Cancellation of NNLO singularities within Local Analytic Sector Subtraction

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Based on

1806.09570
[L. Magnea, E. Maina, G. Pelliccioli, C. Signorile-Signorile, P. Torriellli, S. Uccirati]

2010.14493
[L. Magnea, G. Pelliccioli, C. Signorile-Signorile, P. Torriellli, S. Uccirati]

2209.09123
[**G.B.**, P. Torrielli, S. Uccirati, M. Zaro]

In preparation
[**G.B.**, L. Magnea, G. Pelliccioli, A. Ratti, C. Signorile-Signorile, P. Torriellli, S. Uccirati]

QCD@LHC22
Orsay, 30/12/22

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Introduction to NLO subtraction strategy

FSR
massless QCD

Generalities at NLO

- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

$$\frac{d\sigma_{NLO}}{dX} = \int d\Phi_n \, V \, \delta_{X_n} + \int d\Phi_{n+1} \, R \, \delta_{X_{n+1}}$$

Explicit ϵ poles

Phase space singularities

- *Subtraction algorithm*: introduce **local counterterm** *K* and its integral *I*

$$\int d\Phi_{n+1} \, K \, \delta_{X_n} = \int d\Phi_n \, I \, \delta_{X_n} \qquad\qquad d\Phi_{n+1} = d\Phi_n \, d\Phi_{\mathrm{rad}}$$

- Subtracted NLO cross section numerically integrable in $d = 4$ dimensions

$$\frac{d\sigma_{NLO}}{dX} = \int d\Phi_n \left(V + I \right) \delta_{X_n} + \int d\Phi_{n+1} \left(R \, \delta_{X_{n+1}} - K \delta_{X_n} \right)$$

finite in ϵ

finite in phase space

Strategy of the method

- Partition of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$ (as in FKS) [Frixione, Kunszt, Signer 9512328]

  * Normalised as $$\sum_{i,j\neq i} \mathcal{W}_{ij} = 1 \qquad R = \sum_{i,j\neq i} R\mathcal{W}_{ij}$$

  Minimal approach to disentangle overlapping singularities

  * Single-unresolved configurations

  Sum rules

  Key for integration

  $\mathbf{S}_i$ **soft** parton *i* $$\mathbf{S}_i \sum_{l\neq i} \mathcal{W}_{il} = 1$$

  $\mathbf{C}_{ij}$ **collinear** pair *ij* $$\mathbf{C}_{ij}(\mathcal{W}_{ij} + \mathcal{W}_{ji}) = 1$$

Strategy of the method

- Partition of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$ (as in FKS) [Frixione, Kunszt, Signer 9512328]

- Collect the relevant **IRC limits** for a given sector

$$R\mathcal{W}_{ij} - \Big[\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i\mathbf{C}_{ij}\Big] R\mathcal{W}_{ij} = \text{finite}$$

(Soft + Collinear - Overlap)

Notation

$$\mathrm{L}\, R\,\mathcal{W}_{ij} = (\mathrm{L}\, R)\,(\mathrm{L}\,\mathcal{W}_{ij})$$
$$\text{for } \mathrm{L} = \mathbf{S}_i, \mathbf{C}_{ij}, \ldots$$

Strategy of the method

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$$\text{for}\ \ \mathrm{L} = \mathbf{S}_i, \mathbf{C}_{ij}, \ldots$$

Not yet parametrised

$$\mathbf{S}_i\,R = \mathcal{N}_1\,\delta_{f_i g} \sum_{k,l} \frac{s_{kl}}{s_{ik}s_{il}} B_{kl}(\{k\}_{\cancel{i}})$$

$$\mathbf{C}_{ij}\,R = \mathcal{N}_1 \frac{P_{ij}^{\mu\nu}}{s_{ij}} B_{\mu\nu}(\{k\}_{\cancel{i}\cancel{j}}, k_i + k_j)$$

missing proper
n-body on-shell kinematics!

Strategy of the method

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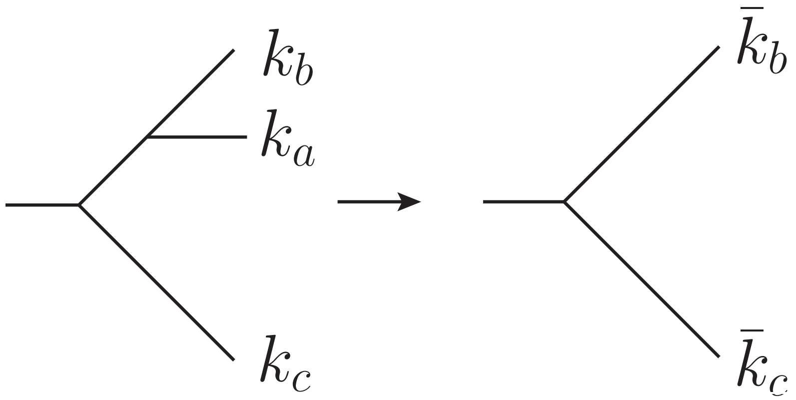
- **CS dipole mapping** [Catani, Seymour 9605323]



  * $\{k_1, ..., k_{n+1}\} \to \{\bar{k}_1, ..., \bar{k}_n\}^{(abc)}$

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{y}{1-y} k_c$$

$$\bar{k}_c^{(abc)} = \frac{1}{1-y} k_c$$

$$y = \frac{s_{ab}}{s_{ab} + s_{ac} + s_{bc}} \, , \quad z = \frac{s_{ac}}{s_{ab} + s_{bc}}$$

  * Phase space factorisation and parametrisation

$$d\Phi_{n+1} = d\Phi_n^{(abc)} \, d\Phi_{\rm rad}^{(abc)} = d\Phi_n(\{\bar{k}\}^{(abc)}) \, d\Phi_{\rm rad}(\bar{s}_{bc}^{(abc)}; y, z, \phi)$$

$$\int d\Phi_{\rm rad}^{(abc)} \propto (\bar{s}_{bc}^{(abc)})^{1-\epsilon} \int_0^\pi d\phi \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz \Big[y(1-y^2) z(1-z) \Big]^{-\epsilon} (1-y)$$

Strategy of the method

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- Collect the relevant **IRC limits** for a given sector
- **CS dipole mapping** [Catani, Seymour 9605323]
- Promotion to **counterterm**: adapt mapping to each kernel

$$K = \sum_{i,j\neq i} \left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij}\right] R \mathcal{W}_{ij}$$

Notation

$$\overline{\mathrm{L}}\, R\, \mathcal{W}_{ij} = \boxed{(\overline{\mathrm{L}}\, R)}\, (\overline{\mathrm{L}}\, \mathcal{W}_{ij})$$

$$\overline{\mathbf{S}}_i\, R = \mathcal{N}_1\, \delta_{f_i g} \sum_{k,l} \frac{s_{kl}}{s_{ik} s_{il}} \bar{B}_{kl}^{(ikl)}$$

$$\overline{\mathbf{C}}_{ij}\, R = \mathcal{N}_1 \frac{P_{ij}^{\mu\nu}}{s_{ij}} \bar{B}_{\mu\nu}^{(ijr)}$$

