The swampland in string theory and the generalized Ricci flow

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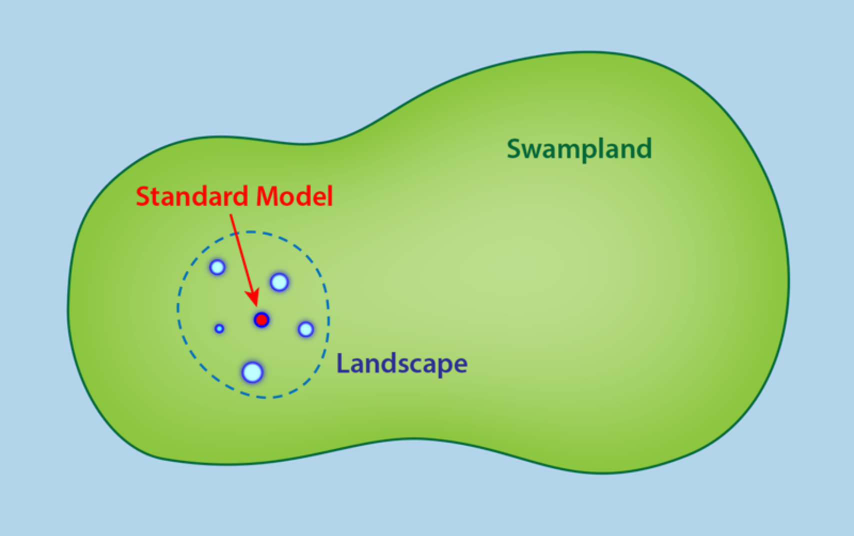


APS/Alan Stonebraker

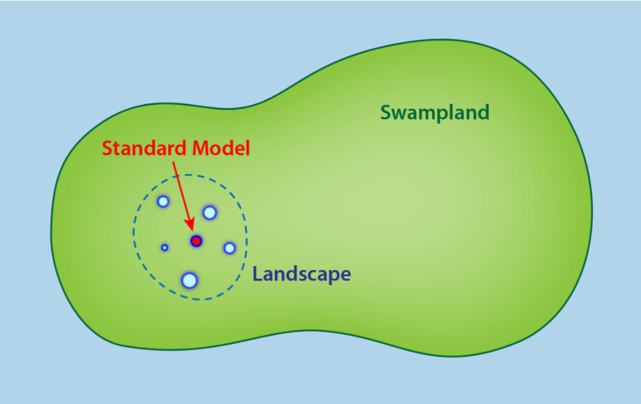
Quantum gravity and the Swampland

Let $\mathscr{M}$ denote the moduli space of a consistent quantum gravity. Choosing a point $p \in \mathscr{M}$ corresponds to fixing the low energy effective Lagrangian for the theory

- Conjecture 0: M is parameterized by inequivalent expectation values of massless scalar fields
- Conjecture 1: Choose any point $p \in M$. For any positive T , there is another point $p \in M$ such that $d(p, p0) > T$.
- Conjecture 2: Compared to the theory at $p_0 \in \mathscr{M}$, the theory at p with $d(p, p0) > T$ has an infinite tower of light particles starting with mass of the order of $e^{-\alpha T}$ for some $\alpha > 0$. In the $T \to \infty$ limit, the number of extra light particles of mass less than a fixed mass scale becomes infinite.
- Conjecture 3: The scalar curvature near the points at infinity is non-positive. (It is strictly negative if the dimension of the moduli space is greater than 1.)
- Conjecture 4: There is no non-trivial 1-cycle with minimum length within a given homotopy class in M.

Ooguri, Vafa, 2006

- Swampland: Effective low-energy physical theories which are not compatible with string theory.
- Landscape: Effective low-energy physical theories which are compatible with string theory.



