On phase-space integrals with Heaviside functions

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Based on: 2111.13594 and 2204.094559

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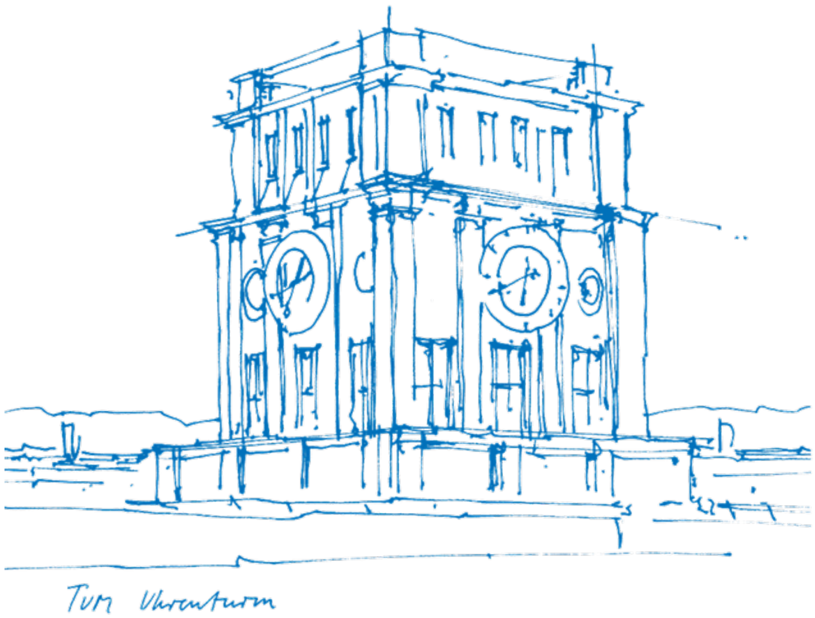
CERN QCD seminar

10th of June 2022

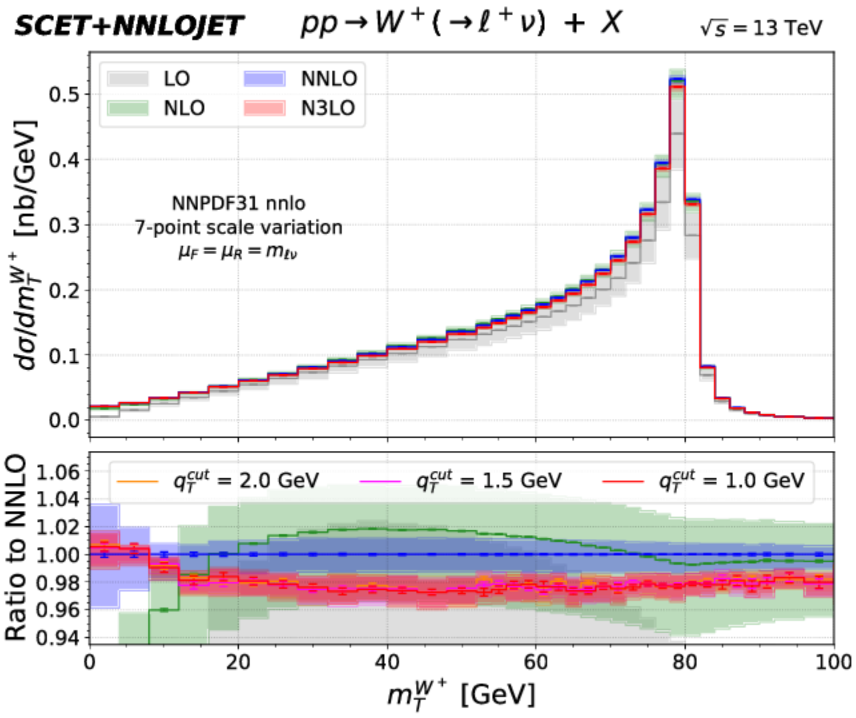


Motivation

Fully-differential description of LHC cross sections have recently reached N3LO in peturbative QCD.

⇒ Higgs production [Chen et al. '21]
⇒ Drell-Yan [Chen et al. '22]

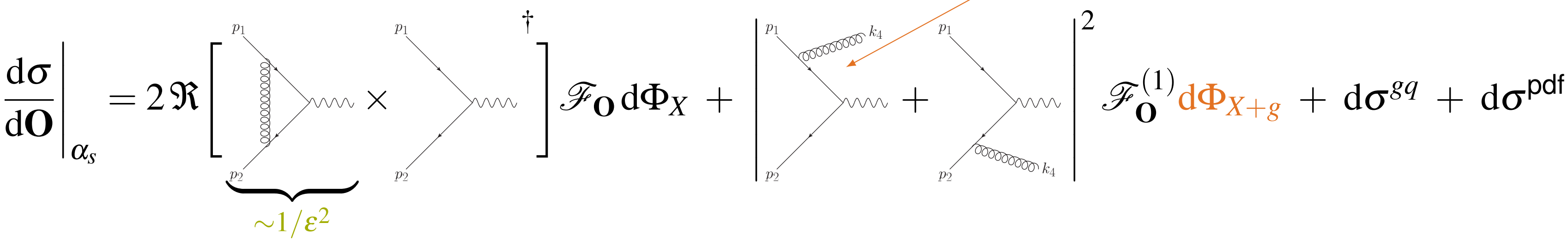
These computations are vital for precise studies at the LHC.



[Chen et al. '22]

Infrared divergences

$$\frac{1}{p_1 \cdot k_4} \sim \frac{1}{E_1 E_4 (1 - \cos\theta_{14})}$$



- soft and collinear divergences arise from
  - loop integrals in virtual corrections
  - phase-space integration over on-shell momenta of final-state partons in real-emission corrections

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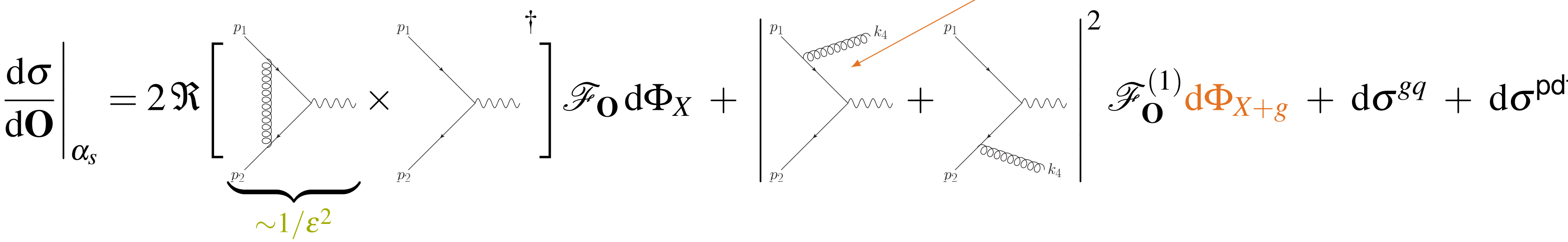

