
Shockwave approximation

High-energy limit: $s = (p_\gamma + p_t)^2 \gg |p_\gamma|^2, M_t^2, |p'_{00}|^2$.
n_1^μ, n_2^μ are light-cone vectors: $n_1^2 = n_2^2 = 0$. Gauge choice: $n_2.\mathcal{A} = 0$.



- Separation of gluon field into "fast gluons" and "slow gluons" with a cut-off defined by an arbitrary rapidity parameter $\eta < 0$:

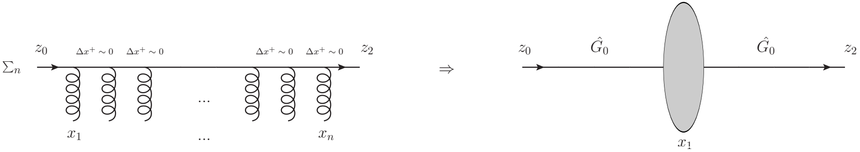
$$\mathcal{A}^\mu(p_g^+, p_g^-, \vec{p}_g) = A^\mu(p_g^+ > e^\eta p_\gamma^+, p_g^-, \vec{p}_g) + b_0^\mu({p_g}^+ < e^\eta p_\gamma^+, p_g^-, \vec{p}_g)$$

- Boost from target rest frame to a frame where $p_\gamma^+ \sim p_t^- \sim \sqrt{\frac{s}{2}}$

$$b_0^\mu(x^+, x^-, \vec{x}) \xrightarrow{\Lambda} b^\mu(x^+, x^-, \vec{x}) = b^-(x^+, \vec{x}) n_2^\mu = \delta(x^+) B(\vec{x}) n_2^\mu$$

with $\Lambda = e^\omega \sim \frac{\sqrt{s}}{M_t}$

Wilson line

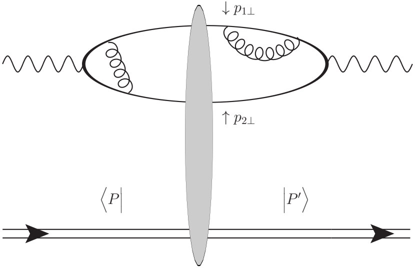


Resummation of interactions with external field $b_\mu(x^+, x^-, \vec{x})$ into a Wilson line :

$$U(\vec{x}_1) = \mathcal{P} \exp \left[ig \int_{-\infty}^{+\infty} dz^+ b^- \left(z^+, \vec{z} \right) \right]$$

$$U(\vec{p}) = \int d^d z_\perp \, e^{i p_\perp \cdot z_\perp} \, U(\vec{z})$$

Factorization in the shockwave approximation



$$\mathcal{M}^{\eta} = \int d^d p_{1\perp} d^d p_{2\perp} \Phi^{\eta}(p_{1\perp}, p_{2\perp}) \left\langle P'\left(p_0'\right) \middle| \left[\mathrm{Tr}\left(U_1^{\eta} U_2^{\eta\dagger}\right) - N_c\right](\vec{p}_1, \vec{p}_2) \middle| P\left(p_0\right) \right\rangle$$

The dipole operator defines as

$$\mathcal{U}_{ij}^{\eta} = 1 - \frac{1}{N_c}\,\mathrm{Tr}\left(U_{\vec{z}_i}^{\eta} U_{\vec{z}_j}^{\eta\dagger}\right)$$

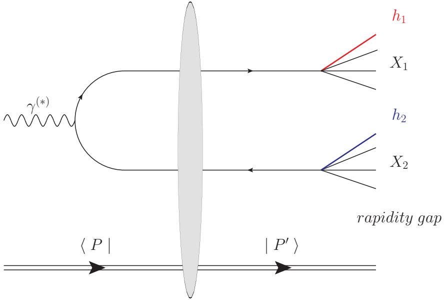
evolves according to the B-JIMWLK [Balitsky, Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner] hierarchy of equations.

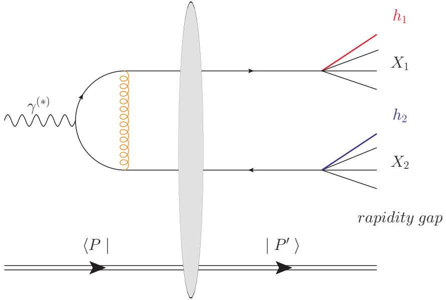
The process of interest

The process studied at NLO :

$$\gamma^{(*)}(p_\gamma) + P(p_0) \to h_1(p_{h1}) + h_2(p_{h2}) + X + P(p_0') \qquad (X = X_1 + X_2)$$

Rapidity gap between $(h_1 h_2 X)$ and $P'(p_0')$.
The photon could be real (photo-production) or virtual (electro-production).