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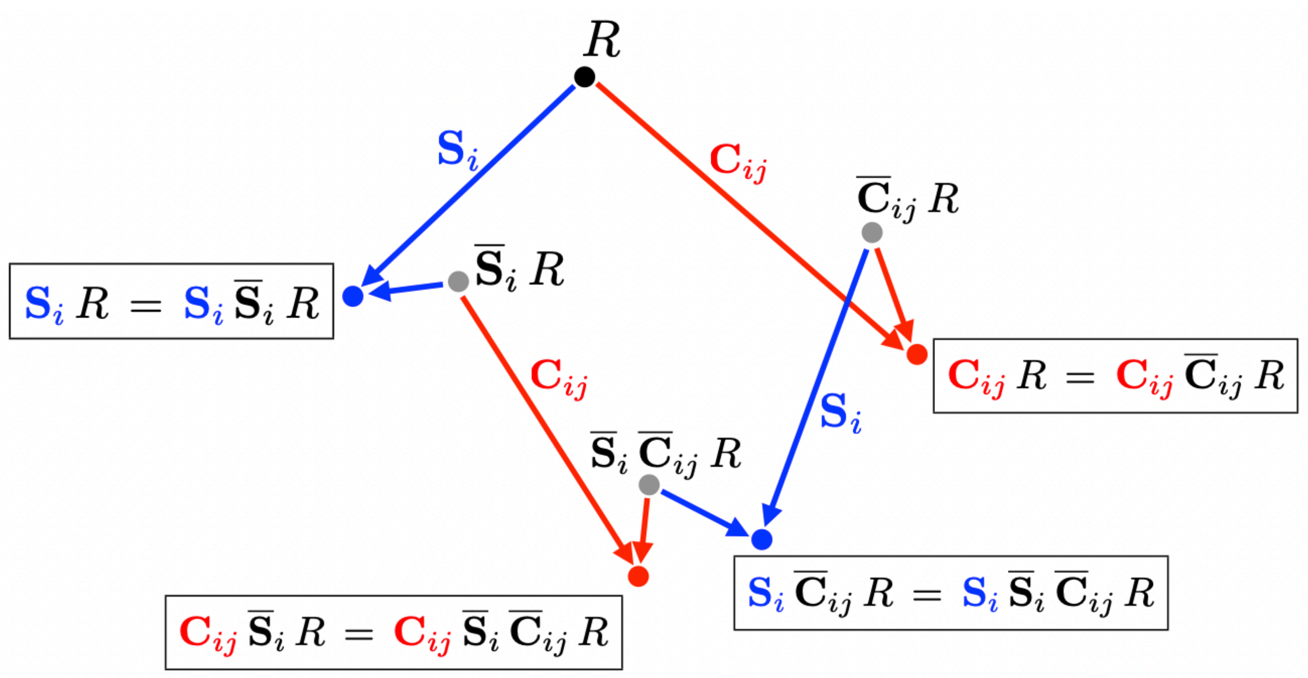
- **Locality** of the cancellation ensured by *consistency relations*

$$\mathbf{S}_i\, R = \mathbf{S}_i \left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} \right) R$$

$$\mathbf{C}_{ij}\, R = \mathbf{C}_{ij} \left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} \right) R$$



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$$\frac{d\sigma_{NLO}}{dX} = \int d\Phi_n \left(V + I\right)\delta_{X_n} + \int d\Phi_{n+1}\left(R\,\delta_{X_{n+1}} - K\,\delta_{X_n}\right)$$

Finite in phase space (integrable in $d = 4$)

$$\mathbf{S}_i\,R = \mathbf{S}_i\left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\right)R$$

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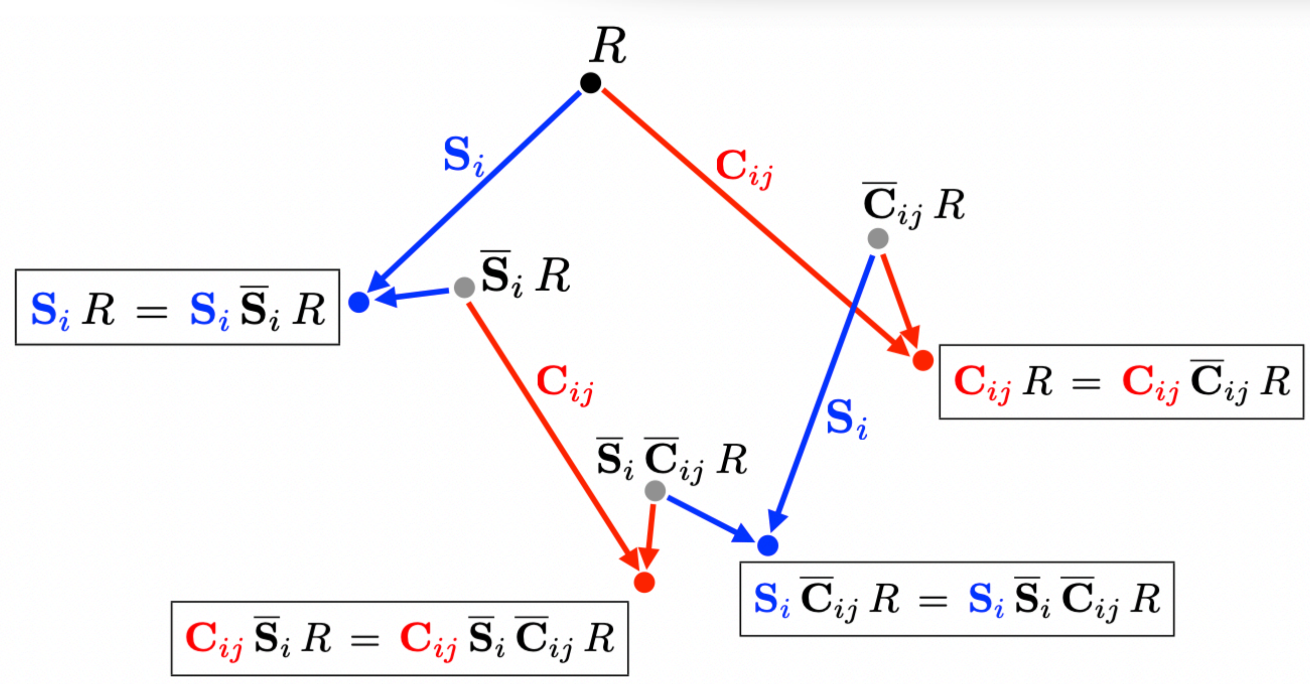

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☑ **Finite** in phase space (integrable in $d = 4$)

- $\mathcal{W}_{ij}$ sum rules + mapping adaptation = **simple analytic** counterterm integration

$$K = \sum_{i,j\neq i}\Big[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\Big]R\mathcal{W}_{ij} = \sum_i\Big[\overline{\mathbf{S}}_i + \sum_{j>i}\overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j)\Big]R$$

Strategy of the method

- Partition of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$ (as in FKS) [Frixione, Kunszt, Signer 9512328]
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- ☑ **Finite** in phase space (integrable in $d = 4$)
- ☑ **Free** from ϵ poles

- $\mathcal{W}_{ij}$ sum rules + mapping adaptation = **simple analytic** counterterm integration

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$$\begin{aligned} I^S &\propto \sum_{k,l}\bar{B}_{kl}^{(ikl)}\frac{1}{\bar{s}_{kl}^{(ikl)}}\int d\Phi_{\text{rad}}(\bar{s}_{kl}^{(ikl)};y,z,\phi)\frac{1-z}{yz} \\ &= \sum_{k,l}\bar{B}_{kl}^{(ikl)}\frac{(4\pi)^{\epsilon-2}}{(\bar{s}_{kl}^{(ikl)})^{\epsilon}}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\Gamma(2-3\epsilon)} \end{aligned}$$

Strategy of the method

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$$\frac{d\sigma_{NLO}}{dX} = \int d\Phi_n \Big(V + I\Big)\delta_{X_n} + \int d\Phi_{n+1}\Big(R\,\delta_{X_{n+1}} - K\,\delta_{X_n}\Big)$$

- ☑ **Finite** in phase space (integrable in $d = 4$)
- ☑ **Free** from ϵ poles

NEW RESULTS

NLO scheme extension to ISR

[**G.B.**, P. Torrielli, S. Uccirati, M. Zaro **2209.09123**]