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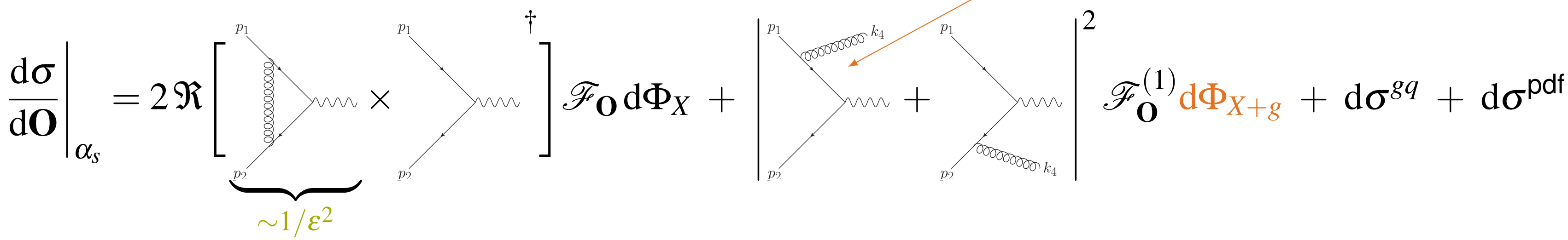

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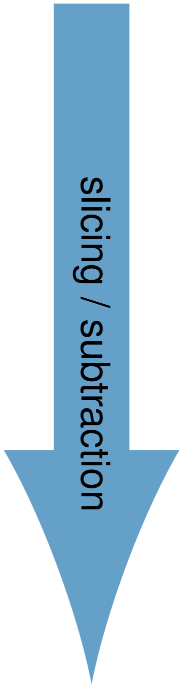
- need to be regulated, extracted and cancelled
- only then can the limit $\varepsilon \to 0$ be taken prior to numerical simulation

$\Rightarrow$ *slicing* and *subtraction* schemes
are used to define contributions that are individually finite

$$\left. d\sigma \right|_{\alpha_s} = d\sigma_{X+1} + d\sigma_X$$



$$\mathscr{I} = \int_0^1 \frac{\mathrm{d}x}{x^{1-n\varepsilon}} f(x)$$

Subtraction schemes

$\Rightarrow$ subtract and add back a term that approximates the divergent behavior

$$\mathscr{I} = \frac{f(0)}{n\varepsilon} + \int_0^1 \mathrm{d}x \frac{f(x) - f(0)}{x} + \mathscr{O}(\varepsilon)$$

☺ local regularisation of divergences

☺ numerically stable

☹ complex construction, tedious to implement

for example

- CoLoRFull [Somogyi et al. '05]
- Antenna [Gehrmann-De Ridder et al. '05]
- STRIPPER [Czakon '10]
- Nested soft-collinear subtraction [Caola et al. '17]
- Local Analytic Sector Subtraction [Magnea et al. '18]

Slicing schemes

$\Rightarrow$ slice phase space into divergent and finite region

$$\mathscr{I} = \left[\frac{1}{n\varepsilon} + \ln(\delta_{\text{cut}})\right] f(0) + \int_{\delta_{\text{cut}}}^1 \frac{\mathrm{d}x}{x} f(x) + \mathscr{O}(\delta, \varepsilon)$$

☹ small cut-off leads to large numerical cancellations and requires precise control of f

☺ achievable at N3LO

for example

- q_T-slicing [Catani et al. ']
- N-jettiness slicing [Boughezal et al. '15]