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The generalized distance conjecture

$$\Delta_g = c \int_{\tau_i}^{\tau_f} \left(\frac{1}{V_M} \int_M \sqrt{g} g^{MN} g^{OP} \frac{\partial g_{MO}}{\partial \tau} \frac{\partial g_{NP}}{\partial \tau} \right)^{1/2} d\tau$$

There must be an infinite tower of states with mass scale (τ) such that

$$m \approx M_p e^{-\alpha|\Delta_g|}$$

The mass scale is setting the natural cut-off space above which the effective field theory description breaks down

Conjecture A. (Kehagias et. al. 2019)

Consider quantum gravity on a family of background metrics $g_{\mu\nu}$ satisfying the Ricci flow equation. There exist an infinite tower of states which become massless when following the flow towards a fixed point at infinite distance.

- We can use Δ_W as a good measure of distance in the background field.
- When adding a dilaton we can use $\Delta_{\mathscr{F}}$

$$\Delta(R) \approx |log\, R|$$

Black hole entropy distance conjecture

Use the size of the horizon

$$\Delta(S) \approx |log\, S|$$

$$m_s \approx S^{-c}$$

Geometric flows

- Geometric evolution equations. What is the long behaviour of the flow?
- A surface in some space (Euclidean, Riemannian,Lorentzian,Minkowski,..) is moving under a geometric flow if there exist a rule by which the surface moves at each point in "time".
- Partial Differential Equations + Riemannian Geometry → Geometric Analysis .

Extrinsic Flows

Curve shortening flow Mullins, '56
Harmonic map heat flow Eells, Sampson,'64
Mean Curvature flow Brakke,'78
Willmore flow Willmore,60's

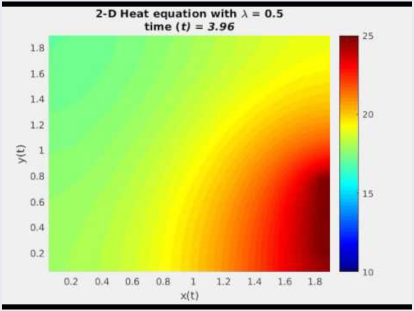
Intrinsic Flows

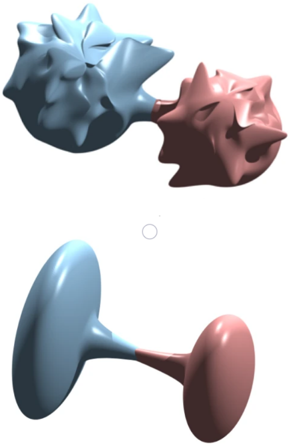
Ricci flow Hamilton,'81
Calabi flow Calabi,'??
Yamabe flow Hamilton,'81

Geometric flows: Curvature and difusion

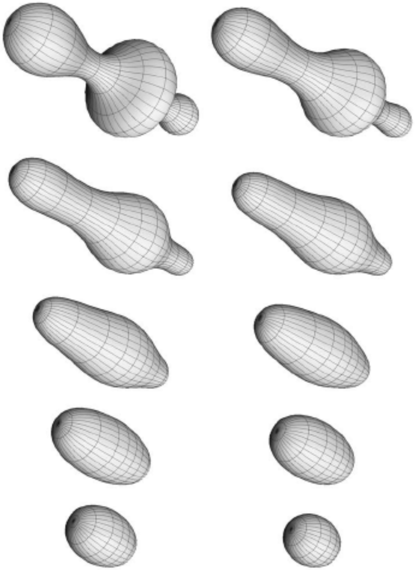
Heat equation $\Longrightarrow$ Difussive equation $\Longrightarrow$ parabolic equation

$$\frac{\partial T(t,x,y)}{\partial t} = \kappa^2 \left(\frac{\partial^2 T(t,x,y)}{\partial^2 x} + \frac{\partial^2 T(t,x,y)}{\partial^2 y} \right)$$

