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POLI

Traversable wormholes supported with arbitrarily small amount of exotic matter

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Theoretical Background

Traversable Wormhole

The line element is

$$\mathrm{ds}^2 = -e^{2\phi}dt^2 + \left(1 - \frac{b}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

Where ϕ is the so called redshift function and b the shape function, both are functions of the radial coordinate only. The red-shift function must be finite since there is not horizon.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu},$$

Sourced by $T^{\mu}_{\nu} = diag(-\rho, p_r, p_t, p_t)$ we get

$$\rho = \frac{1}{8\pi}\frac{b'}{r^2}, \qquad p_r = -\frac{1}{8\pi}\left[\frac{b}{r^3} - 2\left(1 - \frac{b}{r}\right)\frac{\phi'}{r}\right],$$

$$p_t = \frac{1}{8\pi}\left(1 - \frac{b}{r}\right)\left[\phi'' + (\phi')^2 - \frac{b'r - b}{2r^2(1 - b/r)}\phi' - \frac{b'r - b}{2r^3(1 - b/r)} + \frac{\phi'}{r}\right].$$

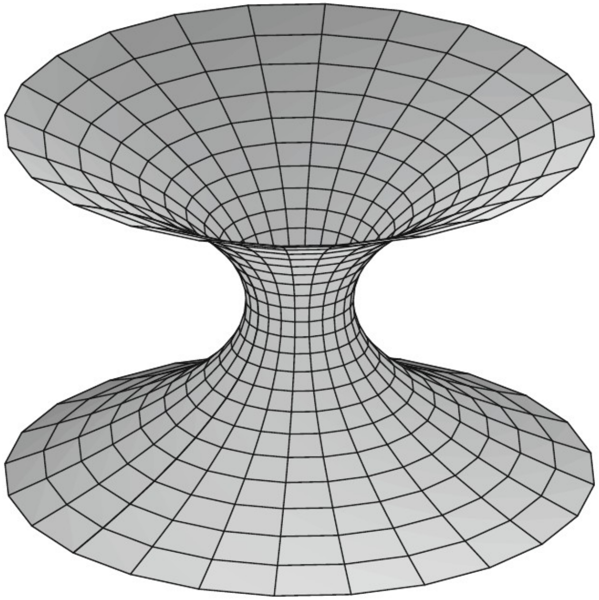
The Mathematics of Embedding

Considering a fixed time and an equatorial slice $(\theta = \frac{\pi}{2}, t = const)$

$$\mathrm{ds}^2 = \frac{dr^2}{1-b/r} + r^2 d\phi^2 = dz^2 + dr^2 + r^2 d\phi^2,$$

which means

$$\frac{dz}{dr} = \pm\left(\frac{r}{b} - 1\right)^{-1/2}.$$



Notice that a throat r_0 is a requirement (minimum radius)
We demand that the solution is asymptotically flat.

$$b > 0, \quad r \in [r_0, \infty) \qquad b(r_0) = r_0$$

$$\lim_{r\to r_0} \frac{dz}{dr} \to \infty \qquad \lim_{r\to\infty} \frac{dz}{dr} = 0$$

At or near the throat, there is the **flaring-out condition**, is a fundamental ingredient.

$$\frac{d^2 r}{dz^2} = \frac{b - b' r}{2b^2} > 0$$

Exotic Matter

The flaring-out condition establishes a profound relation with exotic matter.

$$\xi = -\frac{p_r + \rho}{|\rho|} = \frac{2b^2}{r|b'|}\frac{d^2r}{dz^2} - 2(r-b)\frac{\phi'}{|b'|}$$

At the throat only the first term remains which leads to

$$0 > \rho + p_r,$$

which implies that the throat tension must be greater than the total energy density which violates the null energy condition (NEC). These defines exotic matter.
We can minimize the amount required to construct a traversable wormhole by demanding
A finite quantifier

$$I = \int dV(\rho + p_r) = -\int_{r_0}^{\infty}(1-b')\left[\ln\left(\frac{e^{2\phi}}{1-b/r}\right)\right]dr$$

Humanly Traversable

In order to be alive after the travel, radial and lateral tidal forces must be less than gravity in Earth.

$$\left|\left(1-\frac{b}{r}\right)\left[\phi'' + (\phi')^2 - \frac{rb'-b}{2r(r-b)}\phi'\right]\right||\eta^1| \leq \frac{g_\oplus}{c^2}$$

$$\left|\frac{\gamma^2}{2r^2}\left[\beta^2(b' - \frac{b}{r}) + 2r(r-b)\phi'\right]\right||\eta^2| \leq \frac{g_\oplus}{c^2}$$