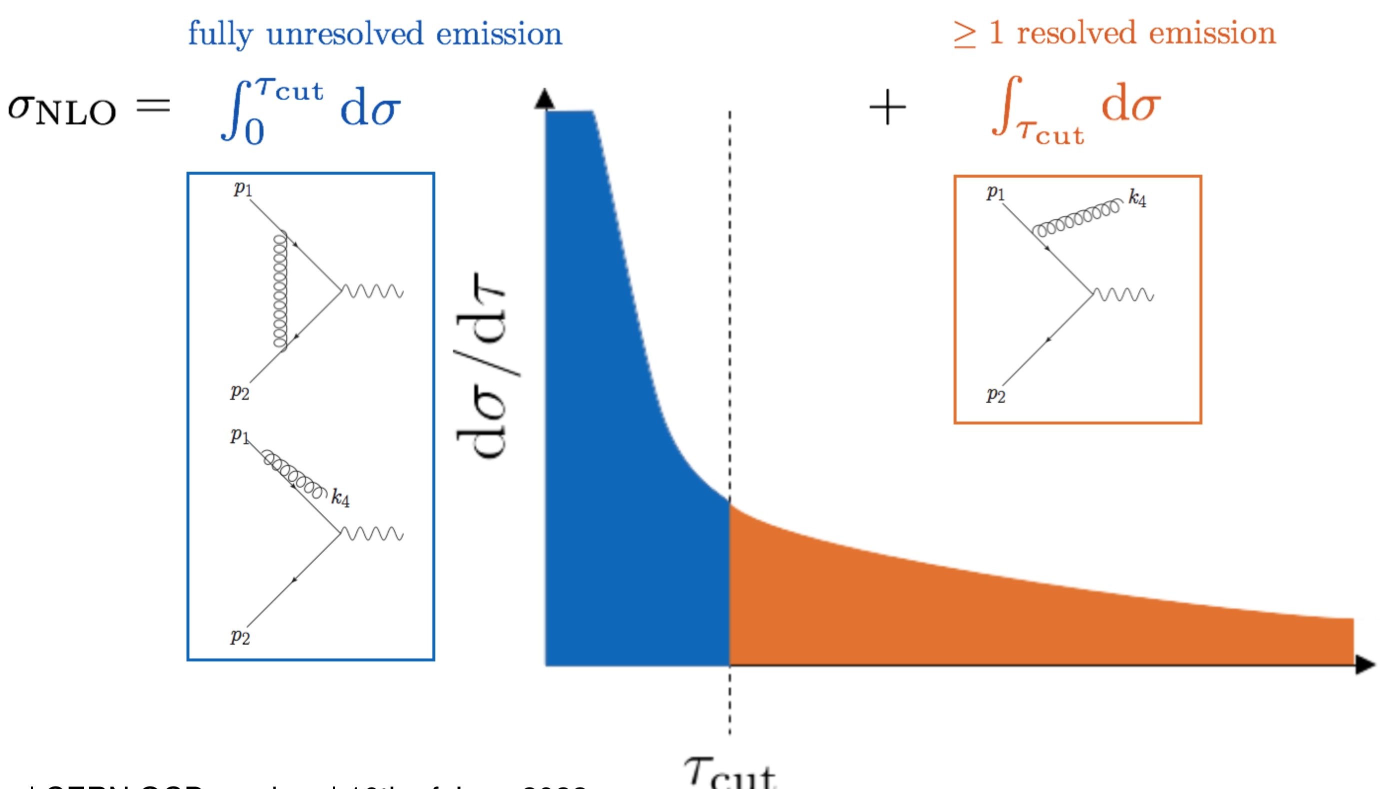
Other schemes

- projection-to-Born [Cacciari et al. '15]
- ...

Zero-jettiness slicing

$$\tau = \sum_j \min_{q \in \{n, \bar{n}\}} \{q \cdot k_j\}$$

- $q \cdot k_j = k_j^0 (1 \mp \cos\theta_j)$ for $q = n, \bar{n}$
- τ vanishes if *all* emitted partons become unresolved (soft / collinear to incoming partons)



Zero-jettiness factorisation

in SCET, the following factorisation formula can be derived [Stewart et al. '09]

$$d\sigma = B \otimes B \otimes S \otimes H \otimes d\sigma_{\text{Born}} + \mathcal{O}(\tau)$$

where the various ingredients are

Beam function

- collinear emission off incoming partons
- process-independent
- known through N3LO [Ebert et al. '20] (partial results in [Behring et al '19])

Soft function

- soft, non-collinear emission
- process-independent
- known through N2LO [Kelley et al. '11] [Monni et al. '11]

Hard function

- virtual corrections
- process-dependent

⇒ factorisation allows for explicit cancellation of IR poles

⇒ computation of $\mathcal{O}(\varepsilon^0)$-contribution to cross section in $\tau \leq \tau_{\text{cut}}$ region

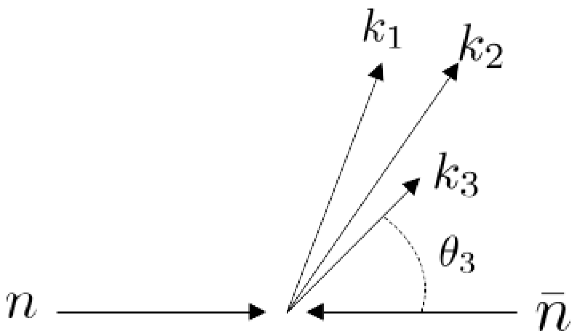
Three-gluon-emission contribution

$$S^{ggg}\Big|_{C_a C_A^2} = 2\int \mathrm{d}\Phi^{nnn}_{\theta\theta\theta}\,|\boldsymbol{J}(k_1,k_2,k_3)|^2 + 6\int \mathrm{d}\Phi^{nn\bar{n}}_{\theta\theta\theta}\,|\boldsymbol{J}(k_1,k_2,k_3)|^2$$

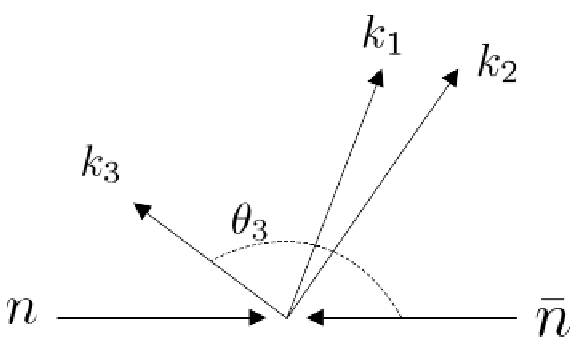
- triple-soft eikonal function $|\boldsymbol{J}|^2$ [Catani et al. '19] contains $\{q\cdot k_i\ ,\ q\cdot k_{ij}\ ,\ q\cdot k_{123}\ ,\ k_i\cdot k_j\ ,\ k^2_{123}\}$, $q=n,\bar{n}$

two distinguishable configurations

1) $\mathrm{d}\Phi^{nnn}_{\theta\theta\theta} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{123}\cdot n]\prod_{i=1}^{3}\theta[k_i\cdot\bar{n} - k_i\cdot n]$



2) $\mathrm{d}\Phi^{nn\bar{n}}_{\theta\theta\theta} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}\cdot n - k_3\cdot\bar{n}]\prod_{i=1}^{2}\theta[k_i\cdot\bar{n} - k_i\cdot n]$
$f_3(k_3\cdot n - k_3\cdot\bar{n})$



where $\widetilde{[\mathrm{d}k]} = \prod_{i=1}^{3}\mathrm{d}^{\mathrm{d}}k_i\,\delta\left[k_i^2\right]$

Integration-by-parts relations

$$0 = \int \prod_{l=1}^{L} \mathrm{d}^{\mathrm{d}} k_l \frac{\partial}{\partial k_i^{\mu}} \left[q^{\mu} \prod_{n=1}^{N} D_n^{-\alpha_n} \right], \quad q \in \{k_1, .., k_L, p_1, \dots, p_E\}, \quad \alpha_n \in \mathbb{Z},$$ [Chetyrkin,Tkachov '81]

- set of linearly independent propagators $\{D_n\}$ defines a topology
- IBP relations relate different integrals within a given topology

$\Rightarrow$ reduce given problem to minimal set of "master integrals" [AIR,FIRE,Reduze,LiteRed,Kira,..]

