Feynman integrals & special functions for $pp \to Hjj$ in NNLO QCD

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in preparation with

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European Research Council
Established by the European Commission

QCD @ LHC 2022,
IJCLab Orsay (France)

30$^{\text{th}}$ November 2022

Michigan State University

Universität Zürich UZH

Introduction

Motivation

- LHC today is a precision machine
- Many measurements statistically limited $\rightrightarrows$ HL-LHC and future colliders
- Theoretical understanding of SM predictions is key to interpret data
- At least NNLO QCD and NLO EW corrections ($\oplus$ parton shower, resummation, ...) must be included to achieve percent level theory uncertainties

"Hard" partonic cross section

$$d\sigma_{h_1 h_2 \to X}(p_1, p_2) = \sum_{i,j} \int \mathrm{d}x_1\, \mathrm{d}x_2\, f_i(x_1,\mu) f_j(x_2,\mu) \boxed{\mathrm{d}\hat{\sigma}_{ij\to X}(x_1 p_1, x_2 p_2, \mu)} + \mathcal{O}(\Lambda_{\mathsf{QCD}}/Q)$$

Intrinsic uncertainty

$$\mathrm{d}\hat{\sigma}_0 \left(1 + \alpha_s \sigma^{(1,0)} + \alpha_s^2 \sigma^{(2,0)} + \alpha \sigma^{(0,1)} + \alpha_s^3 \sigma^{(3,0)} + \alpha\alpha_s \sigma^{(1,1)} + \ldots\right)$$

$$\alpha_s(M_Z) \sim 0.1$$
$$\alpha(M_Z) \sim 0.01$$

Class of processes

Les Houches "wishlist" [2207.02122]

- NLO QCD, EW conceptually solved, in practice $\lesssim 8$ partons
- NNLO QCD beyond $2 \to 2$ remarakably challenging

both technical and conceptual

What kind of functions loop integrals evaluate to?

process	known	desired
⋮	⋮	⋮
$pp \to V + 2j$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$	$\boxed{\mathrm{NNLO_{QCD}}}$ QCD precision
$pp \to V + b\bar{b}$	$\mathrm{NLO_{QCD}}$	$\boxed{\mathrm{NNLO_{QCD}}} + \mathrm{NLO_{EW}}$ $H \to b\bar{b}$ decay [Rene Poncelet's talk]
⋮	⋮	⋮
$pp \to H + 2j$	$\mathrm{NLO_{HTL}} \otimes \mathrm{LO_{QCD}}$ $\mathrm{N^3LO}^{(\mathrm{VBF}^*)}_{\mathrm{QCD}}$ (incl.) $\mathrm{NNLO}^{(\mathrm{VBF}^*)}_{\mathrm{QCD}}$ $\mathrm{NLO}^{(\mathrm{VBF})}_{\mathrm{EW}}$	VBF studies $\boxed{\mathrm{NNLO_{HTL}} \otimes \mathrm{NLO_{QCD}}} + \mathrm{NLO_{EW}}$ $\mathrm{N^3LO}^{(\mathrm{VBF}^*)}_{\mathrm{QCD}}$ $\mathrm{NNLO}^{(\mathrm{VBF})}_{\mathrm{QCD}}$
⋮	⋮	⋮

$$\sigma^{F+X}_{\mathsf{NNLO}} = \sigma^{F+X}_{\mathsf{NLO}} + \int_{\Phi_{F+2}} d\sigma_{\mathsf{RR}} + \int_{\Phi_{F+1}} d\sigma_{\mathsf{RV}} + \int_{\Phi_F} d\sigma_{\mathsf{VV}}$$

[Gloria Bertolotti's talk] IR divergences

Two-loop amplitudes

Structure of analytic loop amplitudes

Rational/algebraic
Feynman rules, particle content
Integral & tensor reduction

Transcendental
Scattering kinematics
Feynman integrals

$$\mathcal{A} = \sum_{\mathbf{i}} \boxed{r_{\mathbf{i}}(\mathbf{s},\epsilon)}\, \boxed{\mathbf{g}^{\mathbf{i}}(\mathbf{s},\epsilon)}$$

Compact analytic form

Function basis

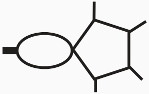
- ~~Redundancy~~
- ~~Dim. reg. artifacts~~
- ✓ Analytic cancellation of IR divergences
- ✓ Enable modern finite-field based methods
- ✓ Amplitudeology friendly