Local analytic sector subtraction at NNLO

FSR
massless QCD

Generalities at NNLO

- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \, VV \, \delta_{X_n}$$

Explicit poles up to ϵ^4

$$+ \int d\Phi_{n+1} \, RV \, \delta_{X_{n+1}}$$

Explicit poles up to ϵ^2

Phase space singularities

$$+ \int d\Phi_{n+2} \, RR \, \delta_{X_{n+2}}$$

Phase space singularities

Generalities at NNLO

- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \underbrace{\left[\left(RV \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} \right) \delta_{X_n} \right]}_{\text{finite in PS}}$$

$$+ \int d\Phi_{n+2} \underbrace{\left[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right]}_{\text{finite in PS}}$$

- Introduce **local counterterms** and its integrals

$$\int d\Phi_{n+2}\, K^{(1)}\, \delta_{X_{n+1}} = \int d\Phi_{n+1}\, I^{(1)}\, \delta_{X_{n+1}}$$

$$\int d\Phi_{n+2}\, K^{(12)}\, \delta_{X_n} = \int d\Phi_{n+1}\, I^{(12)}\, \delta_{X_n}$$

$$\int d\Phi_{n+2}\, K^{(2)}\, \delta_{X_n} = \int d\Phi_n\, I^{(2)}\, \delta_{X_n}$$

$$\int d\Phi_{n+1}\, K^{(\mathrm{RV})}\, \delta_{X_n} = \int d\Phi_n\, I^{(\mathrm{RV})}\, \delta_{X_n}$$

Generalities at NNLO

- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \underbrace{\left(VV + I^{(2)} + I^{(RV)} \right)}_{\text{finite in d=4}} \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \underbrace{\left[\underbrace{\left(RV + I^{(1)} \right)}_{\text{finite in d=4, div. in PS}} \delta_{X_{n+1}} - \underbrace{\left(K^{(\mathrm{RV})} + I^{(12)} \right)}_{\text{finite in d=4, div. in PS}} \delta_{X_n} \right]}_{\text{finite in PS}}$$

$$+ \int d\Phi_{n+2} \underbrace{\left[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right]}_{\text{finite in PS}}$$

- Introduce **local counterterms** and its integrals

$$\int d\Phi_{n+2}\, K^{(1)}\, \delta_{X_{n+1}} = \int d\Phi_{n+1}\, I^{(1)}\, \delta_{X_{n+1}} \qquad \int d\Phi_{n+2}\, K^{(12)}\, \delta_{X_n} = \int d\Phi_{n+1}\, I^{(12)}\, \delta_{X_n}$$

$$\int d\Phi_{n+2}\, K^{(2)}\, \delta_{X_n} = \int d\Phi_n\, I^{(2)}\, \delta_{X_n} \qquad \int d\Phi_{n+1}\, K^{(\mathrm{RV})}\, \delta_{X_n} = \int d\Phi_n\, I^{(\mathrm{RV})}\, \delta_{X_n}$$

Counterterms for RR

- Partition of double-unresolved Φ_{n+2} with **sector functions** $\mathcal{W}_{ijkl}$

* Normalised as $\displaystyle\sum_{i,j\neq i}\sum_{\substack{k\neq i\\ l\neq i,k}}\mathcal{W}_{ijkl}=1$ $\qquad$ $\displaystyle RR=\sum_{i,j\neq i}\sum_{\substack{k\neq i\\ l\neq i,k}}RR\,\mathcal{W}_{ijkl}$

* Uniform double-unresolved configurations

$\mathbf{S}_{ij}$ **double-soft** partons *i* and *j*

$\mathbf{C}_{ijk}$ **triple-collinear** partons *(i,j,k)*

$\mathbf{C}_{ijkl}$ **double-collinear** partons *(i,j)* and *(k,l)*

$\mathbf{SC}_{ijk}$ **soft** parton *i* and **collinear** partons *(j,k)*

Sum rules

$$\mathbf{S}_{ik}\Big(\sum_{b\neq i}\sum_{d\neq i,k}\mathcal{W}_{ibkd}+\sum_{b\neq k}\sum_{d\neq k,i}\mathcal{W}_{kbid}\Big)=1$$

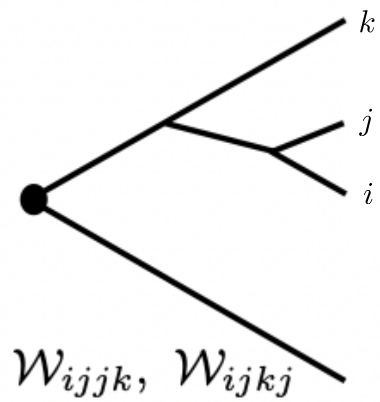
$$\mathbf{C}_{ijk}\sum_{abc\,\in\,\pi(ijk)}\Big(\mathcal{W}_{abbc}+\mathcal{W}_{abcb}\Big)=1$$

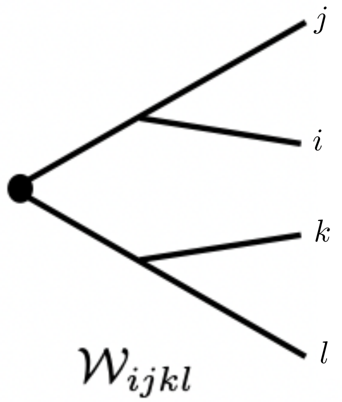
$$\dots$$

Counterterms for RR

- Partition of double-unresolved Φ_{n+2} with **sector functions** $\mathcal{W}_{ijkl}$
- Collect the relevant **IRC limits** for each *topology* $(\tau = ijjk, ijkj, ijkl)$

$$\begin{array}{lllllll}
\mathcal{W}_{ijjk} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ij} & \mathbf{C}_{ijk} & \mathbf{SC}_{ijk} \\
\mathcal{W}_{ijkj} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ik} & \mathbf{C}_{ijk} & \mathbf{SC}_{ijk} & \mathbf{SC}_{kij} \\
\mathcal{W}_{ijkl} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ik} & \mathbf{C}_{ijkl} & \mathbf{SC}_{ikl} & \mathbf{SC}_{kij}
\end{array}$$





Counterterms for RR

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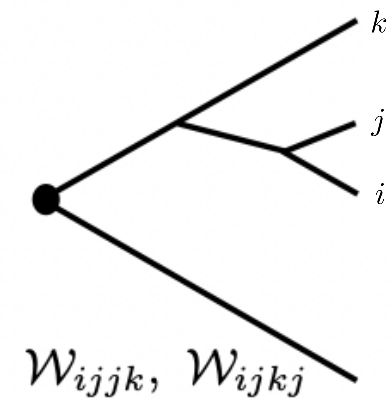
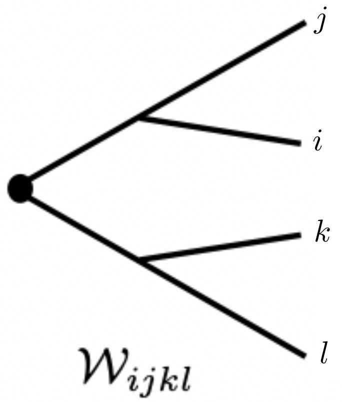
$$\begin{array}{lccccccc} \mathcal{W}_{ijjk} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ij} & \mathbf{C}_{ijk} & \mathbf{SC}_{ijk} & \\ \mathcal{W}_{ijkj} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ik} & \mathbf{C}_{ijk} & \mathbf{SC}_{ijk} & \mathbf{SC}_{kij} \\ \mathcal{W}_{ijkl} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ik} & \mathbf{C}_{ijkl} & \mathbf{SC}_{ikl} & \mathbf{SC}_{kij} \end{array}$$