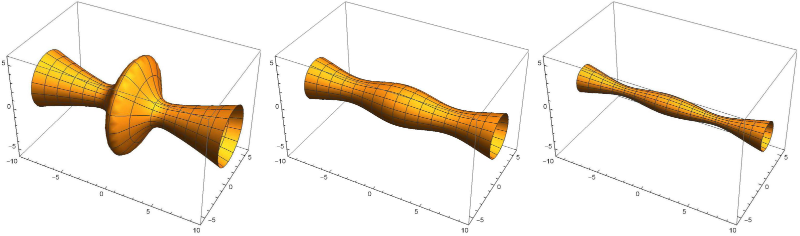




Ni,C.,Lin,Y.,Luo,F.et al 2019



B. Cipra



Lasso Andino, Vsconez 2020

Given a 1-parameter family of metrics $g(\lambda)$ on a Riemann manifold M^n, defined on a "time" interval $I \in \mathbb{R}$, the Ricci flow is defined

$$\frac{\partial g_{ij}}{\partial \lambda} = -2R_{ij}$$

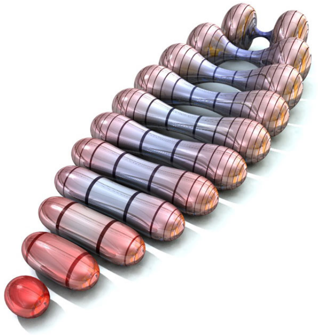
- Ricci flow is not parabolic.
- Define Ricci-DeTurck flow:

$$\frac{\partial g_{ij}}{\partial \lambda} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i$$
$$g(0) = g_0$$

where the 1-form $W = W(\lambda)$ is given by

$$W_j = g_{jk} g^{pq} \left(\Gamma^k_{pq} - \Gamma^k_{pq} \right)$$

The new flow is parabolic.



- Ricci flow allows to control geometric quantities asociated to the metric as it evolves.
- With the evolution equation for the metric we can calculate the evolution for all related quantities:

$$
\begin{aligned}
\frac{\partial}{\partial\lambda}\Gamma^k_{ij} &= -g^{kl}\left(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}\right) \\
\frac{\partial}{\partial\lambda} R &= \Delta R + 2|Rc|^2 \\
\frac{\partial}{\partial\lambda} R_{ij} &= \Delta R_{ij} + 2R_{kijl}R_{kl} - 2R_{ik}R_{jk} \\
\frac{\partial}{\partial\lambda} R_{ijkl} &= \Delta R_{ijkl} + 2\left(B_{ijkl} - B_{ikjl} - B_{iljk}\right) \\
&\quad - \left(R_{ip}R_{pjkl} + R_{jp}R_{ipkl} + R_{kp}R_{ijpl} + R_{lp}R_{ijkp}\right)
\end{aligned}
$$

where

$$B_{ijkl} = -R_{pijq}R_{qlkp}$$

Gradient formulation

- If we define [Perelman, 2002]

$$\mathscr{F}(g,f) := \int_M \left(R + |\nabla f|^2\right) e^{-f} d\mu$$

then the gradient flow for $\mathscr{F}$, under the constraint that $e^{-f} d\mu$ is fixed, is

$$\begin{aligned} \frac{\partial}{\partial \lambda} g_{ij} &= -2\left(R_{ij} + \nabla_i \nabla_j f\right) \\ \frac{\partial}{\partial \lambda} f &= -R - \Delta f \end{aligned}$$

Moreover

$$\frac{\partial}{\partial \lambda} \mathscr{F}(g(\lambda), f(\lambda)) = 2\int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu \geq 0$$

Steady Ricci solitons

$$R_{ij} + \nabla_i \nabla_j f = 0$$

Shrinking Solitons and W-entropy

Theorem Perelman 2002

Let M be a compact manifold. Let $g(\lambda), f(\lambda), \tau(\lambda), \lambda \in [0, \lambda_0]$ be the solution of

$$\begin{aligned} \frac{\partial g_{ij}}{\partial \lambda} &= -2R_{ij} \\ \frac{\partial f}{\partial \lambda} &= -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} \\ \frac{\partial \tau}{\partial \lambda} &= -1 \end{aligned}$$

and

$$\mathscr{W} = (g, u, \tau) = \int_M (\tau(|\nabla log\, u|^2 + R) - log((4\pi\tau)^{n/2} u) - n) u \, d\mu$$

Then

$$\frac{\partial}{\partial \lambda} \mathscr{W}(g, f, \tau) = 2\tau \int_M |R_{ij} + \nabla_i \nabla_j f - \frac{g}{2\tau}|^2 \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} d\mu$$

$\mathscr{W}(g, f, \tau)$ is nondecreasing under Ricci flow.