Parametrization of the matrix element of the dipole operator:

$$\left\langle P'(p_0') \middle| T\left(\mathrm{Tr}\left(U_{\frac{z_\perp}{2}} U^\dagger_{-\frac{z_\perp}{2}}\right) - N_c\right) \middle| P(p_0) \right\rangle \quad \equiv \quad 2\pi\delta\left(p^-_{00'}\right) F(z_\perp).$$

Its Fourier transform is

$$\int d^d z_\perp e^{i(z_\perp \cdot p_\perp)} F(z_\perp) \equiv \mathbf{F}(p_\perp).$$

Hybrid factorisation

Hybrid factorization:

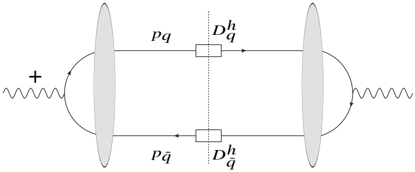
- Collinear factorization: Hard scale with $\Lambda^2_{QCD} \ll \vec{p}^{\,2}_{h_1} \sim \vec{p}^{\,2}_{h_2}$.
  Constraint $\vec{p}^{\,2} \gg \vec{p}^{\,2}_{h_{1,2}}$ with $\vec{p}$, the relative transverse momentum of the two hadrons.
  $\Rightarrow$ Use of one hadron fragmentation function (FF) only to describe hadronization.

- The shockwave factorisation :
  Partonic process at LO and NLO has been calculated in arXiv:1606.00419 [hep-ph].

- We also need $\vec{p}^{\,2}_{h_1} \sim \vec{p}^{\,2}_{h_2} < Q_s^2$ to be in the saturation region.

LO cross-section

Sudakov decomposition for the momenta: $p_i^\mu = x_i p_\gamma^+ n_1^\mu + \frac{\vec{p}^{\,2}}{2x_i p_\gamma^+} n_2^\mu + p_\perp^\mu$.



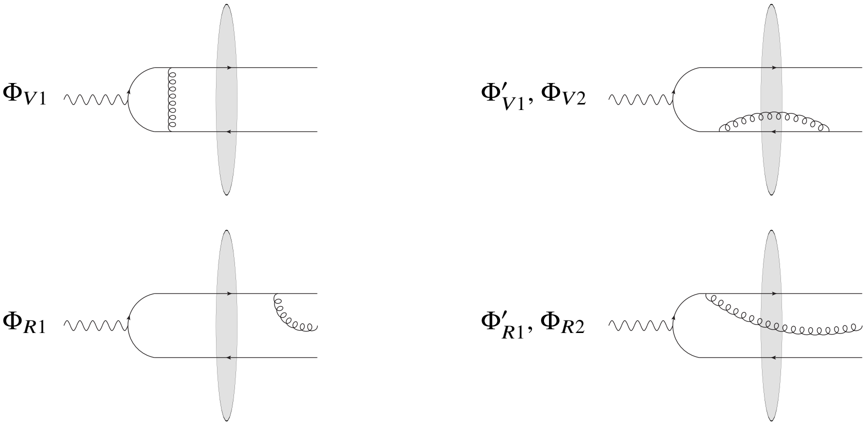
Using collinearity $(p_q^+, \vec{p}_q) = (x_q/x_{h_1})(p_{h_1}^+, \vec{p}_{h_1})$ and $(p_{\bar{q}}^+, \vec{p}_{\bar{q}}) = (x_{\bar{q}}/x_{h_2})(p_{h_2}^+, \vec{p}_{h_2})$:

$$\frac{d\sigma_{0JI}^{h_1 h_2}}{dx_{h_1} dx_{h_2} d^d\vec{p}_{h_1} d^d\vec{p}_{h2}} = \sum_q \int_{x_{h_1}}^1 \frac{dx_q}{x_q} \int_{x_{h_2}}^1 \frac{dx_{\bar{q}}}{x_{\bar{q}}} \left(\frac{x_q}{x_{h1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h2}}\right)^d$$

$$D_q^{h_1}\left(\frac{x_{h_1}}{x_q}\right) D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}}\right) \frac{d\hat{\sigma}_{JI}}{dx_q dx_{\bar{q}} d^d\vec{p}_q d^d\vec{p}_{\bar{q}}} + (h_1 \leftrightarrow h_2)$$

J, I labels the photon polarization for respectively the complex conjugated amplitude and the amplitude.

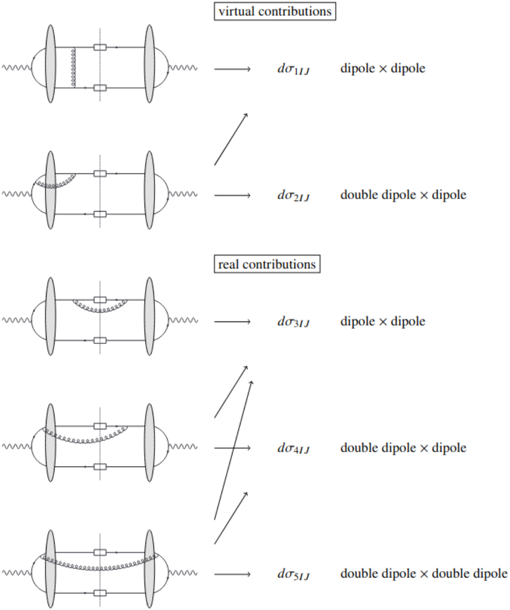
NLO impact factor before fragmentation



Rapidity divergence $x_g \to 0$ in Φ_{V2}. Removed with $\Phi_0 \otimes \mathcal{K}_{B-JIMWLK}$