We also want to spend only an reasonable amount of time in the travel.

$$\Delta t = \int_{r_0}^{r_{st}} \frac{e^{-\phi}dr}{v\sqrt{1-b/r}} < 1 \text{ year},$$

$$\Delta \tau = \int_{r_0}^{r_{st}} \frac{dr}{v\gamma\sqrt{1-b/r}} < 1 \text{ year}$$

Complexity Factor

Luis Herrera defined a new scalar which relates somehow anisotropy and homogeneity of a system.

$$Y_{TF} = 8\pi\Pi - \frac{4\pi}{r^3}\int_0^r \tilde{r}^3\rho' d\tilde{r} \qquad \Pi \equiv p_r - p_t$$

However, as traversable wormholes are defined for $r \in [r_0, \infty)$ the complexity should be modified. The complexity factor for traversable wormholes must be defined as

$$Y_{TF} = 8\pi\Pi - \frac{4\pi}{r^3}\int_{r_0}^r \tilde{r}^3\rho' d\tilde{r}$$

$$Y_{TF} = \left(1 - \frac{b}{r}\right)\left(\frac{\phi'}{r} - \phi'^2 - \phi''] + \phi'\left(1 - \frac{b}{r}\right)^{-1}\left(\frac{rb' - b}{2r^2}\right)\right) + \frac{r_0 b'(r_0) - 3r_0}{2r^3}.$$

Results

Traversable Casimir Wormhole

R. Grattini* constructed a traversable wormhole with red-shift and shape functions

$$\phi = \ln\left(\frac{3r}{3r + r_0}\right), \qquad b = \frac{2r_0}{3} + \frac{r_0^2}{3r},$$

which gives a complexity factor of the form

$$Y_{TF} = -\frac{2r_0(3r^3 + 7rr_0 + r_0^2)}{3r^4(3r + r_0)}.$$

The restrictions of tidal forces are

$$r_0 \gtrsim 10^8 m, \qquad v \lesssim 2.7 r_0 s^{-1}.$$

The quantifier for this system is

$$I = -\frac{4r_0}{\kappa}$$

*R. Garattini, Eur. Phys. J. C 79, 951

Traversable like-Casimir Wormhole

Taking a red-shift and a complexity of the form

$$\phi = \ln\left(\frac{c_0 r}{c_0 r + r_0}\right),$$

$$Y_{TF} = -\frac{r_0(a_4 r^4 + a_3 r^3 r_0 + a_2 r^2 r_0^2 + a_1 r r_0^3 + a_0 r_0^4)}{r^5(c_0 r + r_0)^2}.$$

¿What shape function can solve this problem?

$$b = \frac{1}{30 r^2 r_0^5}\Big(12 a_0 r_0^8 + (15 a_1 - 27 a_0 c_0) r_0^7 r$$
$$+ (20 a_2 - 40 a_1 c_0 + 72 a_0 c_0^2) r_0^6 r^2$$
$$+ (15 + 15 a_3 - 35 a_2 c_0 + 70 a_1 c_0^2 - 126 a_0 c_0^3) r_0^5 r^3$$
$$+ (5 a_4 - 15 a_3 c_0 + 35 a_2 c_0^2 - 70 a_1 c_0^3 + 126 a_0 c_0^4)$$
$$\times \big[12 r^4 r_0^4 + 125 r^5 r_0^3 + 260 r^6 r_0^2 + 210 r^7 r_0$$
$$+ 60 r^8 + r^5 (c_0 r + r_0)^4 \frac{60 c_0}{r_0} \ln \frac{r}{c_0 r + r_0}\big]\Big).$$