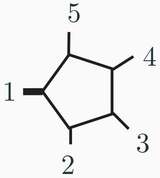
Goal: fully differential cross sections

fast and stable evaluation over whole physical phase space

Setup of the computation

Integral topologies & kinematics

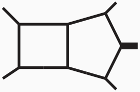




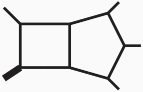
Variables

$p_1^2,\ s_{12},\ s_{23},\ s_{34},\ s_{45},\ s_{15}$

PBmzz



PBzmz



PBzzz

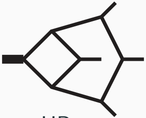
Gram determinants:

$$\Delta_5 = 16\ G(p_1, p_2, p_3, p_4)$$

$$\Delta_3^{(1)} = -4\ G(p_1, p_2 + p_3)$$

$$\Delta_3^{(2)} = -4\ G(p_1, p_2 + p_4)$$

$$\Delta_3^{(3)} = -4\ G(p_1, p_3 + p_4)$$



HBzzz

HBzmz

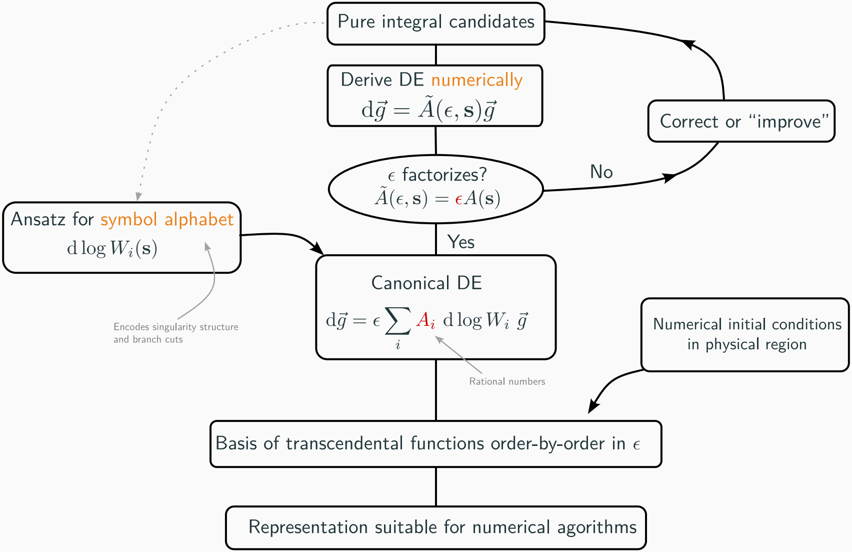
HBmzz

DPmz

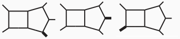
DPzz

Roadmap

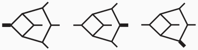
Hope: the result is pure functions of uniform transcendentality [Henn '13]



Canonical DE

Planar 

[Abreu, Ita, Moriello, Page, Tschernow, Zeng '20]

Hexa-box 

[Abreu, Ita, Page, Tschernow '21]

Results through generalized polylogarithms

[Papadopoulos, Tommasini, Wever '15]
[Canko, Papadopoulos, Syrrakos '20] [Syrrakos '20]
[Kardos, Papadopoulos, Smirnov, Syrrakos, Wever '22]

Function basis (planar)

[Badger, Hartanto, Zoia '21] color-ordered

[Chicherin, VS, Zoia '21]

Semi-numerical DE solution

`DiffExp` [Moriello '19] [Hidding '20]
`AMFLow` [Liu, Ma, Wang '17] [Liu, Ma '21]
`SeaSyde` [Armadillo, Bonciani, Devoto, Rana, Vicini '22]
[Hidding, Usovitsch '22]

- Initial (boundary) conditions
- Cross checks
- Numerical data for analytic work

Differential equations

Finding pure MIs

Great progress,
(semi-)automated in many cases

Canonica [Meyer '18] Epsilon [Prausa '17]
Fuchsia [Gituliar, Magerya '17]
DlogBasis [Henn, Mistlberger, V. Smirnov, Wasser '20]
Initial [Dlapa, Henn, Yan '20] [Dlapa, Henn, Wagner '22]
[Dlapa, X. Li, Y. Zhang '21]

Does not quite cut it for our case...

