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Diffeomorphism breaking and background fields in modified gravity

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Work in collaboration with Marco Schreck and Alex Soto [CMR and M. Schreck PRD 104, 124042, 2021; PRD 106, 044050, 2022; and CMR, M. Schreck and A. Soto, PRD 106, 023524, 2022].

Plan of the talk

Introduction: Hamiltonian formulation of General Relativity

- The gravitational action and boundary terms
- The ADM decomposition of spacetime: Lapse and Shift
- Constraints and dynamics

Explicit diffeomorphism violation in the SME gravity sector

- The Hamiltonian formulation
- Extended boundary terms
- Constraints and Hamilton's equations of motion

Final remarks

Motivations

- The standard model of particles is an effective theory which requires an extension at higher energies close to the Planck energy. Many approaches to quantum gravity suggest that new physics in the form of CPT and Lorentz symmetry breaking may arise at the Planck scales.

- Most of the searches have been given within the framework of the Standard-Model Extension (SME) where Lorentz symmetry violation has been less explored in the gravity sector. The presence of nondynamical backgrounds fields may introduce possible issues with the Bianchi identity.

- General Relativity is a constrained system with first class constraints generating symmetries. The Hamiltonian formulation is a well suited to study breaking of local Lorentz and diffeomorphism symmetry.

- A natural question arise about consistency i) the equivalence of the dynamics in the Hamiltonian formulation and ii) are the constraints the projected modified Einstein equations.

The Standard-Model Extension (SME)

[A. Kostelecky and D. Colladay, Phys. Rev. D 55 (1997), 6760-6774 and 58 (1998), 116002].

- The SME is an effective framework to accommodate all possible terms of CPT and Lorentz violation. The Lorentz violating tensors are suggested to arise as expectation values in a more fundamental theory [A. Kostelecky and S. Samuel, Phys.Rev.D 39 (1989) 683].

- It includes extensions to the quantum field theories of the standard model of particles and to General Relativity (GR).

- It has been implemented with operators of mass dimension lower than four (minimal sector) and with operators of dimension higher than four (nonminimal sector).

- Several experimental bounds have been given in all the sectors [A. Kostelecky and D. Colladay Phys.Rev.D 55 (1997) 6760-6774, Phys.Rev.D 58 (1998) 116002].

General Relativity: Boundary term

The action functional for General Relativity contains a Hilbert term and a boundary term

$$S[g] = \underbrace{\int_{\mathcal{M}} d^4x \frac{\sqrt{-g}}{2\kappa} R}_{Hilbert\ term\ S_H} + \underbrace{\frac{\varepsilon}{\kappa} \oint_{\partial\mathcal{M}} d^3y \sqrt{q} K}_{Boundary\ term\ S_B} \,, \tag{1.1}$$

where R is the Ricci scalar in $\mathcal{M}$, K is the trace of the extrinsic curvature on $\partial\mathcal{M}$, $\varepsilon = \pm 1$ (timelike or spacelike hypersurface) and q is the determinant of the induced metric on $\partial\mathcal{M}$.

A variation of the Hilbert term $(\delta(\sqrt{-g}R_{\alpha\beta}g^{\alpha\beta})$ produces

$$\delta S_H = \frac{1}{2\kappa}\int_{\mathcal{M}} G_{\alpha\beta}\delta g^{\alpha\beta}\sqrt{-g}\,d^4x - \frac{1}{\kappa}\oint_{\partial\mathcal{M}} \varepsilon q^{\alpha\beta}\delta g_{\alpha\beta,\mu}n^{\mu}\sqrt{q}\,d^3y\,, \tag{1.2}$$

where $G_{\alpha\beta}$ is the Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\,. \tag{1.3}$$