Shrinking solitons

$$R_{ij} + \nabla_i \nabla_j f = \frac{g}{2\tau}$$

Shrinking Solitons and W-entropy

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Shrinking solitons

$$R_{ij} + \nabla_i \nabla_j f = \frac{g}{2\tau}$$

Why the name **entropy**?

Statistical definition of entropy

Entropy

$$S = k \log W$$

Where k is the Boltzman constant and W is the number of possible microstates corresponding to the macroscopic state of a system.

Statistical Mechanics

$$S = \lim_{N \to \infty} \frac{\log W}{N} = \log Z_\beta - \beta \frac{\partial}{\partial \beta} \log Z_\beta$$

Something Remarkable

- Let M be a closed manifold, g_{ij} and f be the solution of the modified Ricci flow equation and the conjugate heat equation.
- Let's assume that there exists a canonical ensemble with a "density of states" measure for which the partition function is given by

$$\log Z_\beta = \int_M \left(\frac{n}{2} - f\right) (4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu$$

Where $\beta^{-1} = \tau = T - \lambda, \lambda \in (0, \lambda_0)$

W is an entropy

Using the Boltzman entropy formula, Perelman showed that

$$S = -\int_M (\tau(R + |\nabla f|^2 + f - n))(4\pi\tau)^{-\frac{n}{2}} e^{-f} d\mu$$

Therefore

$$W = -S$$

(Carfora and Guenther 2018) "Scale invariant" (α is not a fixed constant) RG-2 flow. Flow coupled to a backwards Fokker-Plack equation.

A probabilistic interpretation

On complete Riemannian manifold. Assume that (h_{ab}, f) is the solution of the Ricci gradient flow system. Then

$$\log Z_\beta = \frac{n}{2}(1 + \log(4\pi\tau)) - H,$$

where

$$H = \int_M u \log u dv = \int_M \left(f + \frac{n}{2}\log(4\pi\tau)\right)\frac{e^{-f}}{(4\pi\tau)^{n/2}}.$$

This H is called the Boltzman-Shannon entropy of the measure udv with $u = \frac{e^{-f}}{(4\pi\lambda)^{n/2}}$

Boltzman-Nash-Shanon entropy of the Gaussian heat kernel measure on $\mathbb{R}^n$

$$H(\gamma_\tau^n) = -\frac{n}{2}(1 + \log(4\pi\tau)).$$

$\mathscr{W}$-entropy

$$W = \frac{d}{d\tau}(\tau(H(\gamma_\tau^n) - H))$$

Defining $F = -\tau \log Z$ the entropy becomes $\mathscr{W} = -\frac{\partial F}{\partial \tau}$, F is called the Helmholtz free energy function. (Xian-Dong Li, 2013)

The entropy formula for the Ricci flow and its geometric applications.
Grigori Perelman 2008

"The interplay of statistical physics and (pseudo)-riemannian geometry occurs in the subject of Black Hole Thermodynamics, developed by Hawking et al. Unfortunately, this subject is beyond my understanding at the moment."

Heterotic Ricci flow

$$\frac{\partial g}{\partial t} = -2\left(R_{ij} + \nabla_i \nabla_j f - \frac{1}{4} H_{ikl} H_j^{kl}\right)$$
$$\frac{\partial B_{ij}}{\partial t} = \nabla^k H_{kij} - H_{kij} \nabla^k f$$
$$\frac{\partial f}{\partial t} = -\Delta f - R + \frac{1}{4}|H|^2$$

