Exercises for tutorial on "Non-linear Dynamics" at the CAS 2011 on Chios

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1 Exercise 1

1.1 Problem:

a) Compute the map:

$$X(L) = ?$$

$$P(L) = X'(L) = ?$$

for a thick sextupole (1D) (length L, strength k) with the equation of motion:

$$x'' = -k \cdot x^2$$

up to order $\mathcal{O}(L^2)$, using the symplectic integration method.

b) Compute the map:

$$X(L) = ?$$

for a thick sextupole (2D) with the Hamiltonian (to give the equation of motion above):

$$H = \frac{1}{3}k(x^3 - 3xy^2) + \frac{1}{2}(p_x^2 + p_y^2)$$

using the Lie transformation method, compare with the solution from a).

1.2 Solution:

a) A solution to order $\mathcal{O}(L^2)$ is given by a thin lens approximation with a single kick in the centre of the element. The map can be written as a "leap-frog" integration:

$$\begin{aligned} x(L) &\approx x_0 + \frac{L}{2}(x'_0 + x'(L)) \\ x'(L) &\approx x'_0 + Lf(x_0 + \frac{L}{2}x'_0) \end{aligned}$$

For the sextupole with:

$$x'' = -k \cdot x^2 = f(x)$$

using the thin lens approximation (type D in the lecture) gives:

$$\begin{aligned} x(L) &= x_0 + x'_0 L - \frac{1}{2}kx_0^2 L^2 - \frac{1}{2}kx_0 x'_0 L^3 - \frac{1}{8}kx_0'^2 L^4 \\ x'(L) &= x'_0 - kx_0^2 L - kx_0 x'_0 L^2 - \frac{1}{4}kx_0'^2 L^3 \end{aligned}$$

Map for thick sextupole of length L in thin lens approximation, accurate to $\mathcal{O}(L^2)$

b) In the case an element is described by a Hamiltonian H, the Lie map of an element of length L and the Hamiltonian H is:

$$e^{-L:H:} = \sum_{i=0}^{\infty} \frac{1}{i!}(-L : H :)^i \tag{1}$$

For example, the Hamiltonian for a thick sextupole is:

$$H = \frac{1}{3}k(x^3 - 3xy^2) + \frac{1}{2}(p_x^2 + p_y^2) \tag{2}$$

To find the transformation we search for:

$$e^{-L:H:}x \quad \text{and} \quad e^{-L:H:}p_x \qquad \text{i.e. for} \tag{3}$$

$$X(L) \;=\; e^{-L:H:}x = \sum_{i=0}^{\infty} \frac{1}{i!}(-L : H :)^i x \tag{4}$$

We can compute:

$$: H :^i x \tag{5}$$

for sufficiently large i:

$$: H :^0 x = x \tag{6}$$

$$: H :^1 x = \left(\frac{\partial H}{\partial x}\frac{\partial x}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial x}{\partial x}\right) \quad = \quad -\,p_x \tag{7}$$

$$: H :^2 x = : H :(-p_x) = \left(\frac{\partial H}{\partial x}\frac{\partial(-p_x)}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial(-p_x)}{\partial x}\right) \quad = \quad -\,k(x^2 - y^2) \tag{8}$$

$$: H :^3 x = : H :(-k(x^2 - y^2)) = \tag{9}$$

$$\left(\frac{\partial H}{\partial x}\frac{\partial(-k(x^2 - y^2))}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial(-k(x^2 - y^2))}{\partial x}\right) \quad = \quad 2kxp_x$$

The same for y to get $2kyp_y$ and we have:

$$: H :^3 x = 2k(xp_x - yp_y) \tag{10}$$

$$.... \tag{11}$$

then we obtain:

$$X(L) \;=\; e^{-L:H:}x = x + p_x L - \frac{1}{2}kL^2(x^2 - y^2) - \frac{1}{3}kL^3(xp_x - yp_y) + ... \tag{12}$$

Comparison with the leap-frog algorithm shows deviation of order $\mathcal{O}(L^3)$.