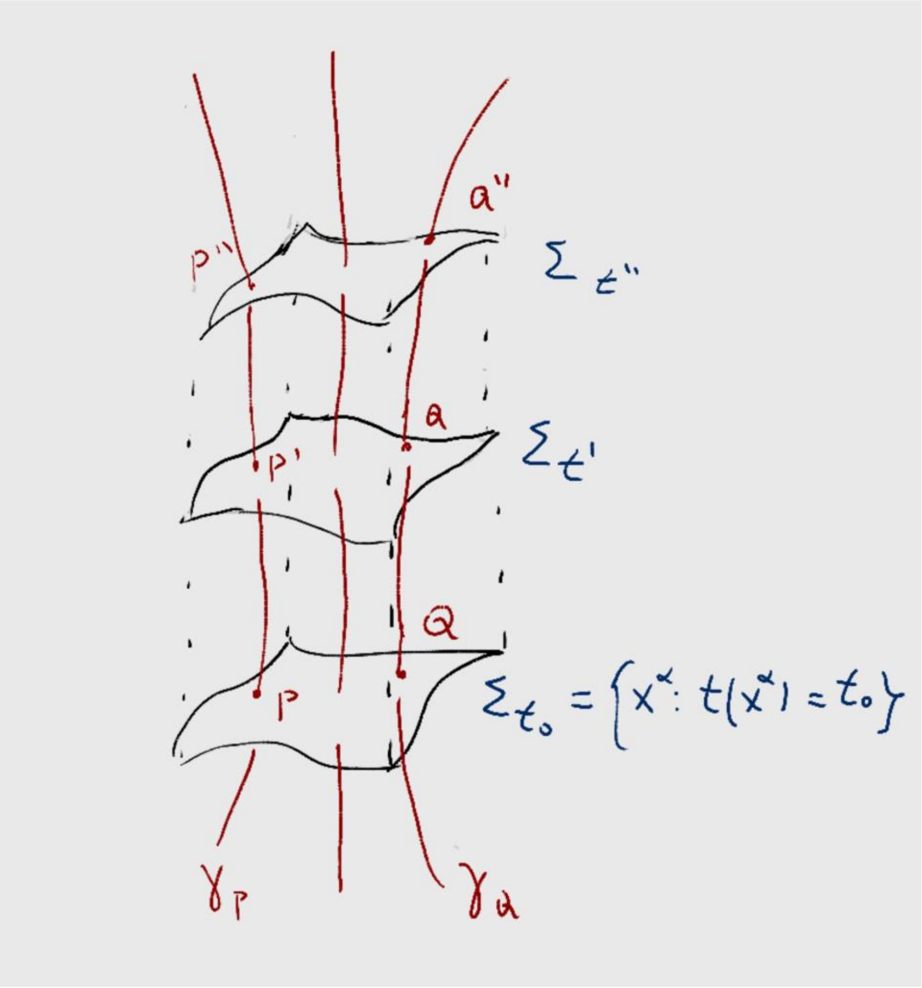
Considering that $K = \nabla_\mu n^\mu$ (∇_μ is the covariant derivative) one can show that the variation of the boundary term is

$$\delta S_B = \frac{1}{\kappa}\oint_{\partial\mathcal{M}} \varepsilon q^{\alpha\beta}\delta g_{\alpha\beta,\mu}n^{\mu}\sqrt{q}\,d^3y\,, \tag{1.4}$$