$\Rightarrow \tilde{\Phi}_{V2} = \Phi_{V2} - \Phi_0 \otimes \mathcal{K}_{B-JIMWLK}$ is finite

NLO cross-section in a nutshell and divergences



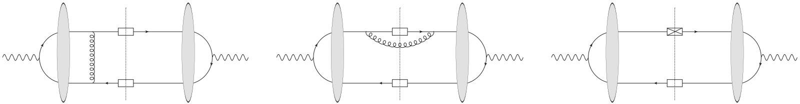
NLO cross-section in a nutshell and divergences

IR divergences to deal with:

- Collinear divergences $\vec{p}_g \propto \vec{p}_q$ or $\vec{p}_g \propto \vec{p}_{\bar{q}}$
- Soft divergences where $x_g \longrightarrow 0$ and $p_{g\perp} = x_g u_\perp \sim x_g \longrightarrow 0$ where $u_\perp$ of order p_T.

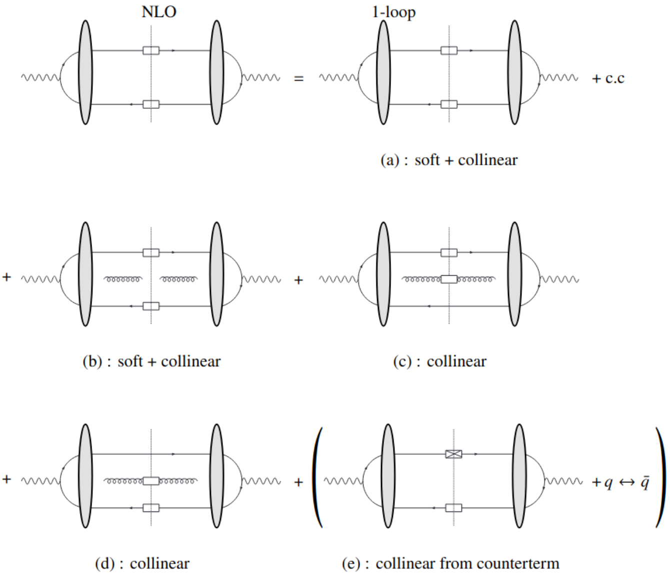
Regularization with dimensional regularization $D = 2 + d = 4 + 2\epsilon$ and longitudinal cut-off $|x_g| > \alpha$.

- We prove the cancellation of divergences between the divergent part of $d\sigma_{3JI}$, counterterms from FF renormalization, and $d\sigma_{1JI}$.
- The finite terms are extracted.



$\downarrow$ $d\sigma_{1IJ}$ | $\downarrow$ $d\sigma_{3IJ}$ | counterterm

NLO cross-section in a nutshell and divergences



Counterterm from renormalization and evolution equation of FF: diagram (e)

bare dressed

$$D_q^{h_1}\left(\frac{x_{h_1}}{x_q}\right) = D_q^{h_1}\left(\frac{x_{h_1}}{x_q},\mu_F\right) - \frac{\alpha_s}{2\pi}\left(\frac{1}{\hat{\epsilon}} + \ln\frac{\mu_F^2}{\mu^2}\right)\int_{\frac{x_{h_1}}{x_q}}^1 \frac{d\beta_1}{\beta_1}$$

$$\times\left[P_{qq}(\beta_1) D_q^{h_1}\left(\frac{x_{h_1}}{x_q\beta_1},\mu_F\right) + P_{gq}(\beta_1)\, D_g^{h_1}\left(\frac{x_{h_1}}{\beta_1 x_q},\mu_F\right)\right]$$

with the usual DGLAP splitting functions

$$P_{qq}(z) = C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]$$

$$P_{gq}(z) = C_F\frac{1+(1-z)^2}{z}$$

+ prescription: $\int_a^1 d\beta \frac{F(\beta)}{(1-\beta)_+} = \int_a^1 d\beta \frac{F(\beta)-F(1)}{1-\beta} - \int_0^a d\beta \frac{F(1)}{1-\beta}$

$\frac{1}{\hat{\epsilon}} = \frac{\Gamma(1-\epsilon)}{(4\pi)^\epsilon \epsilon} \sim \frac{1}{\epsilon} + \gamma_E - \ln(4\pi)$

Counterterm cross-section for LL

$$\frac{d\sigma_{LL}^{h_1h_2}}{dx_{h_1}d_{h_2}d^dp_{h_1\perp}d^dp_{h_2\perp}}\Big|_{\mathrm{ct}} = \frac{4\alpha_{\mathrm{em}}Q^2}{(2\pi)^{4(d-1)}N_c}\sum_q\int_{x_{h_1}}^1 dx_q\int_{x_{h_2}}^1 dx_{\bar{q}}\,x_qx_{\bar{q}}\left(\frac{x_q}{x_{h_1}}\right)^d\left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d\delta(1-x_q-x_{\bar{q}})$$

$$\times\,\mathcal{F}_{LL}\left(-\frac{\alpha_s}{2\pi}\right)\left(\frac{1}{\hat{\epsilon}}+\ln\frac{\mu_F^2}{\mu^2}\right)Q_q^2\left\{\int_{\frac{x_{h_1}}{x_q}}^1\frac{d\beta_1}{\beta_1}\left[C_F\frac{1+\beta_1^2}{(1-\beta_1)_+}D_q^{h_1}\left(\frac{x_{h_1}}{\beta_1x_q},\mu_F\right)D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)\right.\right.$$

$$\left.+P_{gq}(\beta_1)D_g^{h_1}\left(\frac{x_{h_1}}{\beta_1x_q},\mu_F\right)D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)\right]+\int_{\frac{x_{h_2}}{x_{\bar{q}}}}^1\frac{d\beta_2}{\beta_2}\left[P_{gq}(\beta_2)D_q^{h_1}\left(\frac{x_{h_1}}{x_q},\mu_F\right)D_g^{h_2}\left(\frac{x_{h_2}}{\beta_2x_{\bar{q}}},\mu_F\right)\right.$$