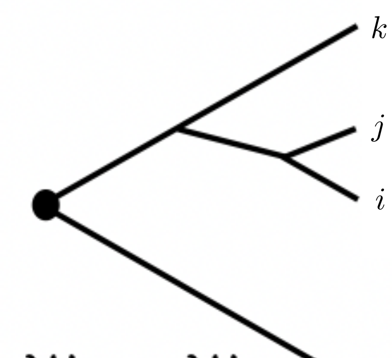
single-unresolved limits

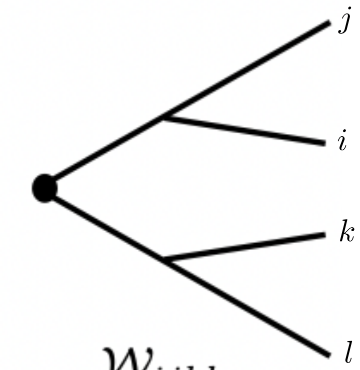
double-unresolved limits

Counterterms for RR

- Partition of double-unresolved Φ_{n+2} with **sector functions** $\mathcal{W}_{ijkl}$
- Collect the relevant **IRC limits** for each *topology* $(\tau = ijjk, ijkj, ijkl)$

$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ij}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	
$\mathcal{W}_{ijkj}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	$\mathbf{SC}_{kij}$
$\mathcal{W}_{ijkl}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijkl}$	$\mathbf{SC}_{ikl}$	$\mathbf{SC}_{kij}$





single-unresolved limits

$$\mathbf{L}^{(1)}_{ij} = \mathbf{S}_i + \mathbf{C}_{ij}(1 - \mathbf{S}_i)$$

double-unresolved limits

$$\mathbf{L}^{(2)}_{ijjk} = \mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})$$
$$\mathbf{L}^{(2)}_{ijkj} = \mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ijk} + \mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})$$
$$\mathbf{L}^{(2)}_{ijkl} = \mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ikl} + \mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})$$

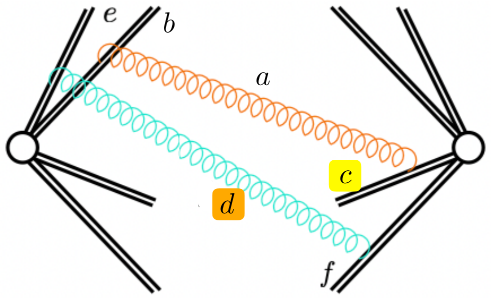
$$\boxed{RR\,\mathcal{W}_\tau - \left[\mathbf{L}^{(1)}_{ij} + \mathbf{L}^{(2)}_{\tau} - \mathbf{L}^{(1)}_{ij}\mathbf{L}^{(2)}_{\tau}\right] RR\,\mathcal{W}_\tau = \text{finite}}$$

Counterterms for RR

- Partition of double-unresolved Φ_{n+2} with **sector functions** $\mathcal{W}_{ijkl}$
- Collect the relevant **IRC limits** for each *topology* $(\tau = ijjk, ijkj, ijkl)$
- **Nested** Catani-Seymour mappings
  - mapping from $(n+2) \to n$ kinematics
  - simple phase-space factorisation
  - parametrisation simplifies kernels expressions

$$\{k\} \to \{\bar{k}\}^{(abc,def)}$$

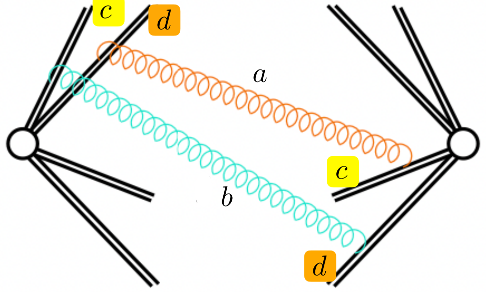
$$d\Phi_{n+2} = d\Phi_n^{(abcd)}\, d\Phi_{\rm rad}^{(abc)}\, d\Phi_{\rm rad}^{(def)}$$



$$\mathbf{S}_{ij}\, RR \supset \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \Bigg[\sum_{\substack{e\neq i,j,c,d \\ f\neq i,j,c,d,e}} \frac{s_{cd}}{s_{ic}s_{id}} \frac{s_{ef}}{s_{je}s_{jf}} \boxed{B_{cdef}(\{k\}_{\not{i}\not{j}})} + 2\, \frac{s_{cd}}{s_{ic}s_{id}} \frac{s_{cd}}{s_{ic}s_{id}} \boxed{B_{cdcd}(\{k\}_{\not{i}\not{j}})} \Bigg]$$

$$\{k\} \to \{\bar{k}\}^{(abcd)}$$

$$d\Phi_{n+2} = d\Phi_n^{(abcd)}\, d\Phi_{\rm rad,2}^{(abcd)}$$



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- **Nested** Catani-Seymour mappings
- Promotion to **counterterms**: adapt mapping to each kernel

$$K^{(\mathbf{1})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}_{ij}^{(\mathbf{1})}\, RR\, \mathcal{W}_{ijkl}$$

single-unresolved limits

$$K^{(\mathbf{2})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}_{ijkl}^{(\mathbf{2})}\, RR\, \mathcal{W}_{ijkl}$$

uniform double-unresolved limits

$$K^{(\mathbf{12})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq ,k}} \overline{\mathrm{L}}_{ij}^{(\mathbf{1})}\, \overline{\mathrm{L}}_{ijkl}^{(\mathbf{2})}\, RR\, \mathcal{W}_{ijkl}$$

strongly-ordered double-unresolved limits

Notation

$$\overline{\mathrm{L}}\, RR\, \mathcal{W}_{ijkl} \;=\; (\overline{\mathrm{L}}\, RR)\, (\overline{\mathrm{L}}\, \mathcal{W}_{ijkl})$$

Counterterms for RR

- Partition of double-unresolved Φ_{n+2} with **sector functions** $\mathcal{W}_{ijkl}$
- Collect the relevant **IRC limits** for each *topology* $(\tau = ijjk, ijkj, ijkl)$
- **Nested** Catani-Seymour mappings
- Promotion to **counterterms**: adapt mapping to each kernel

$$K^{(\mathbf{1})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{1})}_{ij} \, RR \, \mathcal{W}_{ijkl}$$

single-unresolved limits

$$\begin{aligned}
K^{(\mathbf{2})} &= \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl} \, RR \, \mathcal{W}_{ijkl} \\
&= \Bigg\{ \sum_{i,k>i} \overline{\mathbf{S}}_{ik} + \sum_{i,j>i} \sum_{k>j} \overline{\mathbf{C}}_{ijk} \Big(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\Big) \\
&\quad + \sum_{i,j>i} \sum_{\substack{k\neq j \\ k>i}} \sum_{\substack{l\neq j \\ l>k}} \overline{\mathbf{C}}_{ijkl} \Big[1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{jl} \\
&\qquad - \overline{\mathbf{SC}}_{ikl}(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{il}) - \overline{\mathbf{SC}}_{jkl}(1 - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{jl}) \\
&\qquad - \overline{\mathbf{SC}}_{kij}(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}) - \overline{\mathbf{SC}}_{lij}(1 - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jl})\Big] \\
&\quad + \sum_{i,j>i} \sum_{\substack{k\neq i \\ k>j}} \overline{\mathbf{SC}}_{ijk}(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik})(1 - \overline{\mathbf{C}}_{ijk}) \Bigg\} RR
\end{aligned}$$

uniform double-unresolved limits

Collection of universal kernels!