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1. δ-function: reverse unitarity [Anastasiou,Melnikov '02]

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2. θ-function:

Parametric representation

[Caola et al. '18] [Angeles-Martinez et al. '18]
[Bizon et al. '20] [Baranowski '20]

$$\theta[b-a] = \int_0^1 \mathrm{d}z\, \delta[z \cdot b - a]\, b \quad a, b > 0$$

[Chen '20]

$$(2\pi \mathrm{i})\, \theta[D] = \int_{-\infty}^{+\infty} \mathrm{d}x \frac{e^{\mathrm{i}xD}}{x + \mathrm{i}0^+}$$

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Extended notion of IBPs

[Baranowski et al. '21]

$$\frac{\partial}{\partial k}\{g(k)\,\theta[f(k)]\} = \overbrace{g'\theta}^{\text{hom}} + \overbrace{g\,\delta f'}^{\text{inhom}}$$

☹ proliferation of topologies
☹ manual generation of IBP identities
☺ no auxiliary integration

(see also [Luo et al. '19])

Integration-by-parts - example @ NNLO

$$\int \frac{\mathrm{d}\Phi_{\theta\theta}^{nn}}{(k_2 n)\,(k_1 \bar{n})\,(k_{12}\bar{n})}\,, \qquad \mathrm{d}\Phi_{\theta\theta}^{nn} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}n]\,\theta[k_1\bar{n} - k_1 n]\,\theta[k_2\bar{n} - k_2 n]$$

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- IBP relation

$$0 = \int \mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\,\frac{\partial}{\partial k_1^\mu} k_1^\mu \ldots$$

- yields

$$0 = \left[(d-4) - \mathbf{7}^+\mathbf{6}^- - \mathbf{3}^+\mathbf{5}^- + \mathbf{3}^+\right]\mathcal{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$
$$+ \int \mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2 \left[k_1^\mu(\bar{n} - n)_\mu\,\delta(k_1\bar{n} - k_1 n)\right] \times \frac{\theta(k_2\bar{n} - k_2 n)}{(k_1^2)(k_2^2)(\tau - k_{12}n)(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}$$

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- “conventional” IBP relation

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$$+ n\bar{n}\int \mathrm{d}^{\mathrm{d}}k_1\,\mathrm{d}^{\mathrm{d}}k_2\,\frac{1}{(k_1\bar{n} - k_1 n)} \times \frac{\theta(k_2\bar{n} - k_2 n)}{(k_1^2)(k_2^2)(\tau - k_{12}n)(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}$$

- additional inhomogeneous term that requires partial fraction decomposition

Integration-by-parts - example @ NNLO

$$\underbrace{\mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1} = \int \frac{\mathrm{d}\Phi^{nn}_{\theta\theta}}{(k_2 n)\,(k_1\bar{n})\,(k_{12}\bar{n})}}_{\text{topology with } 2\theta\text{-functions}}, \quad \mathrm{d}\Phi^{nn}_{f_1 f_2} = \widetilde{[\mathrm{d}k]}\,\delta[\tau - k_{12}n]\,f_1[k_1\bar{n} - k_1 n]\,f_2[k_2\bar{n} - k_2 n]$$

$$0 = \left[(d-4) - \mathbf{7}^+\mathbf{6}^- - \mathbf{3}^+\mathbf{5}^- + \mathbf{3}^+\right] \mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1}$$

$$0 = n\bar{n}\left[\mathbf{7}^+ - \mathbf{6}^+ + \mathbf{1}^+\mathbf{6}^- - \mathbf{1}^+\right] \mathscr{T}^{\text{ex}}_{1,1,1,0,1,1,1} + \underbrace{\int \frac{\mathrm{d}\Phi^{nn}_{\delta\theta}}{(k_{12}\bar{n})}\left[\frac{1}{(k_1 n)} + \frac{1}{(k_2 n)}\right]}_{\text{topology with } 1\theta\text{-function}}$$

- because of inhomogeneous terms ($\sim \partial_k\theta$), we need new type of topologies:
  - come from all possible ways of replacing $\theta \to \delta$
  - fall into a hierarchical structure: $\{\theta\theta\} \xrightarrow{\partial_k\theta} \{\delta\theta, \theta\delta\} \xrightarrow{\partial_k\theta} \{\delta\delta\}$
  - relations for $\delta\delta$ topologies have no inhomogeneous terms

Computational setup

Mathematica

- derive symmetry relations

$$\int \frac{\mathrm{d}\Phi^{nn}_{\theta\delta}}{(k_1 k_2)\,(k_2 \bar{n})} \overset{k_1 \leftrightarrow k_2}{=} \int \frac{\mathrm{d}\Phi^{nn}_{\delta\theta}}{(k_1 k_2)\,(k_1 \bar{n})}$$

- define list of “seed integrals” for independent sectors
- create IBP relations for seed integrals
- define ordering of integrals that appear in relations
- parse relations to Kira’s “user-defined system of equations” syntax

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Kira⊕FireFly[Maierhöfer et al. ’17][Klappert et al. ’19]

- find MIs
  - consider all topologies $(\theta\theta,\ \delta\theta,\ \theta\delta, \delta\delta)$
  - disregard θ-functions
  - run reduction using “vanilla” Kira
  - corresponds to setting inhomogeneous terms in IBP to zero
- solve “user-defined system of equations”

$$S^{gg}\Big|_{C_a C_A^2} = c_1 I^{nn}_{1,\delta\delta} + \sum_{i=2}^{4} c_i I^{nn}_{i,\delta\theta} + \sum_{i=5}^{7} c_i I^{n\bar{n}}_{i,\theta\theta}$$

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Reduction ✓

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Reduction ✓ Master integrals X

Computational setup

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$$I^{nn}_{4,\delta\theta} = \int \frac{\mathrm{d}\Phi^{nn}_{\delta\theta}}{(k_1 k_2)(k_2 n)(k_{12}\bar{n})}$$

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- regulate both endpoint singularities

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = -2\mathcal{N}_\varepsilon^2 \frac{\Gamma^2(1-2\varepsilon)}{\varepsilon\,\Gamma(1-4\varepsilon)}\left\{\int_0^1 d\xi_2 \left[\frac{F(\xi_2)}{\xi_2^{1+\varepsilon}(1-\xi_2)^{1+2\varepsilon}} - \frac{F(0)}{\xi_2^{1+\varepsilon}} - \frac{F(1)}{(1-\xi_2)^{1+2\varepsilon}}\right] + \frac{F(0)}{-\varepsilon} + \frac{F(1)}{-2\varepsilon}\right\}$$

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- expand with `HypExp` [Huber et al. '05] & integrate with `HyperInt` [Panzer '14]

$$I^{nn}_{4,\delta\theta}\Big|_{\tau\to 1} = \frac{2}{\varepsilon^2} + \frac{\pi^2}{3} - \frac{17\pi^4\varepsilon^2}{90} + \varepsilon^3\left[-6\pi^2\zeta_3 - 26\zeta_5\right] - \varepsilon^4\left[\frac{193\pi^6}{810} + 64\zeta_3^2\right] + \mathcal{O}(\varepsilon^5)$$

