An analysis of the Ruppeiner geometry and the thermodynamics of extremal black holes

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Introduction

This work aims to study the properties and theorems of black hole thermodynamics, focusing on extremal Reissener-Nordström (RN) solution, as well as the geometric formulations made by Ruppeiner about thermodynamical systems, and how we can relate both of these areas.

However, in the Ruppeiner geometry context, we explore how different relations can modify the scalar curvature. These relations are geometry modifications, whether by an analysis of different variables or by approximations of other solutions to reach the RN case. We used natural units $G = \hbar = c = k_B = 1$ and the metric signal is $(-,+,+,+)$.

Thermodynamics

In classical thermodynamics systems, the behavior of the system is described by a small number of parameters. Moreover in the case of black holes, the no-hair theorem allow us to describe all stationary solutions of Einstein-Maxwell equation by: mass M, charge Q and angular momentum J.

Besides the four laws of mechanics of black holes, the extremal and non-extremal solutions can be categorized into two different groups. This idea came up when Hawking, Horowitz and Ross explored the local geometry of extremal RN black holes, which is a trivial topology and imposes a zero entropy, therefore the formula $S = A/4$ is no long validy for this case, see [1]. In this same article, they proposed that extremal RN black holes occur with pair creation. Liberati, Rothman and Sonego showed that incipent RN black holes emit non-thermal radiation, [2].

Summarizing all thermodynamics characteristics of extremal RN:

- $S = 0$
- $\kappa = 0$
- Non-thermal radiation
- Single classical microstate
- Eternal

Ruppeiner Geometry

Looking for the possibility of extracting underlying information to study the behavior of any thermal statistical system, many approximations were made to unify the Riemannian geometry with thermodynamics in the last decades. Ruppeiner proposed in 1979 a model that had a metric based on the hessian matrix of the thermal system and also their intesive and extensive variables, given by the equation

$$ds^2 = g^R_{ij} dM^i dM^j \tag{1}$$

with $g^R_{ij} = -\partial_i \partial_j S(M, N^a)$ and $M^i = (M, N^a)$.

In the context of black hole thermodynamics, we can use this idea to calculate the metric associated with a particular black hole solution. As shown before, the entropy of the Reissner-Nordström black hole is given by S, where M is the extensive variable and Q is the intensive one, usually we denote this set as $S(M, Q)$. Determining the metric by taking the hessian of the entropy function and making some coordinates transformation, we arrive at

$$ds^2 = -\frac{1}{2S} dS^2 + \frac{4S}{1-u^2} du^2 \tag{2}$$

which is a flat metric, since we are dealing with a flatness solution the scalar curvature is zero everywhere. To avoid this scenario, some modifications were taken into account as we evaluate the Ruppeiner metric.

Geometry modification

Shen, Cai, Wang and Su extended the work of Ruppeiner using the Van der Walls-Maxwell model [3]. In this context, the mass (called Enthalpic energy) was rewritten in terms of knowing variables, $\tilde{M} = M - Q\Phi$, now $\tilde{M}$ is the internal energy. Calculating the hessian of the entropy, with respect to $\tilde{M}$, and then computing the scalar curvature,

$$R = -\frac{r_+ - r_-}{r_+(r_+ - 3r_-)^2} \tag{3}$$

We can plot the scalar by r_+ with a fixed r_-, see figure 1. Notice that in the extremal case $R = 0$.