Counterterms for RR

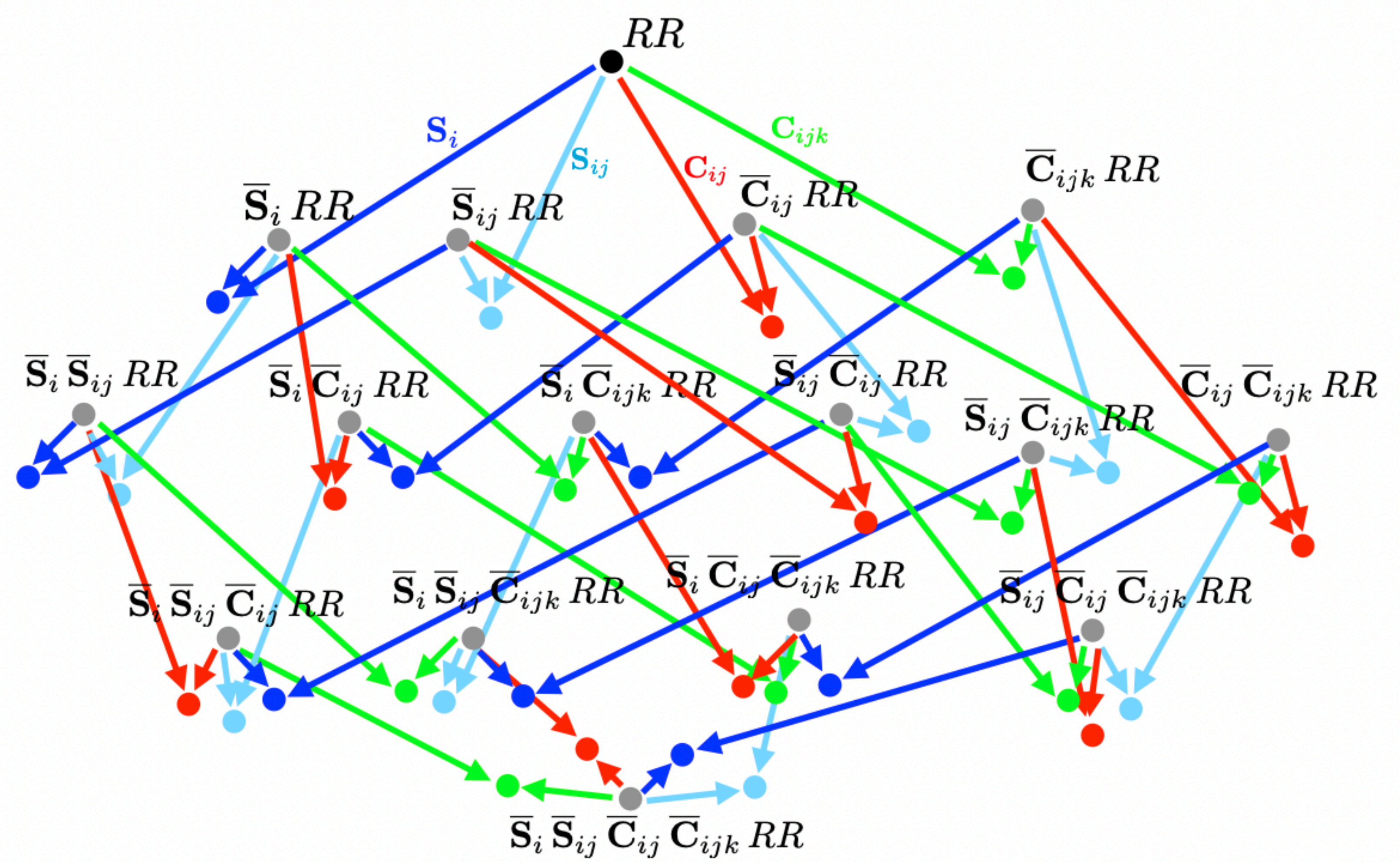
- Partition of double-unresolved Φ_{n+2} with **sector functions** $\mathcal{W}_{ijkl}$
- Collect the relevant **IRC limits** for each *topology* $(\tau = ijjk, ijkj, ijkl)$
- **Nested** Catani-Seymour mappings
- Promotion to **counterterms**: adapt mapping to each kernel

$$K^{(\mathbf{1})},\ K^{(\mathbf{2})},\ K^{(\mathbf{12})}$$

- **Locality** of the cancellation ensured by *consistency relations*

verified sector by sector

$$\overline{\mathbf{S}}_i,\quad \overline{\mathbf{C}}_{ij},\quad \overline{\mathbf{S}}_{ij},\quad \overline{\mathbf{C}}_{ijk}$$



SC limits not displayed.

Counterterms for RR

$$
\begin{aligned}
\frac{d\sigma_{NNLO}}{dX} = & \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})} \right) \delta_{X_n} \\
& + \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_{X_n} \right] \\
& + \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right]
\end{aligned}
$$

Finiteness of double-real correction (integrable in $d = 4$)

NNLO complexity

- $\mathcal{W}_{ijkl}$ sum rules + mapping adaptation = **feasible analytic** counterterms integration

$I^{(\mathbf{1})} = \int d\Phi_{\mathrm{rad}}\, K^{(\mathbf{1})}$ $\qquad$ $I^{(\mathbf{12})} = \int d\Phi_{\mathrm{rad}}\, K^{(\mathbf{12})}$ $\qquad$ $I^{(\mathbf{2})} = \int d\Phi_{\mathrm{rad},2}\, K^{(\mathbf{2})}$

$K^{(2)} \supset \overline{\mathbf{S}}_{ij} RR, \overline{\mathbf{C}}_{ijk} RR$

[Catani, Grazzini 9903516, 9810389]

Counterterm for RV

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})}\right)\delta_{X_n}$$

$$+ \int d\Phi_{n+1}\Big[\left(RV + I^{(1)}\right)\delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)}\right)\delta_{X_n}\Big]$$

$$+ \int d\Phi_{n+2}\Big[RR\,\delta_{X_{n+2}} - K^{(1)}\delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)}\right)\delta_{X_n}\Big]$$

- Apply NLO strategy to **define** and **analytically** integrate in single-unresolved phase space

$$K^{(\mathrm{RV})} \supset \sum_{i,j\neq i}\Big[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i)\Big]RV\,\mathcal{W}_{ij}$$

☑ **Analytically** checked

- $RV + I^{(1)} \to$ free from ϵ poles
- $RV - K^{(\mathrm{RV})} \to$ finite in phase space
- $I^{(1)} - I^{(12)} \to$ finite in phase space
- $K^{(\mathrm{RV})} + I^{(12)} \to$ free from ϵ poles

Counterterm for RV

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})}\right)\delta_{X_n}$$
$$+ \int d\Phi_{n+1}\Big[\left(RV + I^{(1)}\right)\delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)}\right)\delta_{X_n}\Big]$$
$$+ \int d\Phi_{n+2}\Big[RR\,\delta_{X_{n+2}} - K^{(1)}\delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)}\right)\delta_{X_n}\Big]$$

- Apply NLO strategy to **define** and **analytically** integrate in single-unresolved phase space

- $K^{(\mathrm{RV})} \supset \sum_{i,j\neq i}\left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i)\right] RV\,\mathcal{W}_{ij}$
- $I^{(\mathrm{RV})} = \int d\Phi_{\mathrm{rad}}\,K^{(\mathrm{RV})}$

☑ **Analytically** checked

- $RV + I^{(1)} \to$ free from ϵ poles
- $RV - K^{(\mathrm{RV})} \to$ finite in phase space
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NNLO subtraction formula

massless FSR

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})}\right)\delta_{X_n}$$

$$+ \int d\Phi_{n+1}\left[\left(RV + I^{(1)}\right)\delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)}\right)\delta_{X_n}\right]$$

$$+ \int d\Phi_{n+2}\left[RR\,\delta_{X_{n+2}} - K^{(1)}\delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)}\right)\delta_{X_n}\right]$$

☑ **Analytically** verified for an **arbitrary** number of final-state partons

$$\boxed{VV + I^{(2)} + I^{(\mathrm{RV})} \to \text{free from } \epsilon \text{ poles}}$$

NEW RESULTS

[**G.B.**, Ratti, Signorile-Signorile, Torrielli et al. ***In preparation***]