Rely on educated guessing, main guides:

- Integrands with constant leading singularities
  [Arkani-Hamed, Bourjaily, Cachazo, Trnka '10]
- Generalized (unitarity) cuts
- Recycle known simpler results
- Get inspired by symmetries ($\mathcal{N} = 4$ sYM, conformal)
- Guesses are easy to check numerically

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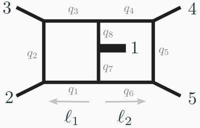
Example: 9 pure integrals (179 in whole family)



$$\rho_i := q_i^2$$

$$\mu_{ij} := \ell_i^{(D-4)} \ell_j^{(D-4)}$$

$$\sqrt{\Delta_5}\sqrt{\Delta_3^{(1)}}\;\mu_{12} \quad , \qquad \frac{\sqrt{\Delta_3^{(1)}}}{\sqrt{\Delta_5}}\,\frac{\partial \mathcal{B}}{\partial \rho_8}$$

$$\sqrt{\Delta_5}\left(2s_{23}\;\mu_{22} + (s_{23}+s_{45}-p_1^2)\;\mu_{12}\right), \quad \begin{matrix} 2\leftrightarrow 5 \\ 3\leftrightarrow 4 \end{matrix}$$

$$\frac{1}{\sqrt{\Delta_5}}\left(2s_{23}\frac{\partial \mathcal{B}}{\partial \rho_3} - (p_1^2+s_{23}-s_{45})\left(\frac{\partial \mathcal{B}}{\partial \rho_7} + \Delta_5\;\mu_{12}\right)\right), \quad \begin{matrix} 2\leftrightarrow 5 \\ 3\leftrightarrow 4 \end{matrix}$$

$$\mathrm{tr}\left(\not{p}_2\not{q}_2\not{q}_3\not{q}_8\not{q}_7\not{q}_6\not{q}_5\not{p}_4\right) \quad , \quad \begin{matrix} 2\leftrightarrow 5 \\ 3\leftrightarrow 4 \end{matrix}$$

$$\sqrt{\Delta_3^{(1)}}\;\mathrm{tr}\left(\not{q}_5\not{p}_5\not{p}_2\not{q}_2\not{q}_3\not{q}_4\right)$$

The alphabet

Hexa-box alphabet [Abreu, Ita, Page, Tschernow '21], no new letters from DP

$$A = \sum_{i=1}^{204} \mathrm{d}\log W_i \; A_i$$

c.f. 31 letters,
1 square root
for massless!

Degree	Letters
linear	27
quadratic	66
qubic	24

$\left.\right\}$ 117 rational

Root	Degree	Perms./ Letters
$\sqrt{\Delta_5}$	4	1
$\sqrt{\Delta_3^{(i)}}$	2	3
$\sqrt{\Sigma_5^{(i)}}$	4	6

$\left.\right\}$ 10 square roots

Algebraic letters, odd under root sign flip

e.g. $W_{118} = \dfrac{p_1^2 - s_{23} + s_{45} + \sqrt{\Delta_3^{(1)}}}{p_1^2 - s_{23} + s_{45} - \sqrt{\Delta_3^{(1)}}}$

Roots	Letters
$\sqrt{\Delta_5}$	32
$\sqrt{\Delta_3^{(i)}}$	12
$\sqrt{\Sigma_5^{(i)}}$	24
$\sqrt{\Delta_5}, \sqrt{\Delta_3^{(i)}}$	3
$\sqrt{\Delta_5}, \sqrt{\Sigma_5^{(i)}}$	6

68 one square root (first three rows)

9 two square roots (last two rows)

$$\Sigma_5^{(1)} = (s_{12}s_{15} - s_{12}s_{23} - s_{15}s_{45} + s_{34}s_{45} + s_{23}s_{34})^2 - 4 s_{23}s_{34}s_{45}(s_{34} - s_{12} - s_{15})$$

Function basis

Basis construction

[Chicherin, VS, Zoia '21] (see also [Chicherin, VS '20] [Badger, Hartanto, Zoia '21])

Pure Feynman integrals

Canonical DE

$$\mathrm{d}\vec{g} = \epsilon\, A\, \vec{g}$$

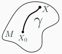
$$A = \sum_i \mathrm{d}\log W_i(\mathbf{s})\; A_i$$

Initial conditions
physics, limits, PSLQ

weight = length = ϵ order

Chen iterated integrals [Chen '77]

$$[W_1, \ldots, W_n]_\gamma = \int_0^1 \mathrm{d}\log W_n(t_n)\ \ldots\ \int_0^{t_2} \mathrm{d}\log W_n(t_1)$$