Complications for three-gluon emission

Note: $nn\bar{n}$ configuration still work in progress

General complexity

- more "level": $\theta\theta\theta$, $\delta\theta\theta$, $\theta\delta\theta$, $\theta\theta\delta$, $\ldots$
- $\mathcal{O}(400)$ topologies
- integral relations

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Rapidity divergences

- appear for certrain observables (e.g. q_T) in the limit of large rapidity and only cancel in *soft+collinear*
- obey a rapidity renormalisation-group formalism [Chiu et al.]
- in case of zero-jettiness, we find individual integrals that are not regulated dimensionally

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Master integrals

- more complicated master integrals

$\Rightarrow$ resort to numerical calculation and PSLQ reconstruction

Integral relations at N3LO

$$\mathrm{d}\Phi^{nnn}_{f_1f_2f_3} \sim \delta[\tau - k_1\cdot n - k_2\cdot n - k_3\cdot n]\, f_1(k_1\cdot\bar{n} - k_1\cdot n) f_2(k_2\cdot\bar{n} - k_2\cdot n) f_3(k_3\cdot\bar{n} - k_3\cdot n)$$
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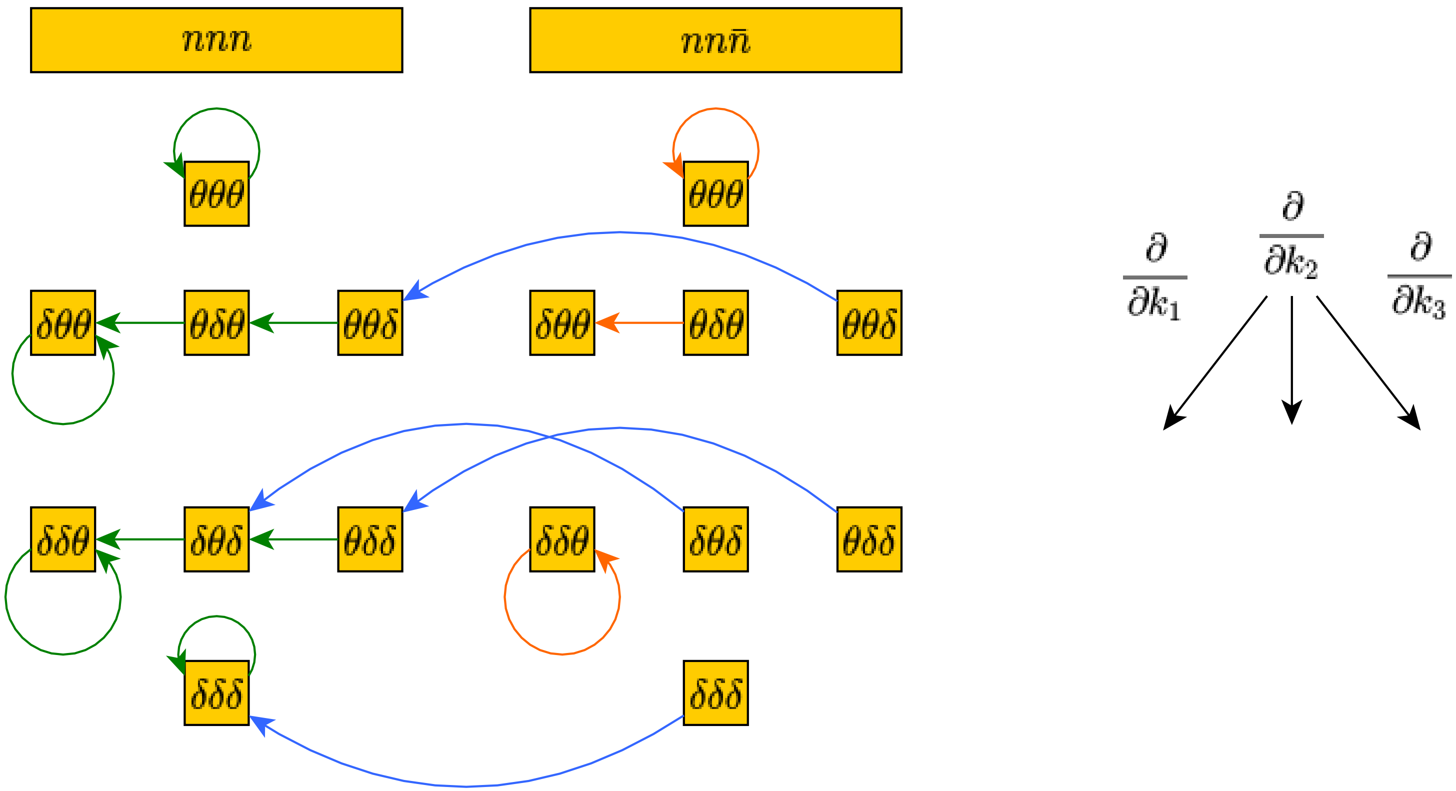