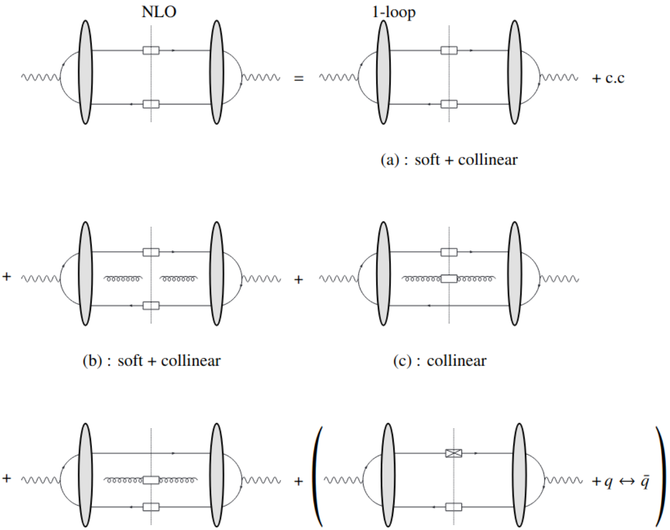
$$\left.\left.+C_F\frac{1+\beta_2^2}{(1-\beta_2)_+}D_q^{h_1}\left(\frac{x_{h_1}}{x_q},\mu_F\right)D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{\beta_2x_{\bar{q}}},\mu_F\right)\right]+3C_FD_q^{h_1}\left(\frac{x_{h_1}}{x_q},\mu_F\right)D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)\right\}$$

$$+(h_1\leftrightarrow h_2)\,.$$

$\mathcal{F}_{LL}$ contains the non-perturbative part: matrix elements of the dipole operators on the target states

$$\mathcal{F}_{LL}=\left|\int d^dp_{2\perp}\frac{\mathbf{F}\left(\frac{x_q}{2x_{h_1}}p_{h_1\perp}+\frac{x_{\bar{q}}}{2x_{h_2}}p_{h_2\perp}-p_{2\perp}\right)}{\left(\frac{x_{\bar{q}}}{x_{h_2}}\vec{p}_{h_2}-\vec{p}_2\right)^2+x_qx_{\bar{q}}Q^2}\right|^2.$$

Diagrams (b), (c), (d)



(a) : soft + collinear

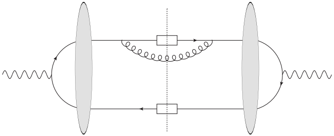
(b) : soft + collinear

(c) : collinear

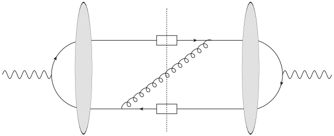
(d) : collinear

(e) : collinear from counterterm

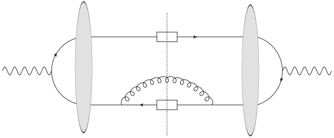
Divergent diagrams in diagram (b), (c), (d)



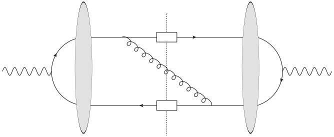
(1) : soft + collinear (qg)



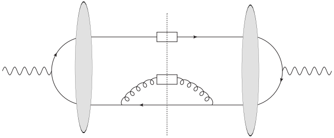
(2) : soft



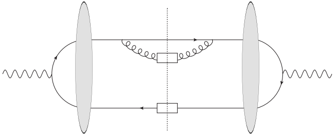
(3) : soft + collinear $(\bar{q}g)$



(4) : soft



(5) : collinear $(\bar{q}g)$



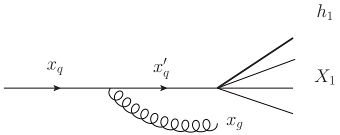
(6) : collinear (qg)

Important points for the calculation of collinear divergences

- Change variables to have longitudinal momentum fraction expressed wrt to the parent parton rather than the photon.
  This is to be able to compare to the counterterm.

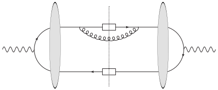
  Example: To extract the collinear divergences from qg splitting, do:

$$x'_q = \beta_1 x_q,$$
$$x_g = (1-\beta_1) x_q.$$



- To disentangle the transverse momentum of the spectator parton and be able to integrate over it without touching the non-perturbative part
  $\Rightarrow$ Fourier transform of the matrix element of the dipole operator.