2 Exercise 2

2.1 Problem:

Starting from the transfer matrix, derive the Lie operators representing:
a) a thick, focusing quadrupole
b) a thick, defocusing quadrupole

2.2 Solution:

a) The matrix for a focusing quadrupole is:

$$\mathcal{M}_s = \begin{pmatrix} \cos L \cdot K & \frac{1}{K} \cdot \sin L \cdot K \\ -K \cdot \sin L \cdot K & \cos L \cdot K \end{pmatrix}$$

b) The matrix for a defocusing quadrupole is:

$$\mathcal{M}_s = \begin{pmatrix} \cosh L \cdot K & \frac{1}{K} \cdot \sinh L \cdot K \\ K \cdot \sinh L \cdot K & \cosh L \cdot K \end{pmatrix}$$

The map is represented like (see lecture):

$$e^{:f:} \quad \leftrightarrow \quad e^{SF} \quad = \quad \exp \begin{pmatrix} b & c \\ -a & -b \end{pmatrix} = a_0 + a_1 \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}$$

Given a quadratic form of the type:

$$f_2 = ax^2 + 2bxp + cp^2$$

we know from the lecture that:

$$e^{SF} \quad = \quad \cos(\sqrt{ac - b^2}) + \frac{\sin(\sqrt{ac - b^2})}{\sqrt{ac - b^2}} \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}$$

For a general 2×2 matrix:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

we get by comparison:

$$\cos(\sqrt{ac - b^2}) = \frac{1}{2} tr(M)$$

and

$$\frac{a}{-m_{21}} = \frac{2b}{m_{11} - m_{22}} = \frac{c}{m_{12}} = \frac{\sqrt{ac - b^2}}{\sin(\sqrt{ac - b^2})}$$

for a Lie form of a map of the type:

$$e^{:f_2:} = e^{:ax^2 + 2bxp + cp^2:}$$

For a focusing quadrupole we find:
$a = k^2 L, \quad b = 0, \quad c = L$ and we have:
$f_2 = -\frac{L}{2}(k^2 x^2 + p^2)$

For a defocusing quadrupole we find:
$a = -k^2 L, \quad b = 0, \quad c = L$ and we have:
$f_2 = \frac{L}{2}(k^2 x^2 - p^2)$

3 Exercise 3

3.1 Problem:

Assume a matrix M of the type:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

described by a generator f. Use the properties of Lie transforms to evaluate the effect of this matrix on the moments x^2, xp, p^2:

$$e^{:f:}x^2 = ?$$

$$e^{:f:}p^2 = ?$$

$$e^{:f:}xp = ?$$

3.2 Solution:

From the matrix M we can directly write:

$$e^{:f:}x = (m_{11}x + m_{12}p)$$

and

$$e^{:f:}p = (m_{21}x + m_{22}p)$$

We know from the lecture some properties of Lie transforms (see lecture) and:

$$e^{:f:}x^2 = (e^{:f:}x)^2$$

therefore:

$$(e^{:f:}x)^2 = (m_{11}x + m_{12}p)^2$$

$$(e^{:f:}x)^2 = m_{11}^2 x^2 \ + \ 2\ m_{11}m_{12}xp \ + \ m_{12}^2 p^2$$

We also can compute:

$$e^{:f:}p^2 = (e^{:f:}p)^2$$

therefore:

$$(e^{:f:}p)^2 = (m_{21}x + m_{22}p)^2$$

$$(e^{:f:}p)^2 = m_{21}^2 x^2 \ + \ 2\ m_{21}m_{22}xp \ + \ m_{22}^2 p^2$$

also for the moment xp:

$$e^{:f:}xp = (e^{:f:}x)(e^{:f:}p) \qquad \text{(see lecture)}$$

$$e^{:f:}xp = m_{11}m_{21}x^2 \ + \ (m_{11}m_{22} \ + \ m_{12}m_{21})xp \ + \ m_{12}m_{22}p^2$$

To summarize the moments we re-write the above in matrix form:

$$\begin{pmatrix} x^2 \\ xp \\ p^2 \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11}^2 & 2m_{11}m_{12} & m_{12}^2 \\ m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & m_{12}m_{22} \\ m_{21}^2 & 2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \circ \begin{pmatrix} x^2 \\ xp \\ p^2 \end{pmatrix}_{s_1}$$