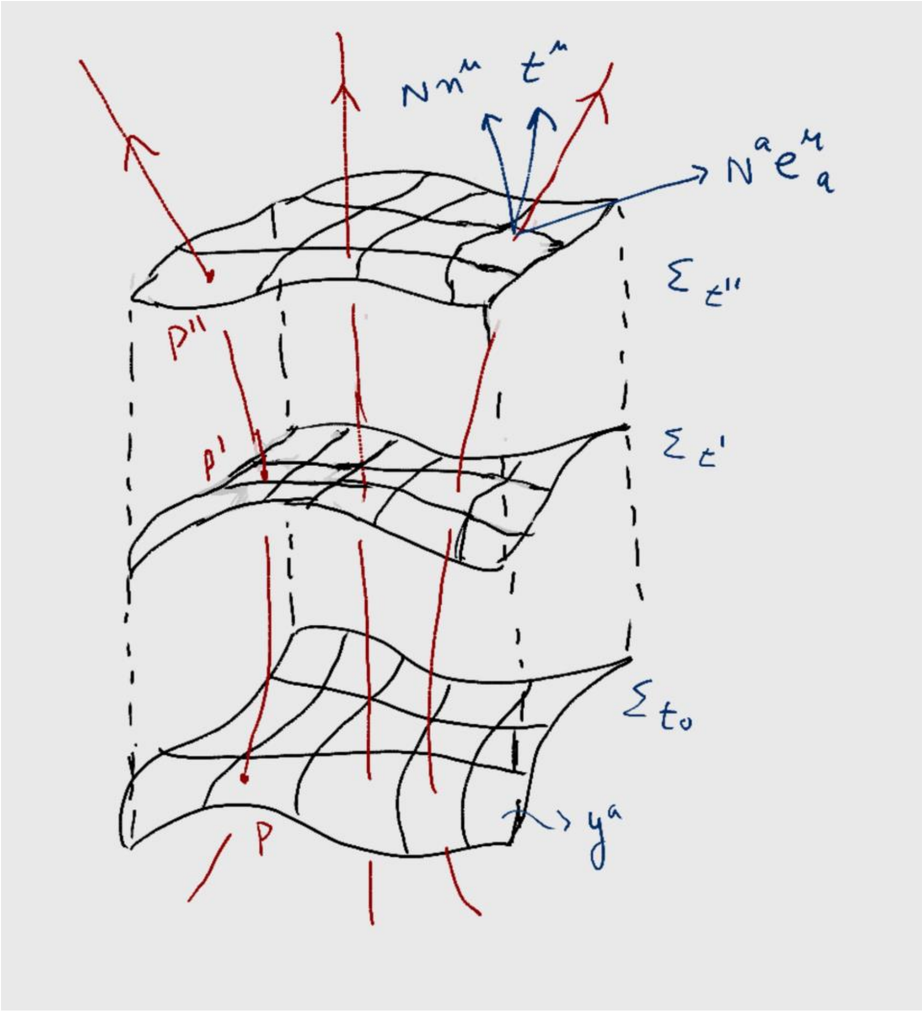
which cancels the second term in (1.1), giving the Einstein equation $G_{\alpha\beta} = 0$.

$3 + 1$ decomposition of spacetime $\mathcal{M}$

- We cover $\mathcal{M}$ with coordinates x^μ.
- We introduce a scalar function $t(x^\mu)$ such that $t = constant$ defines a foliation of $\mathcal{M}$ into spacelike hypersurfaces Σ_t.
- On each hypersurface we introduce coordinates y^a.
- Introduce a congruence of curves such that $y^a(P) = y^a(P') = y^a(P'')$.
- This construction defines a transformation $x^\mu = x^\mu(t, y^a)$.



Foliation of spacetime $\mathcal{M}$



Consider

$$dx^\mu = \left(\frac{\partial x^\mu}{\partial t}\right)_{y^a} dt + \left(\frac{\partial x^\mu}{\partial y^a}\right)_t dy^a \tag{2.1}$$

and we define

$$t^\mu = \left(\frac{\partial x^\mu}{dt}\right)_{y^a}, \qquad e^\mu_a = \left(\frac{\partial x^\mu}{dy^a}\right)_t. \tag{2.2}$$

We can decompose in terms of the normal $n_\alpha = -N\partial_\alpha t$ and tangential vectors

$$t^\alpha = Nn^\alpha + N^a e^\alpha_a. \tag{2.3}$$

Lapse Shift

Considering

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{2.4}$$

we arrive at

$$g_{\mu\nu} = \begin{pmatrix} N_a N^a - N^2 & N_a \\ N_b & q_{ab} \end{pmatrix}. \tag{2.5}$$

Above we have introduced the induced three metric $q_{ab} = g_{\mu\nu}e^\mu_a e^\nu_b$.