Specific case : diagram (1)



$$\left.\frac{d\sigma_{3LL}^{q\bar{q}\to h_1 h_1}}{dx_{h_1} d_{h_2} d^d p_{h_1} d^d p_{h_2}}\right|_{\text{coll qg}}$$

$$= \frac{4\alpha_{\text{em}} Q^2}{(2\pi)^{4(d-1)} N_c} \sum_q \int_{x_{h_1}}^1 \frac{dx'_q}{x'_q} \int_{\alpha}^1 \frac{dx_g}{x_g} \int_{x_{h_2}}^1 \frac{dx_{\bar{q}'}}{x_{\bar{q}'}} \delta(1 - x'_q - x'_{\bar{q}} - x_g)$$

$$\times \left(\frac{x'_q}{x_{h_1}}\right)^d \left(\frac{x'_{\bar{q}}}{x_{h_2}}\right)^d Q_q^2 D_q^{h_1}\left(\frac{x_{h_1}}{x'_q}, \mu_F\right) D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x'_{\bar{q}}}, \mu_F\right) \frac{\alpha_s}{\mu^{2\epsilon}} C_F \frac{d^d p_{g\perp}}{(2\pi)^d}$$

$$\times \int d^d p_{2\perp}\, \mathbf{F}\left(\frac{x'_q}{2x_{h_1}} p_{h_1\perp} + \frac{x'_{\bar{q}}}{2x_{h_2}} p_{h_2\perp} - p_{2\perp} + \frac{p_{g\perp}}{2}\right)$$

$$\times \int d^d p_{2'\perp}\, \mathbf{F}^*\left(\frac{x'_q}{2x_{h_1}} p_{h_1\perp} + \frac{x'_{\bar{q}}}{2x_{h_2}} p_{h_2\perp} - p_{2'\perp} + \frac{p_{g\perp}}{2}\right)$$

$$\times \frac{(dx_g^2 + 4x'_q(x'_q + x_g))x_{\bar{q}}'^2(1 - x'_{\bar{q}})^2}{\left(x'_{\bar{q}}(1 - x'_{\bar{q}})Q^2 + \left(\frac{x'_{\bar{q}}}{x_{h_2}}\vec{p}_{h_2} - \vec{p}_2\right)^2\right)\left(x'_{\bar{q}}(1 - x'_{\bar{q}})Q^2 + \left(\frac{x'_{\bar{q}}}{x_{h_2}}\vec{p}_{h_2} - \vec{p}_{2'}\right)^2\right)\left(x'_q \vec{p}_g - x_g \frac{x'_q}{x_{h_1}}\vec{p}_{h_1}\right)^2}$$

$$+ (h_1 \leftrightarrow h_2)$$

Results for diagram (1)

$$\left.\frac{d\sigma_{3LL}^{q\bar{q}\to h_1h_2}}{dx_{h_1}dx_{h_2}dp_{h_1\perp}d^dp_{h_2\perp}}\right|_{\text{coll. qg div}}$$

$$= \frac{4\alpha_{\text{em}}Q^2}{(2\pi)^{4(d-1)}N_c}\sum_q \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}}\, x_q x_{\bar{q}} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d \delta(1-x_q-x_{\bar{q}})$$

$$\times \int d^dp_{2\perp}\int d^dz_{1\perp}\frac{e^{iz_{1\perp}\cdot\left(\frac{x_q}{2x_{h_1}}p_{h_1\perp}+\frac{x_{\bar{q}}}{2x_{h_2}}p_{h_2\perp}-p_{2\perp}\right)}F(z_{1\perp})}{x_qx_{\bar{q}}Q^2+\left(\frac{x_{\bar{q}}}{x_{h_2}}\vec{p}_{h_2}-\vec{p}_2\right)^2}$$

$$\times \int d^dp_{2'\perp}\int d^dz_{2\perp}\frac{e^{-iz_{2\perp}\cdot\left(\frac{x_q}{2x_{h_1}}p_{h_1\perp}+\frac{x_{\bar{q}}}{2x_{h_2}}p_{h_2\perp}-p_{2'\perp}\right)}F^*(z_{2\perp})}{x_qx_{\bar{q}}Q^2+\left(\frac{x_{\bar{q}}}{x_{h_2}}\vec{p}_{h_2}-\vec{p}_{2'}\right)^2}$$

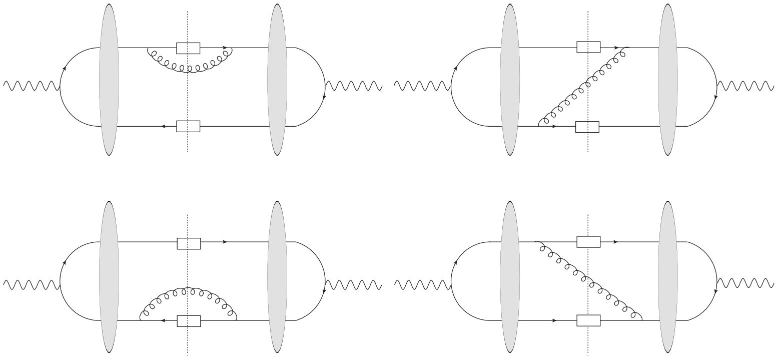
$$\times\frac{\alpha_s}{2\pi}\frac{1}{\hat{\epsilon}}Q_q^2\left[\int_{\frac{x_{h_1}}{x_q}}^1\frac{d\beta_1}{\beta_1}C_F\frac{1+\beta_1^2}{(1-\beta_1)_+}D_q^{h_1}\left(\frac{x_{h_1}}{\beta_1 x_q},\mu_F\right)D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)\right.$$