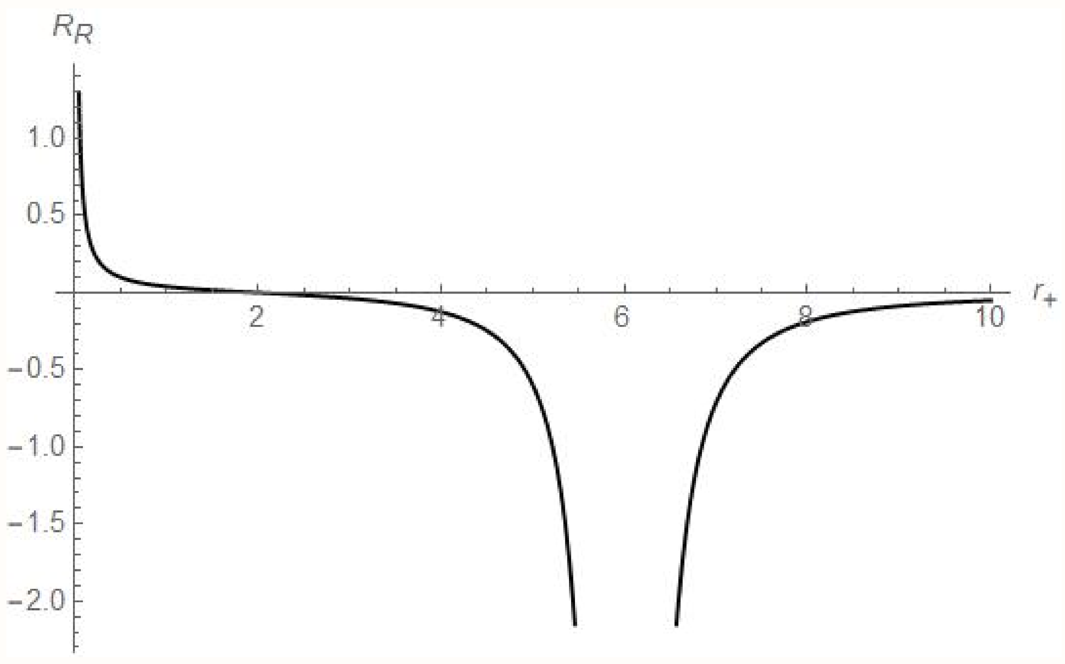

Figure 1: Scalar curvature for $r_- = 2$.

Another helpful modification, made by Mirza and Zamaninasab [4], is to take the Ruppeiner metric of another solution, especially the scalar curvature, and then reduce all the factors to reach the RN solution. Starting from the Kerr-Newman Anti-deSitter solution, computing the scalar curvature, and then making $J \to 0$ and $l \to \infty$, we get

$$R_{(1)} = \frac{r_+^2 + r_- r_+ + 2r_-^2}{r_+(r_+ - r-)(r_+ + r-)^2} \tag{4}$$

We can see the plot of this scalar in figure 2, notice that in extremal case the scalar diverges, however the left and right-hand limits are different.

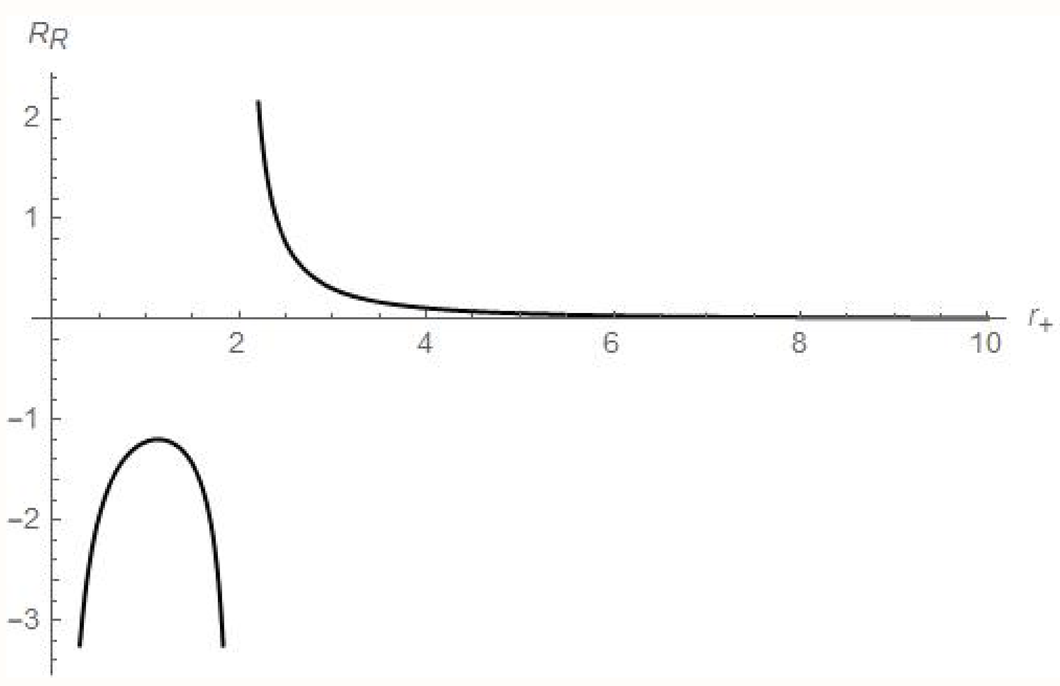

Figure 2: Scalar curvature for $r_- = 2$.