Highlights: Hamiltonian and constraints

- GR is a constrained theory with a Hamiltonian

$$H = \int_{\Sigma_t} \mathrm{d}^3x\, \mathcal{H} = -\int_{\Sigma_t} \mathrm{d}^3x\, \frac{\sqrt{q}}{2\kappa}(N\mathcal{C} + N^i\mathcal{C}_i)\,, \tag{2.6}$$

where the so called Hamiltonian and momentum constraint are

$$C = \frac{2\kappa}{\sqrt{q}}\left[\pi^{ij}\pi_{ij} - \frac{\pi^2}{2}\right] - \frac{\sqrt{q}}{2\kappa}R\,, \qquad C_i = -2D_j\pi^j{}_i\,. \tag{2.7}$$

- Dirac procedure is crucial for this construction, where constraints can be seen to be first class type.

- Both constraints satisfy the diffeomorphism algebra.

- In the first part we have seen that a boundary term is important to avoid higher-order terms.

SME gravity: the u and the s sector

We study the Hamiltonian formulation of SME gravity [K. O'Neal-Ault, Q. G. Bailey and N. A. Nilsson, PRD. 103, 044010, (2021) and CMR and M. Schreck PRD 104, 124042, (2021)].

We focus on the following modified Einstein-Hilbert (EH) action without a cosmological constant

$$S = \int_{\mathcal{M}} \mathrm{d}^4x \frac{\sqrt{-g}}{2\kappa} \left[(1-u)^{(4)}R + s^{\mu\nu\,(4)}R_{\mu\nu} \right] , \tag{3.1}$$

and consider the decomposition

$$s^{\alpha\beta} = q^{\alpha}{}_{\mu} q^{\beta}{}_{\nu} s^{\mu\nu} - (q^{\alpha}{}_{\nu} n^{\beta} + q^{\beta}{}_{\nu} n^{\alpha}) s^{\nu\mathbf{n}} + n^{\alpha} n^{\beta} s^{\mathbf{nn}} . \tag{3.2}$$

Since the middle term can be gauge away by a redefinition of the lapse and shift we have three sectors to analyze: the scalar sector u, the pure space sector s^{ij} and the pure time sector $s^{\mathbf{nn}}$.

The modified Einstein equations for (3.1) are:

$$0 = (1-u)^{(4)}G^{\mu\nu} + \frac{1}{2}(\nabla^\mu\nabla^\nu u + \nabla^\nu\nabla^\mu u) - g^{\mu\nu}\Box u$$
$$- \frac{1}{2}\left(s^{\alpha\beta\,(4)}R_{\alpha\beta}g^{\mu\nu} + \nabla_\alpha\nabla^\mu s^{\alpha\nu} + \nabla_\alpha\nabla^\nu s^{\alpha\mu} - \Box s^{\mu\nu} - g^{\mu\nu}\nabla_\alpha\nabla_\beta s^{\alpha\beta}\right) ,$$

and we are considering an extended GHY boundary term of the form

$$S_{\substack{\text{ext}\\ \text{GHY}}} = \frac{\varepsilon}{2\kappa}\oint_{\partial\mathcal{M}} \mathrm{d}^3y\,\sqrt{q}\,\left[2(1-u)K - s^{\mathbf{nn}}K + K_{ij}s^{ij}\right] ,$$

where the parameter $\varepsilon = \mp 1$ for a spacelike (timelike) boundary $\partial\mathcal{M}$ of the spacetime manifold $\mathcal{M}$ and the integral runs over the coordinates y^a defined on this boundary and

$$S_{\partial\Sigma} = -\frac{1}{2\kappa}\oint_{\partial\Sigma_t} \mathrm{d}^2z\,\sqrt{q}r_l\left[ND^l(2u + s^{\mathbf{nn}})\right] ,$$

with the coordinates z^a given on the boundary of a spacelike hypersurface Σ_t and a suitably normalized vector r_l orthogonal to the boundary.