$$+\int_{\frac{x_{h_1}}{x_q}}^{1-\frac{\alpha}{x_q}}d\beta_1 C_F\frac{2}{1-\beta_1}\left(\frac{c_0^2}{\left(\frac{z_{1\perp}-z_{2\perp}}{2}\right)^2\mu^2}\right)^{\epsilon}D_q^{h_1}\left(\frac{x_{h_1}}{x_q},\mu_F\right)D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)$$

$$\left.-2C_F\ln\left(1-\frac{x_{h_1}}{x_q}\right)D_q^{h_1}\left(\frac{x_{h_1}}{x_q},\mu_F\right)D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)\right]+(h_1\leftrightarrow h_2)\,.$$

Cancellation with counterterm. The second term is to be removed: double-counting with soft contribution. The third term, from the introduction of the + prescription, will be removed with a term in the soft contribution.

Soft limit



Rescaling $\vec{p}_g = x_g \vec{u}$ to isolate the divergences in the form $\int_\alpha^1 \frac{dx_g}{x_g^{3-d}}$ and putting $x_g \to 0$ in the rest of the integrand.

Soft divergence

$$
\left.\frac{d\sigma_{3LL}^{q\bar{q}\to h_1 h_2}}{dx_{h_1} d_{h_2} d^d p_{h_1\perp} d p_{h_2\perp}}\right|_{\text{soft div}} = \frac{4\alpha_{\text{em}} Q^2}{(2\pi)^{4(d-1)} N_c} \sum_q \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}}\; x_q x_{\bar{q}} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d
$$

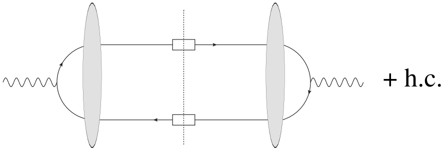
$$
\times\, \delta(1 - x_q - x_{\bar{q}}) Q_q^2 D_q^{h_1}\left(\frac{x_{h_1}}{x_q}, \mu_F\right) D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right) \mathcal{F}_{LL}
$$

$$
\times\, \frac{\alpha_s C_F}{2\pi} \frac{1}{\hat{\epsilon}} \left[-4\ln\alpha + 2\ln x_q + 2\ln\left(1 - \frac{x_{h_1}}{x_q}\right) - 4\epsilon \ln^2 \alpha \right.
$$

$$
\left. -4\epsilon \ln\alpha \ln\left(\frac{\left(\frac{\vec{p}_{h_1}}{x_{h_1}} - \frac{\vec{p}_{h_2}}{x_{h_2}}\right)^2}{\mu^2}\right) + 2\ln x_{\bar{q}} + 2\ln\left(1 - \frac{x_{h_2}}{x_{\bar{q}}}\right)\right] + (h_1 \leftrightarrow h_2)
$$

Cancellation with the residual divergence from the collinear term.

Virtual corrections



1-loop

Cancellation between virtual corrections and soft and with counterterm.

$$\left.\frac{d\sigma_{1LL}^{q\bar{q}\to h_1 h_2}}{dx_{h_1} d_{h_2} d^d p_{h_1\perp} d^d p_{h_2\perp}}\right|_{\text{div}} = \frac{4\alpha_{\text{em}} Q^2}{(2\pi)^{4(d-1)} N_c} \sum_q \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}}\; x_q x_{\bar{q}} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d$$

$$\times\, \delta(1 - x_q - x_{\bar{q}}) Q_q^2 D_q^{h_1}\left(\frac{x_{h_1}}{x_q}, \mu_F\right) D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right) \mathcal{F}_{LL}$$

$$\times\, \frac{\alpha_s}{2\pi} C_F \frac{1}{\hat{\epsilon}} \left[-4\epsilon \ln(\alpha) \ln\left(\frac{\mu^2}{\left(\frac{\vec{p}_{h_2}}{x_{h_2}} - \frac{\vec{p}_{h_1}}{x_{h_1}}\right)^2} \right) + 4\ln(\alpha) \right.$$

$$\left. +4\epsilon \ln^2(\alpha) - 2\ln(x_q x_{\bar{q}}) + 3 \right] + (h_1 \leftrightarrow h_2)$$

Conclusions

- Computation of the NLO cross-section of the diffractive production of a pair of hadrons with large p_T, such that $\Lambda_{QCD}^2 \ll p_T^2$.
- Saturation window: $p_T^2 < Q_s^2$
- Process can be either a photo-production or electro-production. The results are applicable to ultra-peripheral collisions at the LHC (especially at the LHCb) or the EIC.
- Full cancellation of divergences has been observed between real corrections, virtual ones, and counterterm from FF renormalization.
- Expressions of the detailed finite cross-sections are found in the paper, in general kinematics (Q^2, t).
- The phenomenological application of the NLO computation will take time due to the complexity of the analytical results.