6 PARAMETERS!!

Traversable like-Casimir Wormhole

Asymptotically flat

4 PARAMETERS!!

$$b(r_0) = r_0$$

3 PARAMETERS!!

$$I = 0$$

Independent of the throat

1 PARAMETER!!

Exotic matter only around the throat

$$0.950679 < c_0 < 4.86215$$

Traversable like-Casimir Wormhole

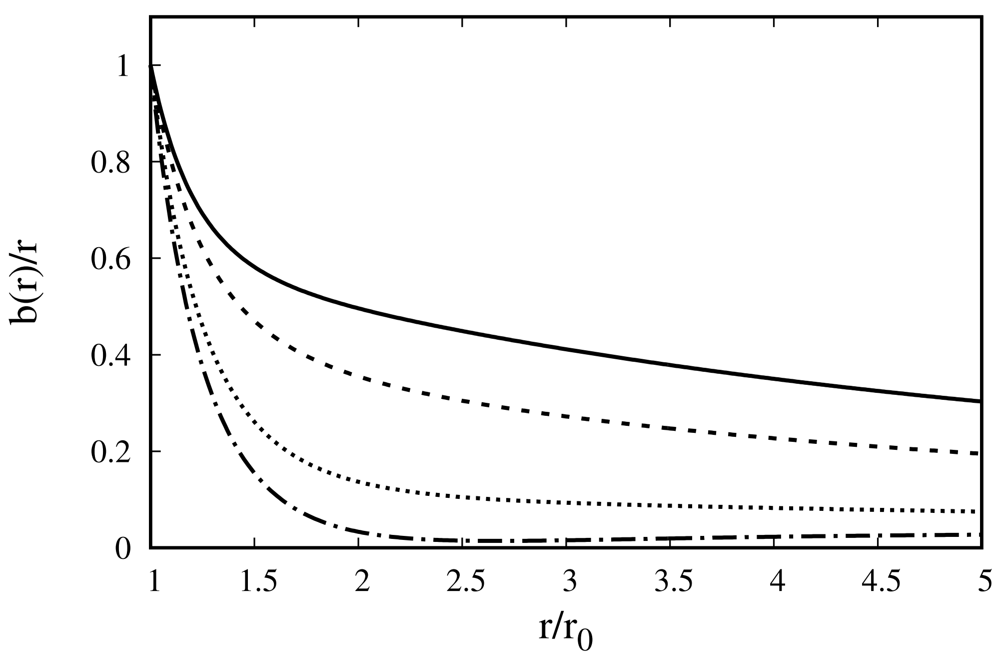
The shape function reads as

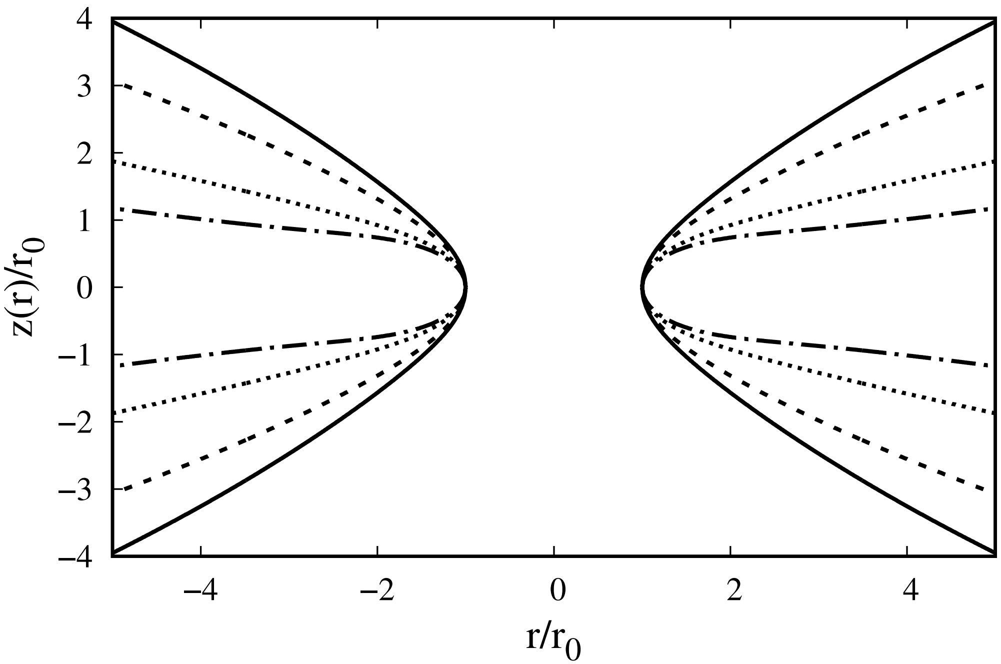
1 PARAMETER!!

$$b(r) = \frac{2r_0}{c_0} + \left(1 - \frac{2}{c_0} - \frac{4\left[4 - 2c_0 + \ln\left(1 + \frac{1}{c_0}\right)(c_0^2 - 2c_0 - 3)\right]}{c_0\left[-3 - 4c_0 + 4c_0(c_0+1)\ln\left(1 + \frac{1}{c_0}\right)\right]}\right)\frac{r_0^2}{r} + \frac{4\left[4 - 2c_0 + \ln\left(1 + \frac{1}{c_0}\right)(c_0^2 - 2c_0 - 3)\right]}{c_0\left[-3 - 4c_0 + 4c_0(c_0+1)\ln\left(1 + \frac{1}{c_0}\right)\right]}\frac{r_0^3}{r^2}$$