NNLO subtraction formula

massless FSR

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \boxed{\left(VV + I^{(2)} + I^{(\mathrm{RV})}\right)} \delta_{X_n}$$

$$+ \int d\Phi_{n+1}\left[\left(RV + I^{(1)}\right)\delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)}\right)\delta_{X_n}\right]$$

$$\begin{aligned}
VV + I^{(2)} + I^{(\mathrm{RV})} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ & \left[I^{(0)} + \sum_j I_j^{(1)} \mathbf{L}_{jr} + \sum_j I_j^{(2)} \mathbf{L}_{jr}^2 + \frac{1}{2}\sum_{j,l\neq j} \gamma_j^{\mathrm{hc}}\,\gamma_l^{\mathrm{hc}}\,\mathbf{L}_{jr'}\mathbf{L}_{lr'}\right]\mathbf{B} \\
& + \sum_j \left[I_{jr}^{(0)} + I_{jr}^{(1)}\,\mathbf{L}_{jr}\right]\mathbf{B}_{jr} - 2\,(1-\zeta_2)\sum_{j,c\neq j,r} \gamma_j^{\mathrm{hc}}\,(2-\mathbf{L}_{cr})\,\mathbf{B}_{cr} \\
& + \sum_{c,d\neq c} \mathbf{L}_{cd}\left[I_{cd}^{(0)} + I_{cd}^{(1)}\,\mathbf{L}_{cd} + \frac{\beta_0}{12}\mathbf{L}_{cd}^2 + (4-\mathbf{L}_{cd})\sum_j \gamma_j^{\mathrm{hc}}\,\mathbf{L}_{jr}\right]\mathbf{B}_{cd} \\
& + \sum_{c,d\neq c}\left[-2+\zeta_2+2\,\zeta_3-\frac{5}{4}\,\zeta_4+2(1-\zeta_3)\,\mathbf{L}_{cd}\right]\mathbf{B}_{cdcd} \\
& + (1-\zeta_2)\sum_{\substack{c,d\neq c\\ e\neq d}} \mathbf{L}_{cd}\,\mathbf{L}_{ed}\,\mathbf{B}_{cded} + \sum_{\substack{c,d\neq c\\ e,f\neq e}} \mathbf{L}_{cd}\,\mathbf{L}_{ef}\left[1-\frac{1}{2}\mathbf{L}_{cd}\left(1-\frac{1}{8}\mathbf{L}_{ef}\right)\right]\mathbf{B}_{cdef} \\
& + \pi\sum_{\substack{c,d\neq c\\ e\neq c,d}}\left[\ln\frac{s_{ce}}{s_{de}}\,\mathbf{L}_{cd}^2 + \frac{1}{3}\ln^3\frac{s_{ce}}{s_{de}} + 2\,\mathrm{Li}_3\left(-\frac{s_{ce}}{s_{de}}\right)\right]\mathbf{B}_{cde}\Bigg\} \\
+ \left(\frac{\alpha_s}{2\pi}\right)\Bigg\{ & \left[\Sigma_\phi - \sum_j \gamma_j^{\mathrm{hc}}\,\mathbf{L}_{jr}\right]\mathbf{V}^{\mathrm{fin}} + \sum_{c,d\neq c}\mathbf{L}_{cd}\left(2-\frac{1}{2}\mathbf{L}_{cd}\right)\mathbf{V}_{cd}^{\mathrm{fin}}\Bigg\} + \mathbf{VV}^{\mathrm{fin}}
\end{aligned}$$

Analytic and compact!

$$\mathrm{L}_{ab} = \log\frac{s_{ab}}{\mu^2}$$

NEW RESULTS

[**G.B.**, Ratti, Signorile-Signorile, Torrielli et al. ***In preparation***]

Status

- ☑ General analytic subtraction formula for massless FSR and ISR at NLO
- ☑ Numerical implementation and validation of NLO subtraction formula
- ☑ General analytic subtraction formula for massless FSR at NNLO

Outlook

- Framework optimisation for relevant phenomenology
  (phase-space integration routine, low-level code implementation, …)
- Numerical implementation of NNLO massless FSR
- Extension of ISR subtraction to NNLO: integrals of complexity similar as massless FSR
- NLO treatment of the massive case; future extension to NNLO

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Thanks
for your attention!

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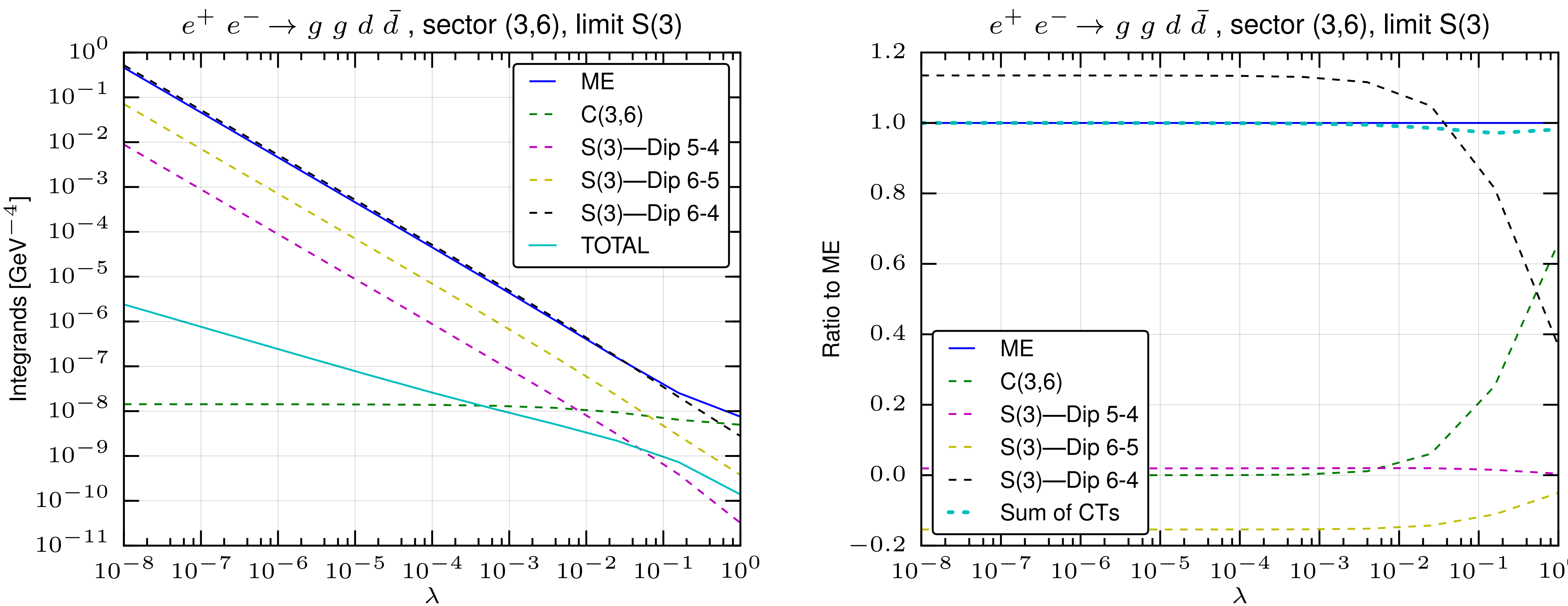
Backup slides

Soft limit: $\lambda \sim E_i^2$



[**G.B.**, P. Torrielli, S. Uccirati, M. Zaro **2209.09123**]

Sector functions

* Example of NLO sector functions $(s_{qi} = 2\,q_{\rm cm}\cdot k_i\,,\, s_{ij} = 2\,k_i\cdot k_j\,,\, s = q_{\rm cm}^2)$

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum\limits_{a,b\neq a}\sigma_{ab}} \qquad \text{with} \qquad \boxed{\sigma_{ij} = \frac{1}{\mathcal{E}_i\,\omega_{ij}}}$$