Thank you for your attention!

LO LL cross-section

$$\frac{d\sigma_{0LL}^{q\bar{q}\to h_1 h_2}}{dx_{h_1}dx_{h_2}d^d p_{h_1\perp}d^d p_{h_2\perp}} = \frac{4\alpha_{\text{em}}Q^2}{(2\pi)^{4(d-1)}N_c}\sum_q \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}}\ x_q x_{\bar{q}} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d$$
$$\times \quad \delta(1-x_q-x_{\bar{q}})Q_q^2 D_q^{h_1}\left(\frac{x_{h_1}}{x_q}\right) D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}}\right)\mathcal{F}_{LL} + (h_1 \leftrightarrow h_2)\,,$$

where

$$\mathcal{F}_{LL} = \left|\int d^d p_{2\perp} \frac{\mathbf{F}\left(\frac{x_q}{2x_{h_1}}p_{h_1\perp} + \frac{x_{\bar{q}}}{2x_{h_2}}p_{h_2\perp} - p_{2\perp}\right)}{\left(\frac{x_{\bar{q}}}{x_{h_2}}\vec{p}_{h_2} - \vec{p}_2\right)^2 + x_q x_{\bar{q}} Q^2}\right|^2 .$$

Another equivalent representation exists for the LO cross-section with, instead of $\mathcal{F}_{LL}$, we use :

$$\tilde{\mathcal{F}}_{LL} = \left|\int d^d p_{1\perp} \frac{\mathbf{F}\left(-\left(\frac{x_q}{2x_{h_1}}p_{h_1\perp} + \frac{x_{\bar{q}}}{2x_{h_2}}p_{h_2\perp} - p_{1\perp}\right)\right)}{\left(\frac{x_q}{x_{h_1}}\vec{p}_{h_1} - \vec{p}_1\right)^2 + x_q x_{\bar{q}} Q^2}\right|^2 .$$

The rest stays the same, apart from this change. This other representation is used when studying divergences from the $\bar{q}$ FF renormalization and the collinear $\bar{q}g$ divergence.

Definition of the non-perturbative functions

- Definition of the matrix element of the dipole operator

$$\left\langle P'\left(p_0'\right)\left|T\left(\mathrm{Tr}\left(U_{\frac{z_\perp}{2}}U^\dagger_{-\frac{z_\perp}{2}}\right)-N_c\right)\right|P\left(p_0\right)\right\rangle \equiv 2\pi\delta\left(p^-_{00'}\right)F\left(z_\perp\right)$$

Its Fourier transform is

$$\int d^d z_\perp e^{i(z_\perp\cdot p_\perp)}F\left(z_\perp\right)\equiv \mathbf{F}\left(p_\perp\right).$$

- Definition of the matrix element of the double dipole operator

$$\left\langle P'\left(p_0'\right)\left|\left(\mathrm{Tr}\left(U_{\frac{z}{2}}U^\dagger_x\right)\mathrm{Tr}\left(U_xU^\dagger_{-\frac{z}{2}}\right)-N_c\,\mathrm{Tr}\left(U_{\frac{z}{2}}U^\dagger_{-\frac{z}{2}}\right)\right)\right|P\left(p_0\right)\right\rangle$$
$$\equiv 2\pi\delta\left(p^-_{00'}\right)\tilde{F}\left(z_\perp,x_\perp\right)$$

with the following Fourier transform

$$\int d^d z_\perp d^d x_\perp e^{i(p_\perp\cdot x_\perp)+i(z_\perp\cdot q_\perp)}\tilde{F}\left(z_\perp,x_\perp\right)\equiv \tilde{\mathbf{F}}\left(q_\perp,p_\perp\right).$$