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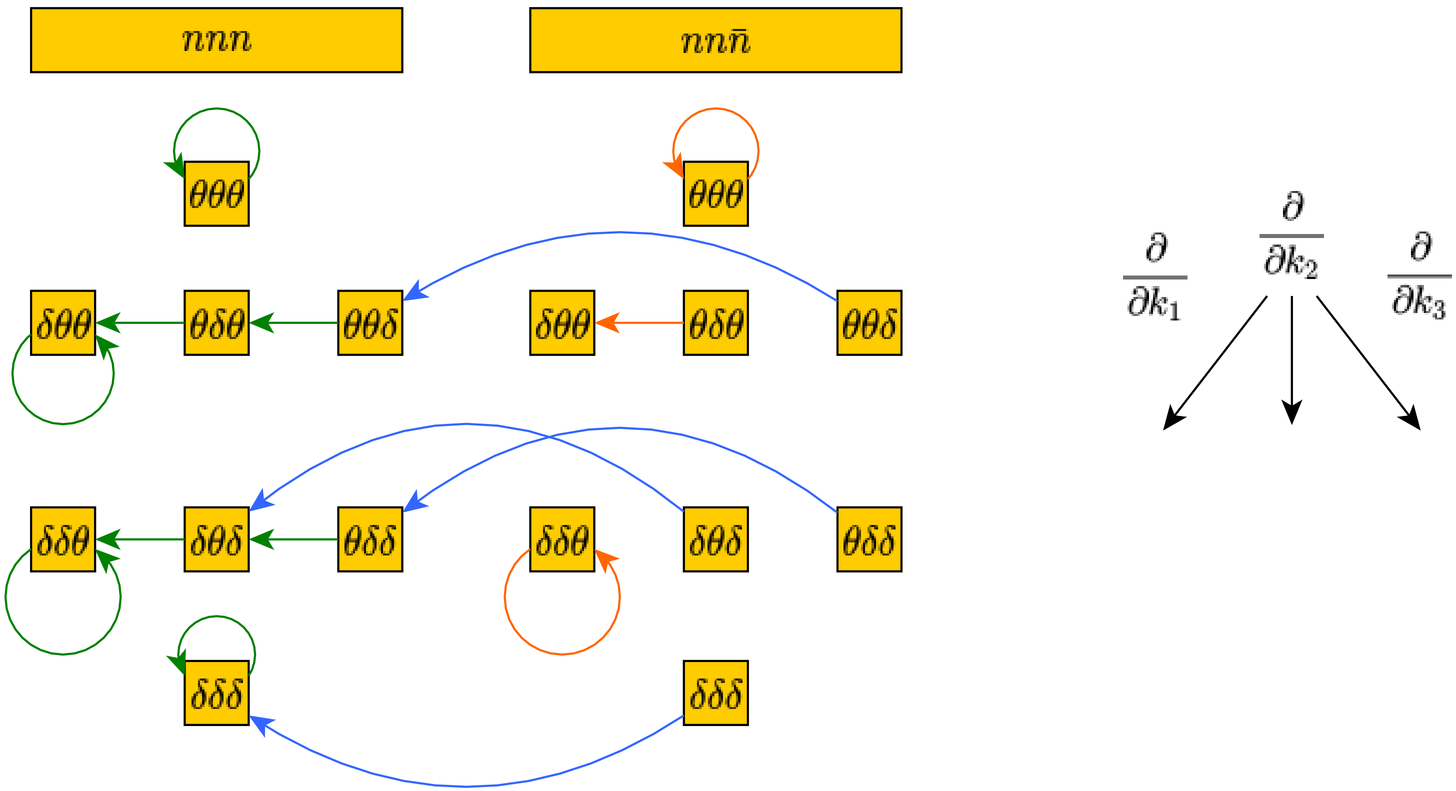

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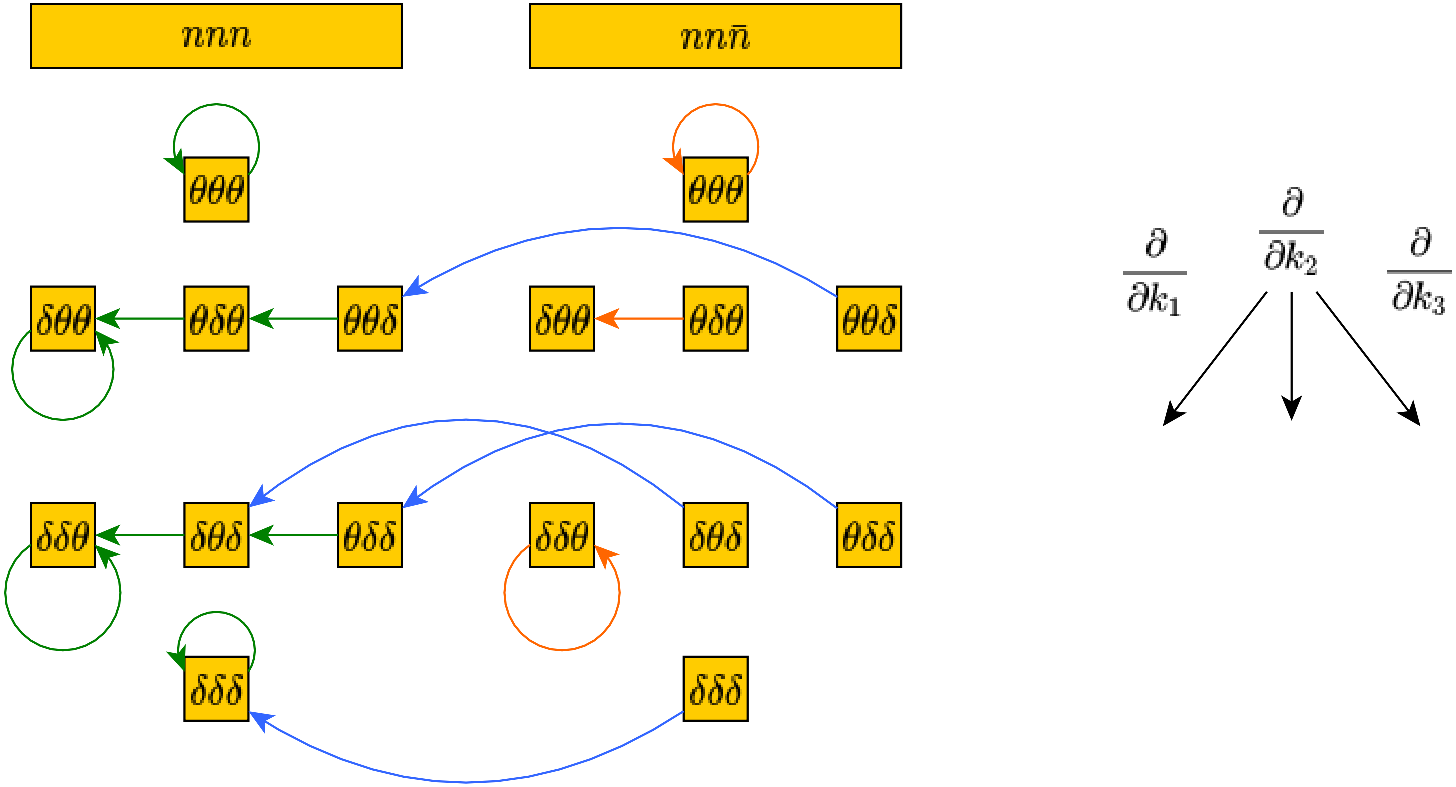