$\mathbf{S}_i$ **soft** parton *i* $\qquad \mathcal{E}_i = \dfrac{s_{qi}}{s} \quad \to 0$

$\mathbf{C}_{ij}$ **collinear** pair *ij* $\qquad w_{ij} = \dfrac{s\,s_{ij}}{s_{qi}s_{qj}} \quad \to 0$

* Example of NNLO sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma} \qquad \text{with} \qquad \boxed{\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i\,w_{ij})^\alpha}\,\frac{1}{(\mathcal{E}_k + \delta_{kj}\,\mathcal{E}_i)\,w_{kl}} \qquad \alpha > 1}$$

$$\sigma = \sum_{\substack{i,\,j\neq i\\ k\neq i,\,l\neq i,k}} \sigma_{ijkl}$$

Slide by Lorenzo Magnea, HP2 2022

N³LO Subtraction

A systematic generalisation to higher orders is possible. At three loops one finds

$$
\begin{aligned}
\frac{d\sigma_{\mathrm{N3LO}}}{dX} = & \int d\Phi_n \left[VVV_n + I_n^{(\mathbf{3})} + I_n^{(\mathbf{RVV})} + I_n^{(\mathbf{RRV},\,\mathbf{2})} \right] \delta_n(X) \\
& + \int d\Phi_{n+1} \bigg[\left(RVV_{n+1} + I_{n+1}^{(\mathbf{2})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{1})} \right) \delta_{n+1}(X) \\
& \qquad - \left(K_{n+1}^{(\mathbf{RVV})} + I_{n+1}^{(\mathbf{23})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{12})} \right) \delta_n(X) \bigg] \\
& + \int d\Phi_{n+2} \bigg\{ \left(RRV_{n+2} + I_{n+2}^{(\mathbf{1})} \right) \delta_{n+2}(X) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{1})} + I_{n+2}^{(\mathbf{12})} \right) \delta_{n+1}(X) \\
& \qquad - \left[\left(K_{n+2}^{(\mathbf{RRV},\,\mathbf{2})} + I_{n+2}^{(\mathbf{13})} \right) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{12})} + I_{n+2}^{(\mathbf{123})} \right) \right] \delta_n(X) \bigg\} \\
& + \int d\Phi_{n+3} \bigg[RRR_{n+3}\, \delta_{n+3}(X) - K_{n+3}^{(\mathbf{1})}\, \delta_{n+2}(X) - \left(K_{n+3}^{(\mathbf{2})} - K_{n+3}^{(\mathbf{12})} \right) \delta_{n+1}(X) \\
& \qquad - \left(K_{n+3}^{(\mathbf{3})} - K_{n+3}^{(\mathbf{13})} - K_{n+3}^{(\mathbf{23})} + K_{n+3}^{(\mathbf{123})} \right) \delta_n(X) \bigg] ,
\end{aligned}
$$

A general formula for $\mathrm{N^kLO}$ subtraction is available, involving $p = 2^{(k+1)} - 2 - k$ counterterms.

NNLO subtraction formula

massless FSR

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \boxed{\left(VV + I^{(2)} + I^{(\mathrm{RV})}\right)} \delta_{X_n}$$

$$+ \int d\Phi_{n+1}\Big[\left(RV + I^{(1)}\right)\delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)}\right)\delta_{X_n}\Big]$$

$$\mathrm{L}_{ab} = \log\frac{s_{ab}}{\mu^2}$$

$$VV + I^{(2)} + I^{(\mathrm{RV})} = \left(\frac{\alpha_s}{2\pi}\right)^2 \bigg\{ \Big[\boxed{I^{(0)}} + \sum_j \boxed{I^{(1)}_j}\,\mathbf{L}_{jr} + \sum_j \boxed{I^{(2)}_j}\,\mathbf{L}^2_{jr} + \ldots$$

$$+ \sum_j \Big[\boxed{I^{(0)}_{jr}} + \boxed{I^{(1)}_{jr}}\,\mathbf{L}_{jr}\Big]\,\mathbf{B}_{jr} - 2\,(1-\zeta_2 \ldots$$

$$+ \sum_{c,d\neq c} \mathbf{L}_{cd}\Big[\boxed{I^{(0)}_{cd}} + \boxed{I^{(1)}_{cd}}\,\mathbf{L}_{cd} + \frac{\beta_0}{12}\,\mathbf{L}^2_{cd} + \ldots$$

$$+ \sum_{c,d\neq c}\Big[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\,\zeta_4 + 2(1 \ldots$$

$$+ (1-\zeta_2)\sum_{\substack{c,d\neq c\\ e\neq d}} \mathbf{L}_{cd}\,\mathbf{L}_{ed}\,\mathbf{B}_{cded} + \sum_{\substack{c,d\neq c\\ e,f\neq e}} \mathbf{L} \ldots$$

$$+ \pi \sum_{\substack{c,d\neq c\\ e\neq c,d}} \Big[\ln\frac{s_{ce}}{s_{de}}\,\mathbf{L}^2_{cd} + \frac{1}{3}\ln^3\frac{s_{ce}}{s_{de}} + 2\,\mathrm{L} \ldots$$

$$+ \left(\frac{\alpha_s}{2\pi}\right)\bigg\{\Big[\Sigma_\phi - \sum_j \gamma_j^{\mathrm{hc}}\,\mathbf{L}_{jr}\Big]\,\mathbf{V}^{\mathrm{fin}} + \sum_{c,d\neq c}\mathbf{L}_{cd} \ldots$$

Analytic and compact!

$$I^{(0)} = N_q^2 C_F^2\left[\frac{101}{8} - \frac{141}{8}\zeta_2 + \frac{245}{16}\zeta_4\right] + N_g N_q C_F\left[C_A\left(\frac{13}{3} - \frac{125}{6}\zeta_2 + \frac{245}{8}\zeta_4\right) + \beta_0\left(\frac{77}{12} - \frac{53}{12}\zeta_2\right)\right]$$
$$+ N_g^2\left[C_A^2\left(\frac{20}{9} - \frac{13}{3}\zeta_2 + \frac{245}{16}\zeta_4\right) + \beta_0^2\left(\frac{73}{72} - \frac{1}{8}\zeta_2\right) + C_A\beta_0\left(-\frac{1}{9} - \frac{11}{3}\zeta_2\right)\right]$$
$$+ N_q C_F\left[C_F\left(\frac{53}{32} - \frac{57}{8}\zeta_2 + \frac{1}{2}\zeta_3 + \frac{21}{4}\zeta_4\right) + C_A\left(\frac{677}{432} + \frac{5}{3}\zeta_2 - \frac{25}{2}\zeta_3 + \frac{47}{8}\zeta_4\right)\right.$$
$$\left. + \beta_0\left(\frac{5669}{864} - \frac{85}{24}\zeta_2 - \frac{11}{12}\zeta_3\right)\right]$$
$$+ N_g\left[C_F C_A\left(-\frac{737}{48} + 11\zeta_3\right) + C_F\beta_0\left(\frac{67}{16} - 3\zeta_3\right) + \beta_0^2\left(\frac{73}{72} - \frac{3}{8}\zeta_2\right)\right.$$
$$\left. + C_A^2\left(-\frac{4289}{216} + \frac{15}{2}\zeta_2 - 14\zeta_3 + \frac{89}{8}\zeta_4\right) + C_A\beta_0\left(\frac{647}{54} - \frac{53}{8}\zeta_2 - \frac{11}{12}\zeta_3\right)\right]$$