Vector subspace, weight-graded

$$\mathbf{G} = \bigoplus_w \mathbf{G}^{(w)}$$

+ shuffle product

$$\mathbf{G}^{w_1} \times \mathbf{G}^{(w_2)} \mapsto \mathbf{G}^{(w_1+w_2)}$$

$$[W_1, \ldots, W_r]_\gamma [W_{r+1}, \ldots, W_n]_\gamma = \sum_{\mathbf{i}\in\text{shuffles}} [W_{i_1}, \ldots, W_{i_n}]_\gamma$$

Basis in $\mathbf{G}^{(w)}$ $\mod$ products

- ✓ complete
- ✓ non-redundant

Initial values

Function basis construction requires algebraic relations between initial values $\vec{g}(X_0)$

Previously

Serious bottleneck [Chicherin, VS, Zoia '21]

- Rely on MPL expressions [Canko, Papadopoulos, Syrrakos '20] to calculate $\vec{g}(X_0)$ to $\mathcal{O}(3000)$ digits
- Pushing the limits of most advanced PSLQ algorithms [Bailey, Broadhurst '01] [Bailey, Borwein, Kimberley, Ladd '17] (~ 400 constants with ~ 2000 digits)

New approach

1. Construct symbol-level function basis, i.e. setting $\vec{g}^{(w)}(X_0)$ with $w > 0$
2. Use their definitions trough MI components to upgrade to iterated integrals
3. Presume that MI expression are polynomials in basis functions and $\langle \mathrm{i}\,\pi \rangle \oplus \langle \pi^2 \rangle \oplus \langle \mathrm{i}\,\pi^3, \zeta_3 \rangle \oplus \langle \pi^4, \mathrm{i}\,\pi\zeta_3 \rangle \implies$ derive constraints
4. Match to numerical evaluation from `AMFlow` to validate and fix remaining rational numbers

✓ Precision for $\vec{g}(X_0)$ need not exceed final target precision

✓ PSLQ fit just one rational number at a time

Non-analyticity within physical region

Consider iterated integral along $\gamma : t \in [0,1] \to \mathcal{P}_{\mathsf{phys}}$, and $W_i(t^\star) = 0$,

$$\int_\gamma \mathrm{d}\log W_i \;\; h = \int_\gamma \frac{\mathrm{d}W_i}{W_i} h \xrightarrow{t \to t^\star} \frac{W'(t)}{t - t^\star} \left(h^{(0)} + h^{(1)}(t - t^\star) + \mathcal{O}\Big((t - t^\star)^2 \Big) \right)$$

Planar scattering

Only linear or quadratic letters vanish in $\mathcal{P}_{\mathsf{phys}}$, poles always canceled, i.e. $h^{(0)} = 0$

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Planar scattering

Only linear or quadratic letters vanish in $\mathcal{P}_{\text{phys}}$, poles always canceled, i.e. $h^{(0)} = 0$

New feature of nonplanar scattering

Square roots of quartic polynomials $\sqrt{\Sigma_5}$ can vanish in $\mathcal{P}_{\text{phys}}$ $\implies$ new types of divergences

1. Integrable square root: $\mathrm{d}\log \frac{a+\sqrt{\Sigma_5}}{a-\sqrt{\Sigma_5}} \xrightarrow{\Sigma_5 \to 0} \frac{\mathrm{d}\Sigma_5}{a\sqrt{\Sigma_5}} \xrightarrow{t \to t^\star} \frac{C}{\sqrt{t - t^\star}} + \ldots$
2. Uncompensated poles: $\mathrm{d}\log \sqrt{\Sigma_5} \xrightarrow{\Sigma_5 \to 0} \frac{\mathrm{d}\Sigma_5}{2\Sigma_5} \xrightarrow{t \to t^\star} \frac{C}{t - t^\star} + \ldots \implies$ log divergence!

- Choose basis functions to localize non-analytic behavior
- Expectation: functions with type 2 divergence cancel out in physical results
- Numerical evaluation more challenging

Basis structure

Weight	P ∪ PB	+HB	+DP	Total
1	11	0	0	11
2	25	10	0	35
3	145	72	0	217
4	675	305	48	1028
#MIs	1361	542	345	2248

Permutation closed
$\sigma\left(f_i^{(w)}\right) \to \sum_j c_{ij} f_j^{(w)} + \ldots$

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Permutation closed
$\sigma\left(f_i^{(w)}\right) \to \sum_j c_{ij} f_j^{(w)} + \dots$

$\mathrm{d}\log\sqrt{\Delta_5}$ likely cancel in finite remainders
[Chicherin, Henn, Papathanasiou '20]

$\mathrm{d}\log\sqrt{\Sigma_5^{(i)}}$ also cancel?