With this modification, it is easy to work with another black hole solution and reduce the variables to reach the scalar associated with the RN solution. Taking the Kerr-Newman solution and reducing, after computing the scalar, the angular momentum $J \to 0$, we get

$$R_{(2)} = -\frac{r_+^2 + r_- r_+ + 2r_-^2}{r_+(r_+ - r-)(r_+ + r-)^2} \tag{5}$$

It is not quite hard to see that $R_{(1)}$ and $R_{(2)}$ are related by a constant, $R_{(1)} = -R_{(2)}$. This relation arises a new problem associated with the Ruppeiner geometry: starting from different solutions and converging to the same at the end of the calculation, the values are different. Without a doubt, we notice that figures 2 and 3 are opposites.

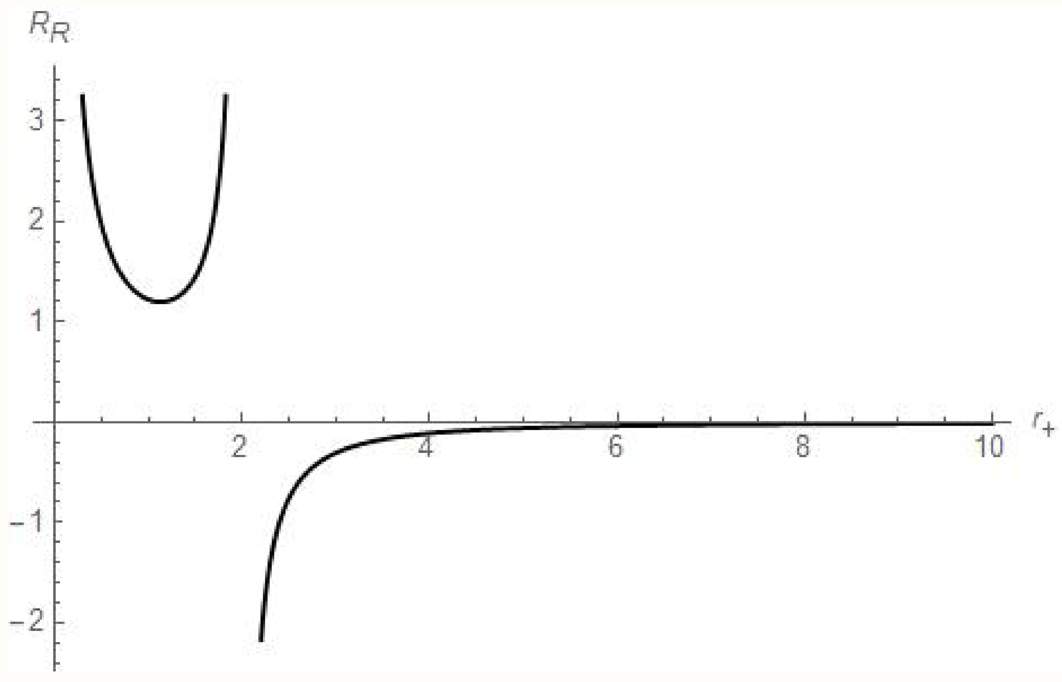

Figure 3: Scalar curvature for $r_- = 2$.

Summary and conclusions

In his article, Medved discussed the relationship between M and $\tilde{M}$ and which case is preferable for describing our system. For a better understanding of the phase space of thermodynamical systems, we choose the internal energy $\tilde{M}$, meanwhile, to explore the underlying properties of the statistical system, we choose the enthalpic energy M, see [5].

For set M, we studied the approximation of the Kerr-Newman and Kerr-Newman Anti-deSitter solutions to RN and observed that the scalar curvature diverges at the extremal point. Recapping the discussion made by Medved, we associate this modification with the underlying properties. We can summarize for $\{M, S, Q\}$:

- Zero degree of freedom
- Unstable surface
- Pair creation x Third law

Furthermore, by applying the modification of the Van der Walls-Maxwell model and reaching the zero scalar curvature at the extremal point, we relate with the properties of the phase space of this system. So we observed that for $\{\tilde{M}, S, \Phi\}$:

- Non-thermal radiation
- Eternal
- Unique law for the extremal case

We concluded that even though the Ruppeiner geometry can not be applied directly to black holes and present some issues for invariance, it was satisfying to relate the properties of extremal black holes with the scalar computed by the geometry. This work opened new ways to explore the thermodynamics of extremal black holes.

References

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