$$I_j^{(1)} = \delta_{f_a\{q,\bar q\}} C_F\left[N_q C_F\left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A\left(\frac{1}{3} - \frac{7}{4}\zeta_2\right) + \frac{2}{3}N_g\beta_0\right.$$
$$\left. + C_F\left(-\frac{3}{8} - 4\zeta_2 + 2\zeta_3\right) + C_A\left(\frac{25}{12} - 3\zeta_2 + 3\zeta_3\right) + \beta_0\left(\frac{1}{24} + \zeta_2\right)\right]$$
$$+ \delta_{f_a g}\left[N_q C_F C_A(10 - 7\zeta_2) - N_q C_F\beta_0\left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A^2\left(\frac{4}{3} - 7\zeta_2\right) + N_g C_A\beta_0\left(\frac{7}{3} + \frac{7}{4}\zeta_2\right)\right.$$
$$\left. - \frac{2}{3}(N_g + 1)\beta_0^2 + \frac{11}{4}C_F C_A - \frac{3}{4}C_F\beta_0 + C_A^2\left(\frac{28}{3} - \frac{23}{2}\zeta_2 + 5\zeta_3\right) - C_A\beta_0\left(\frac{2}{3} - \frac{5}{2}\zeta_2\right)\right]$$

$$I_j^{(2)} = \frac{1}{8}(15\,C_A - 7\,\beta_0 - 15)C_{f_j} - \frac{1}{4}(5\,C_A - 2\,\beta_0)\gamma_j + 2\,\zeta_2\,C_{f_j}^2$$

$$I_{jr}^{(0)} = (-1 + 3\zeta_2 - 2\zeta_3)C_A - \frac{1}{2}(13 + 10\zeta_2 + 2\zeta_3)C_{f_j} + (5 + 2\zeta_3)\gamma_j$$

$$I_{jr}^{(1)} = (1 - \zeta_2)C_A + \frac{1}{2}(4 + 7\zeta_2)C_{f_j} - (2 + \zeta_2)\gamma_j$$

$$I_{cd}^{(0)} = \left(\frac{20}{9} - 2\zeta_2 - \frac{7}{2}\zeta_3\right)C_A + \frac{31}{9}\,\beta_0 + 2\,\Sigma_\phi + 8\,(1 - \zeta_2)\,C_{f_d}$$

$$I_{cd}^{(1)} = -\left(\frac{1}{3} - \frac{1}{2}\zeta_2\right)C_A - \frac{11}{12}\,\beta_0 - \frac{1}{2}\,\Sigma_\phi$$

Double-unresolved phase space

- Catani-Seymour variables $y, z, y', z', x' \in [0,1]$ for mapping $\{k\} \to \{\bar{k}\}^{(abcd)}$:

$$s_{ab} = y'\, y\, s_{abcd}\,, \qquad s_{cd} = (1-y')\,(1-y)\,(1-z)\; s_{abcd}\,,$$
$$s_{ac} = z'\,(1-y')\; y\, s_{abcd}\,, \qquad s_{bc} = (1-y')\,(1-z')\; y\, s_{abcd}\,,$$
$$s_{ad} = (1-y)\left[y'\,(1-z')\,(1-z) + z'z - 2\,(1-2x')\sqrt{y'z'\,(1-z')\; z\,(1-z)}\right] s_{abcd}\,,$$
$$s_{bd} = (1-y)\left[y'z'\,(1-z) + (1-z')\,z + 2\,(1-2x')\sqrt{y'z'\,(1-z')\; z\,(1-z)}\right] s_{abcd}\,,$$

- Phase-space factorisation:

$$d\Phi_{n+2} \quad = \quad d\Phi_n^{(abcd)}\, d\Phi_{\mathrm{rad},2}^{(abcd)}\,,$$

$$\begin{aligned}
\int d\Phi_{\mathrm{rad},2}^{(abcd)} \quad &= \quad \int d\Phi_{\mathrm{rad},2}\left(s_{abcd};\, y, z, \phi, y', z', x'\right) \\
&= \quad N^2(\epsilon)\,(s_{abcd})^{2-2\epsilon}\int_0^1 dx' \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi\; (\sin\phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \\
&\quad \times \left[4\,x'\,(1-x')\; y'\,(1-y')^2\; z'\,(1-z')\; y^2(1-y)^2\, z\,(1-z)\right]^{-\epsilon} \\
&\quad \times \left[x'\,(1-x')\,\right]^{-1/2}(1-y')\; y\,(1-y)\;.
\end{aligned}$$

Analytic integration of double-unresolved counterterms

- Exploit as much as possible symmetries of $d\Phi^{(abcd)}_{\mathrm{rad},2}$:

$$\mathrm{perm}(k_a, k_b, k_c, k_d)\,, \qquad s_{ab} \leftrightarrow s_{cd}\,, \qquad s_{ac} \leftrightarrow s_{bd}\,, \qquad s_{ad} \leftrightarrow s_{bc}\,.$$

- Possible denominator structures reduce to

$$\begin{aligned}
s_{ab} &= y'\, y\, s_{abcd}\,,\\
s_{ac} &= z'\,(1-y')\; y\, s_{abcd}\,,\\
s_{bc} &= (1-y')\,(1-z')\; y\, s_{abcd}\,,\\
s_{cd} &= (1-y')\,(1-y)\,(1-z)\; s_{abcd}\,,\\
s_{bd} &= (1-y)\left[y'z'\,(1-z) + (1-z')\,z + 2\,(1-2w')\,\sqrt{y'z'\,(1-z')\,z\,(1-z)}\right] s_{abcd}\,,\\
s_{ac}+s_{bc} &= (1-y')\; y\, s_{abcd}\,,\\
s_{ad}+s_{bd} &= (y'+z-y'\,z)\,(1-y)\, s_{abcd}\,,\\
s_{ab}+s_{bc} &= (1-z'+z'\,y')\; y\, s_{abcd}\,.
\end{aligned}$$

- Integration measure

$$\begin{aligned}
\int d\Phi^{(abcd)}_{\mathrm{rad},2} \;=\;& N(\epsilon)\,(s_{abcd})^{2-2\epsilon}\int_0^1 dw'\int_0^1 dy'\int_0^1 dz'\int_0^1 dy\int_0^1 dz\,\left[w'\,(1-w')\,\right]^{-1/2-\epsilon}\\
&\times\left[y'\,(1-y')^2\; z'\,(1-z')\; y^2(1-y)^2\, z\,(1-z)\right]^{-\epsilon}(1-y')\; y\,(1-y)\,.
\end{aligned}$$

Analytic integration of double-unresolved counterterms

- Integration measure

$$\int d\Phi^{(abcd)}_{\mathrm{rad},2} = N(\epsilon)\,(s_{abcd})^{2-2\epsilon}\int_0^1 dy\int_0^1 dw'\int_0^1 dz\int_0^1 dy'\int_0^1 dz'\,\left[w'\,(1-w')\,\right]^{-1/2-\epsilon}$$
$$\times\left[y'\,(1-y')^2\,z'\,(1-z')\,y^2(1-y)^2\,z\,(1-z)\right]^{-\epsilon}(1-y')\,y\,(1-y)\,.$$

- Integrate y: fully factorised dependence $\to$ Beta functions.
- Integrate w' (azimuth): at worst one gets rational $\times\,{}_2F_1\left[1,1+\epsilon,1-\epsilon,\frac{y'z'(1-z)}{z(1-z')}\right]$.
- Integrate z: at worst one gets rational $\times\,{}_2F_1\left[1,n+1-\epsilon,1-\epsilon,-\frac{y'z'}{1-z'}\right]$.
- ${}_2F_1\;\to\;$ integral representation in t; integrate in z' and get at worst

$$\int_0^1 dy'\,dt\,t^a\,(1-t)^b\,y'^c\,(1-y')^d\;{}_2F_1\left[n,m-\epsilon,p-2\epsilon,1-ty'\right],\qquad n,m,p\in\mathbb{N}$$

- Expand in ϵ and integrate in $dt\,dy'$.
- Checked against numerical integration (with no symmetries or relabellings encoded).