For the u, $s^{\mathbf{nn}}$, and s^{ij} sector each, the canonical Hamiltonians have the form

$$
\begin{aligned}
H_u = \int_{\Sigma_t} \mathrm{d}^3x \Bigg[& -\frac{\sqrt{q}}{2\kappa} N\left((1-u)R + 2D^i D_i u\right) + \frac{\mathcal{L}_m u}{1-u}\left(\pi - \frac{3}{4}\frac{\sqrt{q}}{\kappa N}\mathcal{L}_m u\right) \\
& + \frac{2\kappa N}{\sqrt{q}(1-u)}\left(\pi^{ij}\pi_{ij} - \frac{\pi^2}{2}\right) - 2(D_i \pi^{ij})N_j \Bigg] ,
\end{aligned}
$$

$$
\begin{aligned}
H_1 = \int_{\Sigma_t} \mathrm{d}^3x \Bigg[& -\frac{\sqrt{q}}{2\kappa} N(R + D^i D_i s^{\mathbf{nn}}) + \frac{\mathcal{L}_m s^{\mathbf{nn}}}{2(1-s^{\mathbf{nn}})}\left(p - \frac{3}{8}\frac{\sqrt{q}}{\kappa N}\mathcal{L}_m s^{\mathbf{nn}}\right) \\
& + \frac{2\kappa N}{\sqrt{q}(1-s^{\mathbf{nn}})}\left(p^{ij}p_{ij} - \frac{p^2}{2}\right) - 2(D_i p^{ij})N_j \Bigg] ,
\end{aligned}
$$

$$
\begin{aligned}
H_2 = \int_{\Sigma_t} \mathrm{d}^3x \Bigg[& -\frac{\sqrt{q}}{2\kappa} N\left(R + s^{ij}R_{ij} - D_j D_i s^{ij}\right) + \left(P_{ij} - \frac{P}{2} q_{ij}\right)\mathcal{L}_m s^{ij} \\
& + \frac{2\kappa N}{\sqrt{q}}\left(P^{ij}P_{ij} - (1 - s^i{}_i)\frac{P^2}{2} - 2s^{ij}(P_{ij}P - P_i{}^k P_{kj})\right) \\
& - 2(D_i P^{ij})N_j \Bigg] + \mathcal{O}[(s^{ij})^2] .
\end{aligned}
$$

The idea is to project the modified Einstein tensor and compare with:

- The constraints obtained using the Dirac procedure+Boundary terms.
- The Hamilton-Jacobi equations of motion

$$\dot{q}_{ij} = \{q_{ij}, H\}\,, \tag{1a}$$

$$\dot{\pi}^{ij} = \{\pi^{ij}, H\}\,. \tag{1b}$$

The first set of equations corresponds to a geometrical identity, whereas the second set describes the dynamics of the theory under consideration.

Let $(J_2)^{\mu\nu} := J^{\mu\nu}|_{s^{\mathbf{nn}}=s^{i\mathbf{n}}=0}$ be the left-hand side of the modified Einstein equations for the purely spacelike sector composed of the coefficients s^{ij}. Considering their projection into Σ_t results in

$$
\begin{aligned}
(\vec{\boldsymbol{q}}^*\mathbf{J}_2)^{ij} = {} & \frac{2\kappa}{N\sqrt{q}}\dot{P}^{ij} + R^{ij} - \frac{R}{2}q^{ij} + \frac{1}{N}(q^{ij}D_kD^kN - D^iD^jN) \\
& + \frac{1}{2}\left[-q^{ij}(s^{kl}R_{kl} - D_kD_ls^{kl}) - D_kD^is^{kj} - D_kD^js^{ki} + D_kD^ks^{ij}\right] \\
& + \frac{1}{2}\left(2q^{ij}\left[a_kD_ls^{kl} + s^{kl}(D_ka_l + a_ka_l)\right] - a_k(D^is^{kj} + D^js^{ki}) + a_kD^ks^{ij}\right) \\
& + \frac{\kappa^2}{q}\Bigg(\left[(1 - s^k{}_k)P^2 - 2P^{kl}P_{kl} + 4s^{kl}(PP_{kl} - P_k{}^mP_{lm})\right]q^{ij} \\
& \qquad + 8\Bigg[P^{ik}P_k{}^j + s^{ik}P_{kl}P^{lj} + s^{jk}P_{kl}P^{li} - s^{ik}PP_k{}^j - s^{jk}PP_k{}^i \\
& \qquad + s^{kl}(P^i{}_kP_l{}^j - P_{kl}P^{ij})\Bigg] + 2P^2s^{ij} - 4(1 - s^k{}_k)PP^{ij}\Bigg) \\
& + \frac{\kappa}{N\sqrt{q}}\left[2\left(P^i{}_k\mathcal{L}_ms^{jk} + P^j{}_k\mathcal{L}_ms^{ik}\right) - \left(P\mathcal{L}_ms^{ij} + P^{ij}q_{kl}\mathcal{L}_ms^{kl}\right)\right].
\end{aligned}
$$