Integral relations at N3LO

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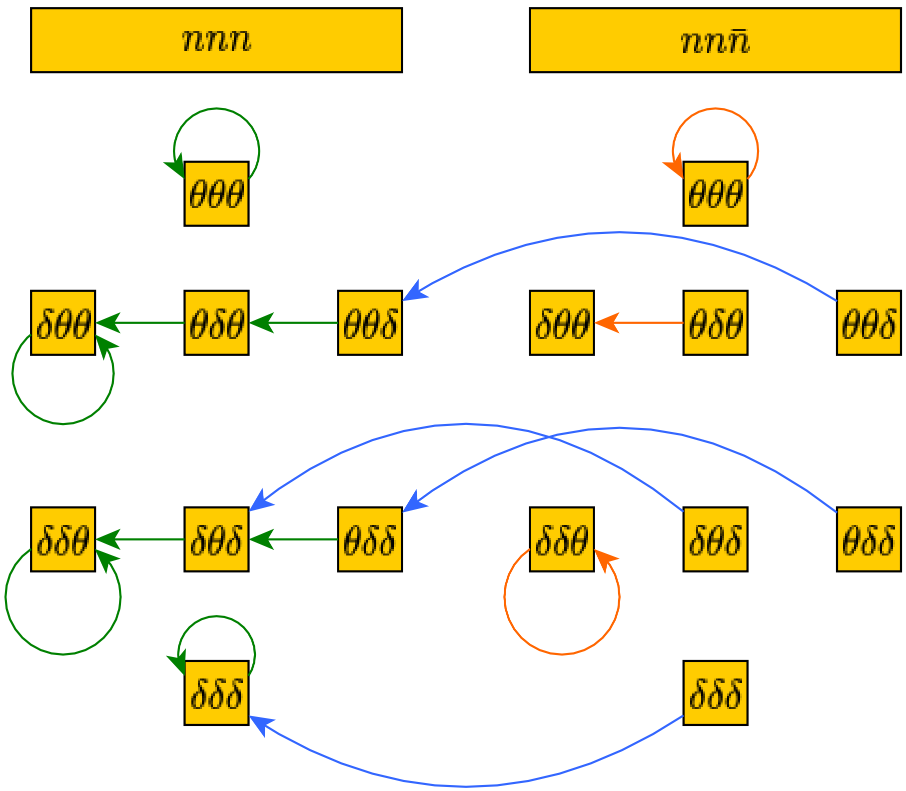

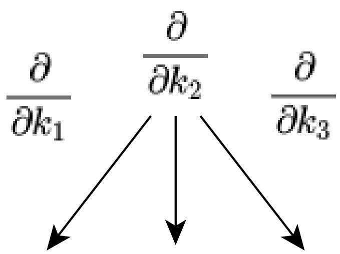

Rapidity divergences at N3LO

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- arise, when $k_i \cdot n \gg k_i \cdot \bar{n}$ or $k_i \cdot n \gg k_i \cdot \bar{n}$
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Introduce analytic regulator

$$\int \mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} \longrightarrow \lim_{\nu \to 0} \int \mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} (k_1 n)^{\nu} (k_2 n)^{\nu} (k_3 n)^{\nu}$$

$$\int \mathrm{d}\Phi^{nn\bar{n}}_{f_1 f_2 f_3} \longrightarrow \lim_{\nu \to 0} \int \mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} (k_1 n)^{\nu} (k_2 n)^{\nu} (k_3 \bar{n})^{\nu}$$

- IBP relations

$$\frac{\partial}{\partial k_1^{\mu}} : \quad 0 = I`\theta_1 + I(\bar{n} - n)^{\mu} \delta_1 \quad \longrightarrow \quad 0 = \left[ I`\theta_1 + I(\bar{n} - n)^{\mu} \delta_1 + \nu \, n^{\mu} \frac{I\theta_1}{k_1 \cdot n} \right] (\ldots)^{\nu}$$

- IBP reduction in limit $\nu \to 0$

$$S^{ggg} = \sum_k c_k I_k + \nu \sum_k \bar{c}_k \bar{I}_k(\nu \approx 0) + \mathcal{O}(\nu)$$

Rapidity divergences at N3LO

- arise, when $k_i \cdot n \gg k_i \cdot \bar{n}$ or $k_i \cdot n \gg k_i \cdot \bar{n}$
  - for $f_i = \delta$, $k_i \cdot n = k_i \cdot \bar{n}$
  - for $f_i = \theta$, $k_i \cdot n < k_i \cdot \bar{n}$ (except $nn\bar{n}$: $k_3 \cdot \bar{n} < k_i \cdot n$)
- break dimensional regularization and are **hard to detect**
  - original integrals and master integrals are *finite*
  - *only* numerical checks showed that reduction relations are wrong

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- IBP reduction in limit $\nu \to 0$

“$\nu = 0$”-reduction $\quad \sim 1/\nu$

$$S^{ggg} = \sum_k c_k I_k + \nu \sum_k \bar{c}_k \bar{I}_k (\nu \approx 0) + \mathcal{O}(\nu)$$

Rapidity divergences at N3LO

Example

- consider the following master integral

$$\bar{I}^{\nu} = \int \frac{\mathrm{d}\Phi^{nnn,\nu}_{\theta\delta\theta}}{(k_1 k_3)(k_1 n)(k_{12}\bar{n})(k_3\bar{n})}$$

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- use $\delta[\alpha_2 - \beta_2]$
- introduce fraction $\xi_i = \beta_i/\alpha_i\,, i = 1,3$

$$\bar{I}^{\nu} = \int \mathrm{d}\Omega \int_0^1 \prod \mathrm{d}\beta_i\, \mathrm{d}\xi_1\, \mathrm{d}\xi_3 \left(\frac{\beta_1^2\beta_3^2}{\xi_1\xi_3}\right)^{-\varepsilon} \frac{\beta_2^{-2\varepsilon}\,\delta[1-\beta_{123}]\,(\beta_1\beta_2\beta_3)^{\nu}}{(\xi_1+\xi_3-2\sqrt{\xi_1\xi_3}\boldsymbol{e}_1^{\perp}\boldsymbol{e}_2^{\perp})\beta_1(\beta_1+\xi_1\beta_2)\beta_3}$$

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- consider the region $\beta_1 \sim \xi_1 \sim \xi_3 \sim \lambda \to 0$

$$\bar{I}^{\nu} = \int \mathrm{d}\Omega \int \frac{\mathrm{d}\lambda}{\lambda}\left(\frac{\lambda^2}{\lambda\lambda}\right)^{-\varepsilon} \lambda^{\nu} \times \cdots \sim \frac{1}{\nu}$$

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$\Rightarrow$ $1/\nu$-pole gets multiplied with reduction coefficient $c \sim \nu$