The divergent partonic LL cross-section

$$
\begin{aligned}
d\hat{\sigma}_{3LL}|_{div} = {} & \frac{4\alpha_{\rm em}Q^2}{(2\pi)^{4(d-1)}N_c} Q_q^2 dx_q' dx_{\bar{q}}' \delta(1-x_q'-x_{\bar{q}}'-x_g) d^d p_{q\perp} d^d p_{\bar{q}\perp} \frac{\alpha_s C_F}{\mu^{2\epsilon}} \frac{dx_g}{x_g} \frac{d^d p_{g\perp}}{(2\pi)^d} \\
& \times \int d^d p_{1\perp} d^d p_{2\perp} \delta(p_{q1\perp}+p_{\bar{q}2\perp}+p_{g\perp}) \mathbf{F}\left(\frac{p_{12\perp}}{2}\right) \\
& \times \int d^d p_{1'\perp} d^d p_{2'\perp} \delta(p_{q1'\perp}+p_{\bar{q}2'\perp}+p_{g\perp}) \mathbf{F}^*\left(\frac{p_{1'2'\perp}}{2}\right) \\
& \times \left\{ \frac{\left(dx_g^2+4x_q'(x_q'+x_g)\right)}{\left(Q^2+\frac{\vec{p}_{\bar{q}2}^{\,2}}{x_{\bar{q}}'(1-x_{\bar{q}}')}\right)\left(Q^2+\frac{\vec{p}_{\bar{q}2'}^{\,2}}{x_{\bar{q}}'(1-x_{\bar{q}}')}\right)(x_q'\vec{p}_g-x_g\vec{p}_q)^2} \right. \\
& - \frac{\left(2x_g-dx_g^2+4x_q'x_{\bar{q}}'\right)\left(x_q'\vec{p}_g-x_g\vec{p}_q\right)\cdot\left(x_{\bar{q}}'\vec{p}_g-x_g\vec{p}_{\bar{q}}\right)}{\left(Q^2+\frac{\vec{p}_{\bar{q}2'}^{\,2}}{x_{\bar{q}}'(1-x_{\bar{q}}')}\right)\left(Q^2+\frac{\vec{p}_{q1}^{\,2}}{x_q'(1-x_q')}\right)(x_q'\vec{p}_g-x_g\vec{p}_q)^2\left(x_{\bar{q}}'\vec{p}_g-x_g\vec{p}_{\bar{q}}\right)^2} \\
& + \frac{\left(dx_g^2+4x_{\bar{q}}'(x_{\bar{q}}'+x_g)\right)}{\left(Q^2+\frac{\vec{p}_{q1}^{\,2}}{x_q'(1-x_q')}\right)\left(Q^2+\frac{\vec{p}_{q1'}^{\,2}}{x_q'(1-x_q')}\right)(x_{\bar{q}}'\vec{p}_g-x_g\vec{p}_{\bar{q}})^2} \\
& \left. - \frac{\left(2x_g-dx_g^2+4x_q'x_{\bar{q}}'\right)\left(x_q'\vec{p}_g-x_g\vec{p}_q\right)\cdot\left(x_{\bar{q}}'\vec{p}_g-x_g\vec{p}_{\bar{q}}\right)}{\left(Q^2+\frac{\vec{p}_{q1'}^{\,2}}{x_q'(1-x_q')}\right)\left(Q^2+\frac{\vec{p}_{\bar{q}2}^{\,2}}{x_{\bar{q}}'(1-x_{\bar{q}}')}\right)(x_q'\vec{p}_g-x_g\vec{p}_q)^2\left(x_{\bar{q}}'\vec{p}_g-x_g\vec{p}_{\bar{q}}\right)^2} \right\}
\end{aligned}
$$

The first and third terms are associated with collinear divergences. All of them contribute to the soft divergence.

Finite term from diagram (1)

$$
\left.\frac{d\sigma_{3LL}^{q\bar{q}\to h_1h_2}}{dx_{h_1}dx_{h_2}dp_{h_1\perp}d^dp_{h_2\perp}}\right|_{\text{coll. qg fin}}
= \frac{4\alpha_{\text{em}}Q^2}{(2\pi)^{4(d-1)}N_c}\sum_q\int_{x_{h_1}}^1 dx_q\int_{x_{h_2}}^1 dx_{\bar{q}}\; x_q x_{\bar{q}}\delta(1-x_q-x_{\bar{q}})\left(\frac{x_q}{x_{h_1}}\right)^d\left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d
$$

$$
\times\int d^dp_{2\perp}\int d^dz_{1\perp}\frac{e^{iz_{1\perp}\cdot\left(\frac{x_q}{2x_{h_1}}p_{h_1\perp}+\frac{x_{\bar{q}}}{2x_{h_2}}p_{h_2\perp}-p_{2\perp}\right)}F(z_{1\perp})}{x_qx_{\bar{q}}Q^2+\left(\frac{x_{\bar{q}}}{x_{h_2}}\vec{p}_{h_2}-\vec{p}_2\right)^2}
$$

$$
\times\int d^dp_{2'\perp}\int d^dz_{2\perp}\frac{e^{-iz_{2\perp}\cdot\left(\frac{x_q}{2x_{h_1}}p_{h_1\perp}+\frac{x_{\bar{q}}}{2x_{h_2}}p_{h_2\perp}-p_{2'\perp}\right)}F^*(z_{2\perp})}{x_qx_{\bar{q}}Q^2+\left(\frac{x_{\bar{q}}}{x_{h_2}}\vec{p}_{h_2}-\vec{p}_{2'}\right)^2}
$$

$$
\times\frac{\alpha_sC_F}{2\pi}\left\{\int_{\frac{x_{h_1}}{x_q}}^1\frac{d\beta_1}{\beta_1}Q_q^2D_q^{h_1}\left(\frac{x_{h_1}}{\beta_1x_q},\mu_F\right)D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)\right.
$$

$$
\times\left[\ln\left(\frac{c_0^2}{\left(\frac{z_{1\perp}-z_{2\perp}}{2}\right)^2\mu^2}\right)\frac{1+\beta_1^2}{(1-\beta_1)_+}+\frac{(1-\beta_1)^2+2(1+\beta_1^2)\ln\beta_1}{(1-\beta_1)}\right]
$$

$$
\left.-2\ln\left(1-\frac{x_{h_1}}{x_q}\right)\ln\left(\frac{c_0^2}{\left(\frac{z_{1\perp}-z_{2\perp}}{2}\right)^2\mu^2}\right)D_q^{h_1}\left(\frac{x_{h_1}}{x_q},\mu_F\right)D_{\bar{q}}^{h_q}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)\right\}+(h_1\leftrightarrow h_2)\,.
$$

with $c_0 = 2e^{-\gamma_E}$