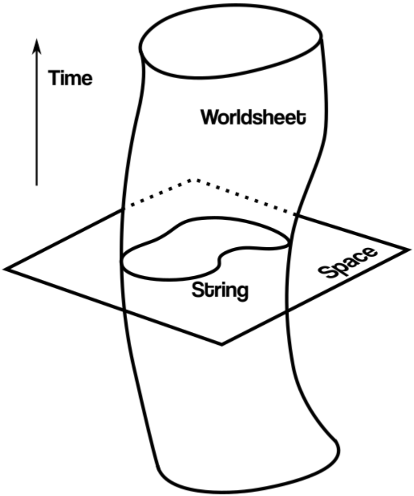
Generalized $\mathscr{F}$ entropy

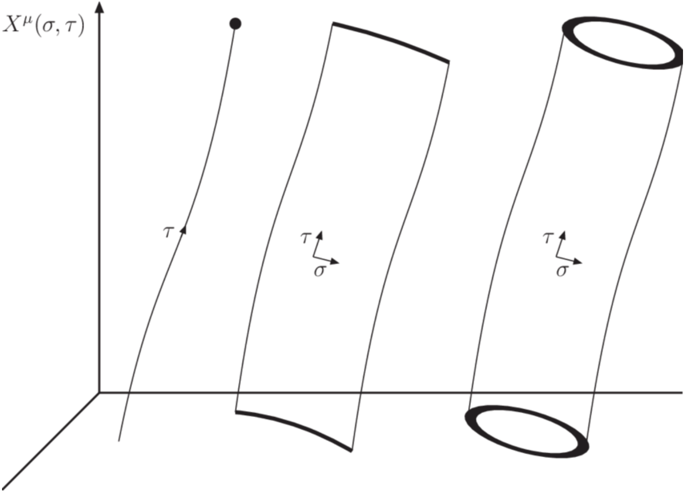
$$F(g_{ij}, B_{ij}, f) = \int_M \left(R + |\nabla f|^2 - \frac{1}{2}|H|^2\right) e^{-f} dV$$

Proposition

The generalized entropy $\mathscr{F}$ is monotonous under the Heterotic Ricci flow

$$\frac{dF}{dt} = \int_M \left(2|R_{ij} - \frac{1}{4} H_{ikl} H_j^{kl} + \nabla_i \nabla_j f|^2 + \frac{1}{2}|\nabla^k H_{kij} - H_{kij} \nabla^k f|^2\right) e^{-f} dV$$


Time
Worldsheet
String
Space



Marques N.

Non-linear sigma models

- Let (M, g) be a Riemannian manifold and let Σ be a two dimensional Riemannian surface with metric γ . We define a map $\phi : \Sigma \to M$ and the functional

$$S(\phi) = \int \frac{1}{\alpha} g_{ij}(\phi(x)) \partial^{\mu} \phi^{i}(x) \partial^{\nu} \phi^{j}(x) \gamma_{\mu\nu} dx$$

with $\alpha > 0$.

- The Kind of classical theories described by the above functional are called non-linear sigma models. (Σ, γ) is called the worldsheet and (M, g) is the target space.
- In the process of quantizing the action the world-sheet nonlinear sigma model renormalization group appears. This process of quantization requires the introduction of a cutoff Λ which parametrizes a family of quantum field theories.

The RG-2 flow

The second order approximation in λ to the RG flow is

$$\underbrace{\frac{\partial g_{ij}}{\partial \lambda} = -2R_{ij}}_{\textit{Ricci flow}} - \frac{\alpha}{2} R_{iklm} R_{j}^{\ klm}$$

If M is a smooth $n-$ dimensional manifold, the space of all Riemannian metrics $\mathscr{M} = \mathscr{M}(M)$ is a infinite dimensional smooth manifold modeled on the space of symmetric $(0,2)-$ tensor fields with compact support. A smooth Riemannian metric can be defined by

$$G_g(h,k) = \int_M tr(g^{-1}hg^{-1}k)dV$$

Remember the Shannon-Boltzman entropy

$$\frac{d}{dt}S_{BS} = \mathscr{P}$$

$$S_{BH} = \int_0^t \mathscr{P}(s)ds$$

We can associate a Gaussian probability measure $\beta(t)$ to every metric solution $g_{ij}(t)$,

$$\Delta^{\mathscr{P}}_{g(t_1),g(t_2)} = \mathscr{W}_2(\beta(t_1), \beta(t_2))\, t_1, t_2 \in [0, T)$$

where $\mathscr{W}_2$ is the Wasserstein distance of order 2.

$$lim_{t\to\infty}\Delta_{g(t_1),g(t_2)} = \frac{1}{\sqrt{2\pi}} e^{\int_0^{t_0} \mathscr{P}(s)ds}$$

Ongoing work.......

Thank you!