$$
-\left\{P^{ij},\int_{\Sigma_t}\mathrm{d}^3x\,\frac{\sqrt{q}}{2\kappa}NR\right\} = -\frac{\sqrt{q}}{2\kappa}\left(NG^{ij}+q^{ij}D_lD^lN-D^iD^jN\right)\,,
$$

$$
\begin{aligned}
-\left\{P^{ij},\int_{\Sigma_t}\mathrm{d}^3x\,\frac{N\sqrt{q}}{2\kappa}(s^{kl}R_{kl}-D_lD_ks^{kl})\right\} = \frac{N\sqrt{q}}{4\kappa}\Big(&-2q^{ij}\left[a_kD_ls^{kl}+s^{kl}(D_ka_l+a_ka_l)\right]\\
&+a_k(D^is^{kj}+D^js^{ki})-a_kD^ks^{ij}\\
&+q^{ij}(s^{kl}R_{kl}-D_kD_ls^{kl})+D_kD^is^{kj}\\
&+D_kD^js^{ki}-D_kD^ks^{ij}\Big)\,,
\end{aligned}
$$

$$
\begin{aligned}
\left\{P^{ij},\int_{\Sigma_t}\mathrm{d}^3x\,\left(P_{kl}-\frac{P}{2}q_{kl}\right)\mathcal{L}_ms^{kl}\right\} = \;&\frac{1}{2}(P^{ij}q_{kl}\mathcal{L}_ms^{kl}+P\mathcal{L}_ms^{ij})\\
&-(P^i{}_k\mathcal{L}_ms^{kj}+P^j{}_k\mathcal{L}_ms^{ki})\,,
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\left\{P^{ij},\int_{\Sigma_t}\mathrm{d}^3x\,\frac{2\kappa N}{\sqrt{q}}\left[P^{ij}P_{ij}-\frac{1}{2}(1-s^k{}_k)P^2-2s^{kl}(PP_{kl}-P_k{}^nP_{ln})\right]\right\}\\
&=-\frac{\kappa N}{2\sqrt{q}}\Big(\left[(1-s^k{}_k)P^2-2P^{kl}P_{kl}+4s^{kl}(PP_{kl}-P_k{}^mP_{lm})\right]q^{ij}\\
&\qquad+8\Big[P^{ik}P_k{}^j+s^{ik}P_{kl}P^{lj}+s^{jk}P_{kl}P^{li}-s^{ik}PP_k{}^j-s^{jk}PP_k{}^i\\
&\qquad+s^{kl}(P^i{}_kP_l{}^j-P_{kl}P^{ij})\Big]\Big)+2P^2s^{ij}-4(1-s^k{}_k)PP^{ij}\,.
\end{aligned}
$$

Final Remarks

- We have developed the Hamiltonian formulation of canonical SME gravity.
- We have generalized to include background-dependent boundary terms, which allow to get rid of higher-order terms.
- We have shown the equivalence of covariant and Hamiltonian formalisms with respect to constraints and dynamics.
- There are many aspects that need to be understood in particular the constraint structure of the theory.