$$0.950679 < c_0 < 4.86215$$

Traversable like-Casimir Wormhole

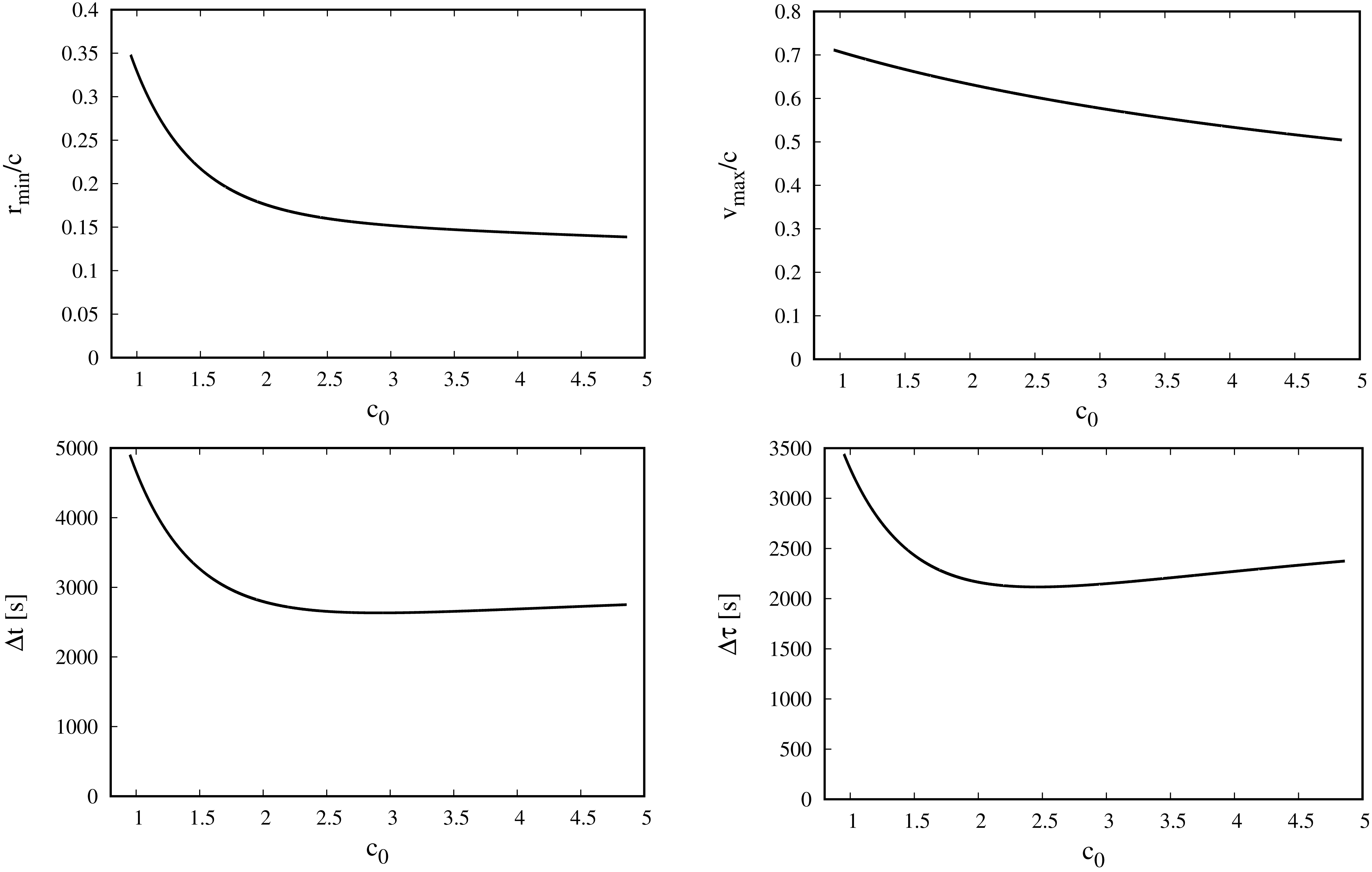




$$c_0 = 0.96(solid), \quad c_0 = 1.5(dashed),$$

$$c_0 = 3(dotted), \quad c_0 = 4.5(dash - dotted)$$

Humanly Traversable



Conclusions

Conclusions

- The first definition of complexity for a wormhole.

- The solution could be characterized with only one parameter.

- The wormhole geometry is traversable fulfilling: the flaring-out condition, tidal acceleration of the order of the Earth gravitational acceleration, finite time to travel from a spatial station to the throat, and a minimal amount of exotic matter.

- The quantifier does not depends on the size of the throat but is arbitrarily small which means that, we can construct the traversable wormhole with a minimal quantity of exotic matter.

Final Remarks

- Quasi Normal Modes of this wormhole model and it's stability. It's already done!