Master integrals at N3LO

- fall into two categories: with and without the propagator $k_{123}^2 = 2\,[k_1k_2 + k_1k_3 + k_2k_3]$
  - without k_{123}^2: compute along the lines discussed for NNLO
$$\sim \int \frac{1}{(k_1k_2)(k_1k_3)\,\ldots}$$
  - with k_{123}^2: difficult to compute analytically

Numerical computation of MI (see also [Liu et al. '17])

- add parameter m^2 to propagator
$$I = \int \frac{\mathrm{d}\Phi_{f_1f_2f_3}^{nnn,\nu}}{\left[k_{123}^2\right]\,\ldots} \quad\longrightarrow\quad J = \int \frac{\mathrm{d}\Phi_{f_1f_2f_3}^{nnn,\nu}}{\left[k_{123}^2 + m^2\right]\,\ldots}$$
- using the IBP technology for θ-functions, derive differential equation
$$\partial_{m^2}\boldsymbol{J} = \widehat{K}\,\boldsymbol{J}$$
- compute boundary conditions in $m^2 \to \infty$, where the "bad" propagator simplifies
- solve differential equation numerically, compute the limit $m \to 0$, and recover I from Taylor-branch in
$$J = \sum c_{n_1n_2n_3}(m^2)^{n_1+n_2\varepsilon}\ln^{n_3}(m^2)$$
- result with 2000 digits, allows to reconstruct analytic expression with PSLQ

Master integrals at N3LO - boundary constants

Computing master integrals in configuration nnn
in the limit $m \to \infty$ leads to three different regions

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$$\int \frac{(\alpha_i\beta_i)^{-\varepsilon}\,\delta[\tau-\beta_{123}]\,\delta[\alpha_1-\beta_1]\,\delta[\alpha_2-\beta_2]\,\delta[\alpha_3-\beta_3]}{\left[m^2\ +\ \alpha_1\beta_2+\alpha_2\beta_1-\sqrt{\alpha_1\beta_1\alpha_2\beta_2}\cos\phi_{12}\ +\ 2\leftrightarrow 3\ +\ 1\leftrightarrow 3\right]} \sim \left[m^2\right]^{-1}$$

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- two δ-functions: α_1 unconstrained

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- no MIs with three θ-functions (zero δ functions)

Result for same-hemisphere contribution

$$
\begin{aligned}
S_3^{nnn} = & \frac{24}{\varepsilon^5} + \frac{308}{3\varepsilon^4} + \frac{1}{\varepsilon^3}\left(-12\pi^2 + \frac{3380}{9}\right) + \frac{1}{\varepsilon^2}\left(-1000\zeta_3 + \frac{440\pi^2}{9} + \frac{10048}{9}\right) \\
& + \frac{1}{\varepsilon}\left(-\frac{2377\pi^4}{45} + \frac{440\zeta_3}{3} + \frac{7192\pi^2}{27} + \frac{253252}{81}\right) \\
& + \left(-28064\zeta_5 + \frac{1972\zeta_3\pi^2}{3} - \frac{638\pi^4}{15} + 4224\,\mathrm{Li}_4\left(\frac{1}{2}\right) + 3696\zeta_3\ln(2) - 176\pi^2\ln^2(2) + 176\ln^4(2)\right. \\
& \left. + \frac{13208\zeta_3}{3} + \frac{78848\pi^2}{81} + 96\ln(2) + \frac{1925074}{243}\right) \\
& + \varepsilon\left(2304\ \zeta_{-5,-1} - 4464\zeta_5\ln(2) + 25784\zeta_3^2 - \frac{67351\pi^6}{567} - 6336 G_R(0,0,r_2,1,-1)\right. \\
& - 6336 G_R(0,0,1,r_2,-1) - 3168 G_R(0,0,1,r_2,r_4) - 6336 G_R(0,0,r_2,-1)\ln(2) + \frac{268895\zeta_5}{3} \\
& - 45056\,\mathrm{Li}_5\left(\frac{1}{2}\right) - 45056\,\mathrm{Li}_4\left(\frac{1}{2}\right)\ln(2) + 176\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\pi - 1056\zeta_3\,\mathrm{Li}_2\left(\frac{1}{4}\right) - 3982\zeta_3\pi^2 \\
& - 21824\zeta_3\ln^2(2) + 2112\zeta_3\ln(2)\ln(3) - 1584\,\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\ln(3) - \frac{4400\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\pi^3}{27} + \frac{88\pi^4\ln(2)}{45} \\
& - \frac{616\pi^4\ln(3)}{27} + \frac{11264\pi^2\ln^3(2)}{9} - \frac{22528\ln^5(2)}{15} + 8576\,\mathrm{Li}_4\left(\frac{1}{2}\right) + 7504\zeta_3\ln(2) + \frac{4174\pi^4}{27} \\
& - \frac{1072\pi^2\ln^2(2)}{3} + \frac{1072\ln^4(2)}{3} + \frac{554032\zeta_3}{27} - 32\pi^2\ln(2) + \frac{730378\pi^2}{243} - 384\ln^2(2) + 832\ln(2) \\
& \left. + \frac{1408681}{81} + \sqrt{3}\left(192\,\Im\left\{\mathrm{Li}_3\left(\frac{\exp(i\pi/3)}{2}\right)\right\} + 160\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln(2) - 16\pi\ln^2(2) - \frac{560\pi^3}{81}\right)\right) + \mathcal{O}(\varepsilon^2)
\end{aligned}
$$

- up to $\mathcal{O}(\varepsilon^0)$ only zeta values and $\mathrm{Li}_4\left(\frac{1}{2}\right)$
- at $\mathcal{O}(\varepsilon)$, we find multiple zeta value $\zeta_{-5,-1}$, GPLs with sixth root of unity, ...

Conclusion

- extended notion of IBP relations & reverse unitarity enables us to reduce phase-space integrals that contain Heaviside functions
- the method also provides the means to derive differential equations
- with this, we computed the same-hemisphere configuration of the three-gluon-emission contribution to the zero jettiness function

Outlook

- computation of second configuration ($nn\bar{n}$) currently underway
- remaining contributions are
  - RRR: $gq\bar{q}$-emission
  - RRV: one-loop corrections to gg- and $q\bar{q}$-emission
  - RVV: two-loop correction to g-emission
- other observables, e.g. 1-jettiness
- other objects containing Heaviside functions, e.g. soft contributions in subtraction schemes