7 functions diverge at $\Sigma_5^{(3)} = 0$

Weight 3

$$1,\dots,145,146,\dots,207,\overbrace{208,\dots,213}^{\mathrm{d}\log\sqrt{\Sigma_5^{(i)}}},\underbrace{214,\dots,217}_{\mathrm{d}\log\sqrt{\Delta_5}}$$

Weight 4

$$1,\dots,112,\underbrace{113,}_{\mathrm{d}\log\sqrt{\Delta_5}}114,\dots,664,\underbrace{665,\dots,675,}_{\mathrm{d}\log\sqrt{\Delta_5}}$$

$$676,\dots,920,\overbrace{921,\dots,945,946,\dots,950}^{\mathrm{d}\log\sqrt{\Sigma_5^{(i)}}},\underbrace{951,\dots,974,\overbrace{975,\dots,980}^{\mathrm{d}\log\sqrt{\Sigma_5^{(i)}}}}_{\mathrm{d}\log\sqrt{\Delta_5}},981,\dots,1028$$

Numerical evaluation

Weights 1 and 2

Well-defined combinations of $\log$, Li_2 functions

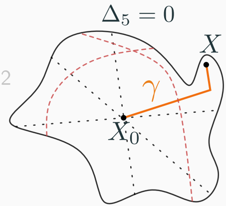
$$f_{13}^{(2)} = \mathrm{Li}_2\left(1 - \underbrace{\frac{s_{15} - s_{23} - s_{34}}{s_{15}}}_{>0}\right)$$

Weights 3 and 4

$$f_i^{(3)} = \int_0^1 \sum_j \frac{\partial \log W_j(t)}{\partial t} \boxed{h_{i,j}^{(2)}}\, \mathrm{d}t$$

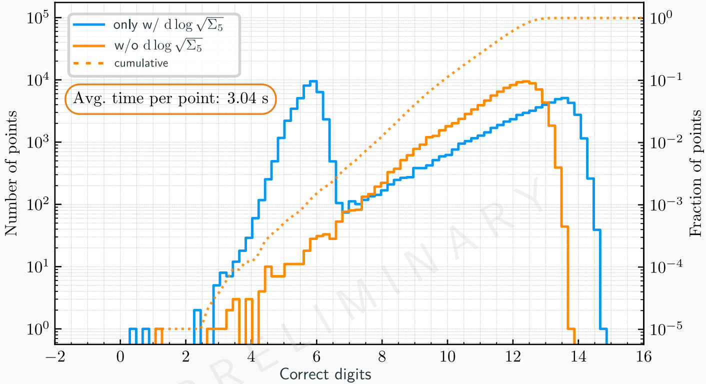
$$f_i^{(4)} = \int_0^1 \sum_{j,k} \frac{\partial \log W_j(t)}{\partial t} \log \frac{W_k(1)}{W_k(t)} \boxed{h_{i,jk}^{(2)}}\, \mathrm{d}t$$

weight 2



- Numerical one-fold integration [Caron-Huot, Henn '14] of analytic integrands $\implies$ exponential convergence [Takahasi, Mori '73]
- No crossing of physical thresholds $\implies$ no analytic continuation needed
- Dedicated series expansions around (spurious) singularities

Numerical performance



- Sample over physical phase space for NLO Wjj production at the LHC
- Evaluate all functions on each point, plot the worst accuracy per point
- Timing for all functions on one CPU
- Worse stability for functions with $\mathrm{d}\log\sqrt{\Sigma_5}$ (hopefully irrelevant!)

Very promising performance!

Soon available in `PentagonFunctions++`
https://gitlab.com/pentagon-functions/PentagonFunctions-cpp

Conclusions

Conclusions & Outlook

Conclusions

- A complete set of special functions describing double virtual corrections for two-loop five-point one-mass processes is available.
- Enables application of modern techniques for analytic calculation of two-loop amplitudes.
- Ready for cross section calculations.

Outlook

- Paper to appear soon.
- Possibility of calculating NNLO QCD corrections for a large class of $2 \to 3$ processes is open. Hopefully exciting phenomenology in near future!
- Important contribution towards N^{3}LO QCD corrections for Vj, $V\gamma$ production.

Acknowledgments

This work has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme, *Novel structures in scattering amplitudes* (grant agreement No. 725110).

This work has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme grant agreement 101019620 (ERC Advanced Grant TOPUP).

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Backup

Numerical performance: conservative precision rescue